18.03: Differential Equations

For practical information, see syllabus on Stellar / Learning Modules

§1.1. Overview

A differential equation is an equation with derivatives and the unknown is a function.

For now we will study $\boxed{\text{Ordinary Differential Equations}}$ (ODE for short) where the unknown is a function of 1 variable

Here is an example:

Example 1: $\boxed{y'(t) = 68 - y(t).}$

Here $y(t)$ is the unknown function.
and $y'(t)$ is its derivative.
We also write the equation as

$$\boxed{\frac{dy}{dt} = 68 - y}$$

Here is what we can do with differential equations:

1 Modeling: write the equation modeling a given situation

In this course we only do simple modeling, more realistic cases belong to physics, chemistry, econ etc.

2 Finding all solutions to a given equation, i.e. finding a formula ~~for~~ the general solution.

3 Finding the solution satisfying additional initial or boundary conditions.

4 Understanding the qualitative behavior of a solution.

Much of the course will focus on steps 2 - 3 but there will be some 1 and 4 as well.

Let me show what result steps 1 - 4 are supposed to yield on Example 1.

For now I will skip many details (that's what the rest of the course is about)

1 Modeling: one situation that the equation $y' = 68 - y$ models is cooling of soup where

t = time (say in minutes) and

$y(t)$ = temperature of soup at time t (in °Fahrenheit)

In practice, we will be asked to obtain the equation from the description of the situation, in our example using Newton's Law of Cooling.

2 <u>General solution</u>: in our example it turns out to be given by the formula

$$\boxed{y(t) = 68 + ce^{-t}}$$

where c is an arbitrary constant. Putting in different values of c gives different solutions, e.g.

$c = 5 \longrightarrow y(t) = 68 + 5e^{-t}$ is a solution

Let's check actually that this function solves the equation: if $y = 68 + 5e^{-t}$ then $y' = -5e^{-t}$, so $y' = 68 - y$ indeed

Here's another solution:

$c = 0 \longrightarrow y(t) = 68$

3 So which solution should we choose? We fix a unique solution by imposing the <u>initial condition</u>

$$y(0) = 212$$

i.e. the soup is boiling at time 0.

Plugging in the initial condition into the formula ~~for~~ for the general solution, we get $212 = y(0) = 68 + c \cdot e^{-0} = 68 + c$

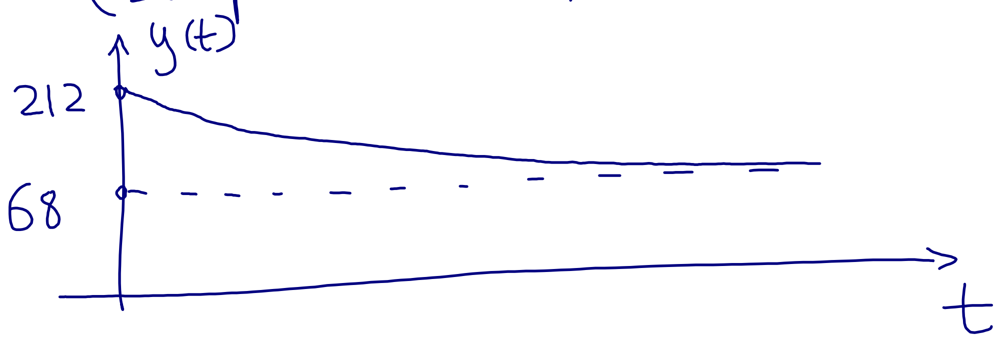
So $c = 212 - 68 = 144$.

So our solution is $\boxed{y(t) = 68 + 144 e^{-t}}$.

④ For qualitative behavior, let's study the <u>limit</u> of $y(t)$ as $t \to \infty$:

$$y(t) \xrightarrow[t \to \infty]{} 68 \quad \text{as} \quad e^{-t} \xrightarrow[t \to \infty]{} 0.$$

So y converges to the equilibrium point (soup eventually gets cold)



Note that the same equation can model different systems.

For example, we could use it to model warming of ice cream. Just change the initial condition to $y(0) = 10$ (say) and get $c = -58$, $y(t) = 68 - 58 e^{-t}$