18.125 Homework 6
due Wed Mar 16 in class

1. (1 pt) Assume that $f \in L^1(\mathbb{R}^N; \lambda_{\mathbb{R}^N})$ has the following property:

$$\int_{\mathbb{R}^N} f(x)\varphi(x)\,dx = 0$$

for each continuous compactly supported $\varphi : \mathbb{R}^N \to \mathbb{R}$. (Here compactly supported means that there exists a compact set $K \subset \mathbb{R}^N$ such that $\varphi = 0$ outside of K.) Show that $f = 0$ almost everywhere.

2. (1 pt) Assume that μ is a finite measure on $\mathbb{R}$. Show that μ can be written as a sum of two measures $\mu_1 + \mu_2$, where μ_1 has a continuous distribution function and μ_2 is purely atomic, namely there exist at most countably many $x_j \in \mathbb{R}$, $\rho_j \geq 0$ such that

$$\mu_2(A) = \sum_{j\colon x_j \in A} \rho_j, \quad A \in \mathcal{B}_{\mathbb{R}}.$$

3. (2 pts) Do Exercise 2.2.38.

4. (2 pts) Do Exercise 2.2.39 (correction: change $F(b_n)$ to $F(b_n-)$).

5. (3 pts) This exercise gives the construction of Cantor measures on Cantor sets.

For each $\theta \in (0,1)$ and a closed interval $I = [a,b]$, define closed intervals

$$J_-(I,\theta) = \Big[a, \frac{a+b}{2} - \theta\frac{b-a}{2}\Big], \quad J_+(I,\theta) = \Big[\frac{a+b}{2} + \theta\frac{b-a}{2}, b\Big]$$

so that I is a nonoverlapping union of $J_-(I,\theta)$, $J_+(I,\theta)$, and the middle part of length $\theta|I|$. Fix a sequence

$$\theta_j \in (0,1), \quad j = 1, 2, \dots$$

For each string $\alpha_1 \dots \alpha_n$ with $\alpha_1, \dots, \alpha_n \in \{-,+\}$, and $\emptyset$ denoting the empty string, define the intervals $I(\alpha_1 \dots \alpha_n)$ inductively by

$$I(\emptyset) = [0,1], \quad I(\alpha_1 \dots \alpha_n) = J_{\alpha_n}\big(I(\alpha_1 \dots \alpha_{n-1}), \theta_n\big).$$

For instance, $I(-+) = J_+(J_-([0,1],\theta_1),\theta_2)$. (It might be helpful to visualize these intervals as lying on a tree, with $I(\alpha_1 \dots \alpha_{n-1})$ being the parent of $I(\alpha_1 \dots \alpha_n)$.) Define the Cantor set:

$$\mathcal{C} = \bigcap_{n=1}^{\infty} \mathcal{C}_n, \quad \mathcal{C}_n = \bigcup_{\alpha_1,\dots,\alpha_n \in \{-,+\}} I(\alpha_1 \dots \alpha_n).$$

(a) Show that $\mathcal{C}$ is a closed uncountable set such that $\mathbb{R} \setminus \mathcal{C}$ is dense in $\mathbb{R}$. Show that $\mathcal{C}$ has Lebesgue measure zero if and only if the series $\sum_n \theta_n$ diverges.

(b) Show that there exists unique finite Borel measure μ on $\mathbb{R}$ such that

$$\mu(\mathbb{R} \setminus \mathcal{C}) = 0; \quad \mu(I_{\alpha_1 \dots \alpha_n}) = 2^{-n} \text{ for all } \alpha_1, \dots, \alpha_n \in \{-,+\}.$$

(Hint: construct the distribution function F_μ instead.) Show that F_μ is continuous.

(c) Show that μ is absolutely continuous with respect to the Lebesgue measure λ when $\sum_n \theta_n$ converges and μ is singular with respect to λ when $\sum_n \theta_n$ diverges. (Hint: for the first part, the density is a multiple of the indicator function of $\mathcal{C}$.)

6. (1 pt) Do Exercise 3.3.16.