18.125 Homework 12
due Wed May 4 in class

1. (2 pts) Do Exercise 7.1.9.

2. (1 pt) Do Exercise 7.1.11.

3. (1 pt) Do Exercise 7.2.12.

4. (1 pt) Do Exercise 7.2.13.

5. (1 pt) Do Exercise 7.2.14.

6. (1 pt) Do Exercise 7.3.31.

7. (3 pts) This exercise outlines an alternative proof of Fourier inversion formula for the special class of Schwartz functions on $\mathbb{R}$. For $f \in C^\infty(\mathbb{R})$, we say that f is a Schwartz function, and write $f \in \mathscr{S}(\mathbb{R})$, if for each multiindices $j, k \in \mathbb{N}_0$, with d_x denoting differentiation,

$$\sup_{x \in \mathbb{R}} |x^j d_x^k f(x)| < \infty.$$

(a) Derive the identities (for the first one, integrate by parts)

$$\widehat{d_x f}(\xi) = -2\pi i \xi \hat{f}(\xi), \quad d_\xi \hat{f}(\xi) = 2\pi i \, \widehat{x f}(\xi)$$

for all $f \in \mathscr{S}(\mathbb{R})$ and use them to show that

$$f \in \mathscr{S}(\mathbb{R}) \implies \hat{f} \in \mathscr{S}(\mathbb{R}).$$

(b) Assume that $f \in \mathscr{S}(\mathbb{R})$ and $f(0) = 0$. Show that there exists $g \in \mathscr{S}(\mathbb{R})$ such that

$$f(x) = x g(x).$$

(Hint: to show that g is smooth at 0, apply the fundamental theorem of calculus to the function $t \in [0, 1] \mapsto f(tx)$ to find $g(x) = \int_0^1 f'(tx)\, dt$.) Deduce from here that

$$\int_{\mathbb{R}} \hat{f}(\xi)\, d\xi = 0. \tag{1}$$

(c) Now let $f \in \mathscr{S}(\mathbb{R})$ be arbitrary and fix $x \in \mathbb{R}$. Applying (1) to the function $y \mapsto f(x + y) - e^{-\pi y^2} f(x)$ and using Lemma 7.3.5, deduce the Fourier inversion formula:

$$f(x) = \int_{\mathbb{R}} e^{-2\pi i x \xi} \hat{f}(\xi)\, d\xi.$$