18.03: Differential Equations

For practical information, see the course web site: http://math.mit.edu/~dyatlov/18.03/

§1.1. Overview

A differential equation is an equation with derivatives and the unknown is a function.

For now we will study $\boxed{\text{Ordinary Differential Equations}}$ (ODE for short) where the unknown is a function of 1 variable

Here is an example:

Example 1: $\boxed{y'(t) = 68 - y(t)}$.

Here $y(t)$ is the unknown function and $y'(t)$ is its derivative.

We also write the equation as

$$\boxed{\frac{dy}{dt} = 68 - y}$$

Here is what we can do with differential equations:

1. Modeling: write the equation which models a given situation
   In this course we only do simple modeling, more realistic cases belong to physics, chemistry, econ etc.
2. Finding all solutions to a given equation, i.e. finding a formula for the general solution.
3. Finding the solution satisfying additional initial or boundary conditions.
4. Understanding the qualitative behavior of a solution.
5. Solving differential equations numerically (this is what usually happens in real life)

Much of the course will focus on steps 2-3 but there will be some 1 and 4-5 as well

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Let me show what result steps 1-4 are supposed to yield on Example 1.
For now I will skip many details (that's what the rest of the course is about)

1. Modeling: one situation that the equation

$$y' = 68 - y$$

models is cooling of soup where
t = time (say in minutes) and
$y(t)$ = temperature of soup at time t (in °Fahrenheit)

In practice, we will be asked to obtain the equation from the description of the situation, in our example using Newton's Law of Cooling.

2 <u>General solution</u>: in our example it turns out to be given by the formula

$$\boxed{y(t) = 68 + ce^{-t}}$$

where c is an arbitrary constant.

Putting in different values of c gives different solutions, e.g.

$c = 5 \longrightarrow y(t) = 68 + 5e^{-t}$ is a solution

Let's check actually that this function solves the equation: if $y = 68 + 5e^{-t}$ then $y' = -5e^{-t}$, so $y' = 68 - y$ indeed

Here's another solution:

$c = 0 \longrightarrow y(t) = 68$

3 So which solution should we choose? We fix a unique solution by imposing <u>the initial condition</u>

$$y(0) = 212$$

i.e. the soup is boiling at time 0.

Plugging in the initial condition into the formula ~~for~~ for the general solution, we get $212 = y(0) = 68 + C \cdot e^{-0} = 68 + C$

So $C = 212 - 68 = 144$.

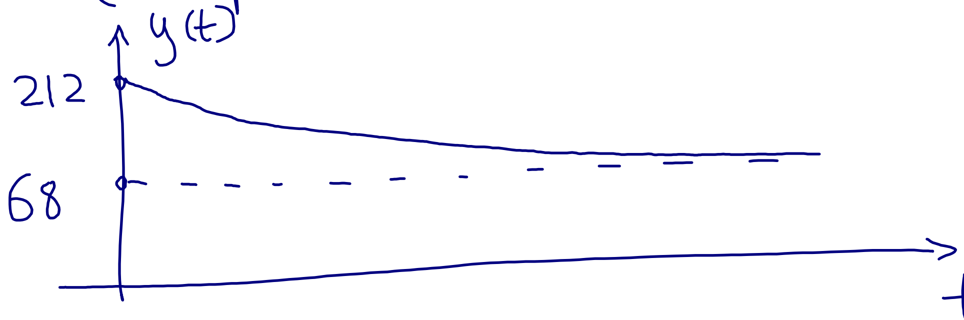
So our solution is

$$\boxed{y(t) = 68 + 144e^{-t}.}$$

4 For qualitative behavior, let's study the limit of $y(t)$ as $t \to \infty$:

$$y(t) \xrightarrow[t\to\infty]{} 68 \quad \text{as } e^{-t} \xrightarrow[t\to\infty]{} 0.$$

So y converges to the equilibrium point (soup eventually gets cold)



Note that the same equation can model different systems.

For example, we could use it to model warming of ice cream. Just change the initial condition to $y(0) = 10$ (say) and get $C = -58$, $y(t) = 68 - 58e^{-t}$

§1.1. Taxonomy of ODEs

Let's look at a few examples:

#	Equation	Name/meaning	General solution
1	$y' = 68 - y$	Cooling Population growth ...	$y = 68 + ce^{-t}$
2	$y'' + y = 0$	Harmonic oscillator Springs, dashpot ...	$y = C_1 \cos t + C_2 \sin t$ (C_1, C_2 any constants)
3	$y' = -ty$	Quantum harmonic oscillator	$y = Ce^{-t^2/2}$
4	$y' = y^2$	Doomsday (I made it up...)	$y = \frac{1}{C-t}$ or $y = 0$

Now a few definitions & general observations:

- The order of an ODE is the longest number of times the unknown function $y(t)$ is differentiated

  Note that in the above examples the general solution depends on k constants where k is the order.

- A linear homogeneous equation is one of the form

  $$a_k(t)y^{(k)}(t) + \dots + a_1(t)y'(t) + a_0(t)y(t) = 0$$

where $a_0, \dots, a_k$ are given functions

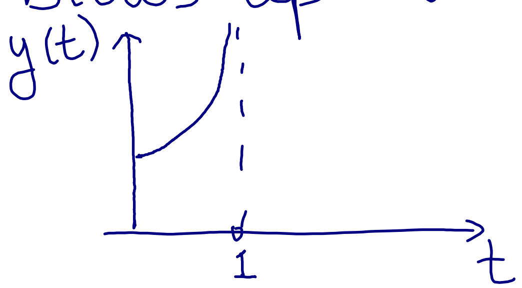
- A <u>linear inhomogeneous</u> equation has the form

$$a_k(t)\,y^{(k)}(t) + \cdots + a_1(t)\,y'(t) + a_0(t)\,y(t) = b(t)$$

where $a_0, \ldots, a_k, b$ are given functions

- For linear equations, we say they have constant coefficients if
  $a_0, \ldots, a_k$ are constants
  (but b does not need to be constant)
- Nonlinear equations might have solutions which blow up.

e.g. Example 4: $y' = y^2$,
one solution is $y(t) = \dfrac{1}{1-t}$,
blows up at $t = 1$ ("doomsday")



For the above examples:

#	Equation	Order	Adjectives
1	$y' = 68 - y$	1	linear inhomogeneous constant coefficient
2	$y'' + y = 0$	2	linear homogeneous constant coefficient
3	$y' = -ty$	1	linear homogeneous variable coefficient
4	$y' = y^2$	1	nonlinear