18.125 Homework 5

due Wed Mar 9 in class

1. (1 pt) Give an example of a sequence of Lebesgue measurable functions $f_n : [0,1] \to \mathbb{R}$ such that $f_n \nearrow 0$, yet $\int_{\mathbb{R}} f_n(x)\,dx$ does not converge to 0.

2. (1 pt) Find a sequence of Lebesgue measurable functions $f_n : [0,1] \to [0,1]$ such that

$$\int_0^1 \underline{\lim}_{n\to\infty} f_n(x)\,dx \;<\; \underline{\lim}_{n\to\infty} \int_0^1 f_n(x)\,dx.$$

3. (2 pts) Assume that $J \subset \mathbb{R}^N$ is a rectangle and $f : J \to \mathbb{R}$ a Riemann integrable function. Construct a sequence of partitions (i.e. non-overlapping, finite, exact covers) $\mathcal{P}_k$ of J such that $\mathcal{P}_{k+1}$ is a refinement of $\mathcal{P}_k$ and the mesh size $\|\mathcal{P}_k\|$ converges to zero. Define the following functions on J:

$$f_k(x) = \begin{cases} \inf_I f(x), & \text{if } x \in I^\circ \text{ for some } I \in \mathcal{P}_k, \\ 0, & \text{otherwise;} \end{cases}$$

$$g_k(x) = \begin{cases} \sup_I f(x), & \text{if } x \in I^\circ \text{ for some } I \in \mathcal{P}_k, \\ 0, & \text{otherwise.} \end{cases}$$

Show that $f_k(x) \nearrow f(x)$ and $g_k(x) \searrow f(x)$ for Lebesgue almost every x. (Hint: bound the Lebesgue measure of the set $\{x \mid \lim_{k\to\infty}(g_k(x) - f_k(x)) > \frac{1}{m}\}$ for each m, by considering k for which $\mathcal{U}(f;\mathcal{P}_k) - \mathcal{L}(f;\mathcal{P}_k) < \frac{1}{m^2}$.) Use this to show that f is Lebesgue integrable on J and its Riemann and Lebesgue integrals coincide.

4. (2 pts) Let V be the set of all Lebesgue measurable functions $f : [0,1] \to \mathbb{R}$. Show that there exists no topology $\mathcal{T}$ on V such that $f_n \to f$ in the topology $\mathcal{T}$ if and only if $f_n(x) \to f(x)$ for Lebesgue almost every x. (Hint: assume such topology $\mathcal{T}$ exists. Take a sequence of functions $f_n(x)$ such that f_n converges to 0 in measure, but not almost everywhere. Show that there exists a neighborhood $\mathscr{U}$ of 0 with respect to $\mathcal{T}$ and a subsequence $f_{n_k} \notin \mathscr{U}$, and use Theorem 3.2.10 to reach a contradiction.)

5. (2 pts) Do Exercise 3.2.19.

6. (2 pts) Do Exercise 3.2.20.