18.125 Homework 4

due Wed Mar 2 in class

1. (2 pts) Let E_1, E_2 be topological spaces and endow $E_1 \times E_2$ with the product topology, namely open sets in $E_1 \times E_2$ are unions (not necessarily countable) of products $A_1 \times A_2$ where $A_j \subset E_j$ are open sets. Let $\mathcal{B}_{E_1}, \mathcal{B}_{E_2}, \mathcal{B}_{E_1 \times E_2}$ be the corresponding Borel σ-algebras. Show that (see the beginning of §3.1.1)

$$\mathcal{B}_{E_1} \times \mathcal{B}_{E_2} \ \subset \ \mathcal{B}_{E_1 \times E_2}.$$

Hint: one part of a possible proof uses the σ-algebra

$$\{A_1 \subset E_1 \mid A_1 \times A_2 \in \mathcal{B}_{E_1 \times E_2} \text{ for all open } A_2 \subset E_2\}.$$

2. (1 pt) Let $f, g : (E, \mathcal{B}) \to \overline{\mathbb{R}}$ be two nonnegative measurable functions. (Note we are using $\overline{\mathbb{R}}$ here.) Show that the function $f \cdot g$ is measurable directly (without appealing to the composition argument of §3.1.1) by writing each sublevel set $\{f \cdot g < y\}$ as a countable union of intersections of sublevel sets of f and g. Do the same for the function $\max(f, g)$.

3. (1 pt) Show that the map

$$(\overline{\mathbb{R}} \times \overline{\mathbb{R}}; \mathcal{B}_{\overline{\mathbb{R}}} \times \mathcal{B}_{\overline{\mathbb{R}}}) \to (\overline{\mathbb{R}}; \mathcal{B}_{\overline{\mathbb{R}}}), \quad (x, y) \mapsto x \cdot y$$

is measurable. (You may use Lemma 3.1.1.)

4. (2 pts) Assume that (E, B, μ) is a finite measure space and $f : E \to \overline{\mathbb{R}}$ is a nonnegative measurable function whose range $f(E)$ is a countable set. Prove that the Lebesgue integral of f is given by the sum of the series

$$\int_E f\, d\mu = \sum_{a \in f(E)} a \cdot \mu\big(\{f = a\}\big).$$

5. (2 pts) Do Exercise 3.1.13.

6. (2 pts) Do Exercise 3.1.14.