18.125 Homework 9

due Wed Apr 13 in class

1. (2 pts) Assume that $M \subset \mathbb{R}^N$ is a hypersurface and (Ψ, U) is a coordinate chart for M. Assume also that $U' \subset \mathbb{R}^{N-1}$ is an open set and $\Phi : U' \to U$ is a diffeomorphism, and put $\Psi' := \Psi \circ \Phi$; then (Ψ', U') is another coordinate chart for M. Without using the surface measure λ_M, show that for each Borel measurable nonnegative $f : M \to \mathbb{R}$,

$$\int_U (f \circ \Psi) J_\Psi \, d\lambda_{\mathbb{R}^{N-1}} = \int_{U'} (f \circ \Psi') J_{\Psi'} \, d\lambda_{\mathbb{R}^{N-1}}$$

where $J_\Psi, J_{\Psi'}$ are defined by (5.2.12). (Hint: use Jacobi's formula and the fact that $J_\Psi(y) = \sqrt{\det(d\Psi(y)^T d\Psi(y))}$.)

2. (2 pts) Do Exercise 6.1.6.

3. (2 pts) Do Exercise 6.1.7.

4. (2 pts) Do Exercise 6.1.9.

5. (1 pt) Let $(E, \mathcal{B}, \mu)$ be a measure space and $f : E \to \mathbb{R}$ a measurable function. Show that for all $p \in [1, \infty)$,

$$\big(\|f\|_{L^p}\big)^p = \int_0^\infty p t^{p-1} \mu\big(\{x \in E \colon |f(x)| \geq t\}\big) \, dt.$$

6. (1 pt) Assume that $(E, \mathcal{B}, \mu)$ is a finite measure space and $f : E \to \mathbb{R}$ is a bounded measurable function. It is easy to see that $f \in L^p(E, \mu)$ for all p. Show that for any sequence $p_j \in [1, \infty]$ converging to some $p \in [1, \infty]$, we have $\|f\|_{L^{p_j}} \to \|f\|_{L^p}$.