The Central Limit Theorem, Stirling's Formula, and All That

Euler's Gamma Function: For $\gamma > 0$, set

$$\Gamma(\gamma) = \int_0^\infty t^{\gamma-1} e^{-t}\, dt. \tag{1}$$

Using integration by parts, one has

$$\Gamma(\gamma+1) = \int_0^\infty t^\gamma e^{-t}\, dt = -t^\gamma e^{-t}\Big|_0^\infty + \gamma \int_0^\infty t^{\gamma-1} e^{-t}\, dt$$

and therefore

$$\Gamma(\gamma+1) = \gamma\Gamma(\gamma). \tag{2}$$

In particular, because $\Gamma(1) = 1$ and $\Gamma(n+1) = n\Gamma(n)$, it follows, by induction on $n \geq 1$, that

$$\Gamma(n) = (n-1)! \quad \text{for } n \geq 1, \tag{3}$$

where we have adopted the convention that $0! = 1$. Evaluation of $\Gamma(\gamma)$ for non-integer γ's is more challenging. For example,

$$\Gamma\big(\tfrac{1}{2}\big) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt \overset{\tau=(2t)^{\frac{1}{2}}}{=} 2^{\frac{1}{2}} \int_0^\infty e^{-\frac{\tau^2}{2}}\, d\tau = 2^{-\frac{1}{2}} \int_{-\infty}^\infty e^{-\frac{\tau^2}{2}}\, d\tau,$$

and

$$\left(\int_{-\infty}^\infty e^{-\frac{\tau^2}{2}}\, d\tau\right)^2 = \iint\limits_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dxdy = \int_0^{2\pi} \left(\int_0^\infty re^{-\frac{r^2}{2}}\, dr\right) d\theta = 2\pi.$$

Hence,

$$\int_{-\infty}^\infty e^{-\frac{t^2}{2}}\, dt = \sqrt{2\pi} \quad \text{and} \quad \Gamma\big(\tfrac{1}{2}\big) = \sqrt{\pi}. \tag{4}$$

Random Variables: We say that $f : \mathbb{R} \longrightarrow [0,\infty)$ is a *probability density* if $\int_{-\infty}^\infty f(x)\, dx = 1$. Given a probability density f and $-\infty \leq a < b \leq \infty$, we say that the *random variable* X has density f if the probability $\mathbb{P}(a \leq X \leq b)$ that X lies in the interval $[a,b]$ is given by $\int_a^b f(x)\, dx$. More generally, if $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ and X has density f, the *expected value* $\mathbb{E}[\varphi(X)]$ of the random variable $\varphi(X)$ is given by $\int_{-\infty}^\infty \varphi(x) f(x)\, dx$. Note that $\mathbb{P}(a \leq X \leq b)$ is the expected value of $\mathbf{1}_{[a,b]}(X)$, where $\mathbf{1}_{[a,b]}(x)$ equals 1 or 0 depending on whether x is or is not in the interval $[a,b]$.

Two important quantities associated with a random variable x are its expectation value $\mathbb{E}[X]$ and its *variance*

$$\mathrm{Var}(X) \equiv \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] = \mathbb{E}[X^2] - \mathbb{E}[X]^2, \tag{5}$$

where the last equality is a consequence of

$$\mathbb{E}\big[(X-\mathbb{E}[X])^2\big] = \mathbb{E}\big[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2\big] = \mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2.$$

The expected value of X should be thought of as the most *typical* value of X, and its variance is a measure of the amount by which X differs from its expected value. When X has density f,

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} xf(x)\,dx \quad \text{and}\\ \operatorname{Var}(X) &= \int_{-\infty}^{\infty} \big(x-\mathbb{E}[X]\big)^2 f(x)\,dx = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \left(\int_{-\infty}^{\infty} xf(x)\,dx\right)^2.\end{aligned} \tag{6}$$

An important observation, known as *Chebychev's inequality*, is that

$$\mathbb{P}\big(|X-\mathbb{E}[X]|\geq R\big) \leq R^{-2}\operatorname{Var}(X). \tag{7}$$

To prove (7), observe that

$$\begin{aligned}\mathbb{P}\big(|X-\mathbb{E}[X]|\geq R\big) &= \mathbb{E}\Big[\mathbf{1}_{[R,\infty)}\big(\big|X-\mathbb{E}[X]\big|\big)\Big] \leq \mathbb{E}\left[\frac{(X-\mathbb{E}[X])^2}{R^2}\mathbf{1}_{[R,\infty)}\big(\big|X-\mathbb{E}[X]\big|\big)\right]\\ &\leq \mathbb{E}\left[\frac{(X-\mathbb{E}[X])^2}{R^2}\right] = \frac{\operatorname{Var}(X)}{R^2}.\end{aligned}$$

Gamma and Normal Random Variables: Clearly, for each $\gamma>0$, the function

$$f_\gamma(x) = \begin{cases} \Gamma(\gamma)^{-1}x^{\gamma-1}e^{-x} & \text{if } x>0\\ 0 & \text{if } x\leq 0\end{cases}$$

is a probability density. A random variable for which f_γ is the density function is said to be a γ-Gamma random variable. If X_γ is such a random variable, then (cf. (2))

$$\mathbb{E}[X_\gamma] = \frac{1}{\Gamma(\gamma)}\int_0^\infty x^\gamma e^{-x}\,dx = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)} = \gamma,$$

and similarly

$$\mathbb{E}[X_\gamma^2] = \frac{\Gamma(\gamma+2)}{\Gamma(\gamma)} = \frac{(\gamma+1)\Gamma(\gamma+1)}{\Gamma(\gamma)} = (\gamma+1)\gamma.$$

Hence,

$$\mathbb{E}[X_\gamma] = \gamma = \operatorname{Var}(X_\gamma). \tag{8}$$

In particular,

$$\bar{X}_\gamma \equiv \frac{X_\gamma-\gamma}{\gamma^{\frac12}} \implies \mathbb{E}[\bar{X}_\gamma]=0 \quad \text{and} \quad \operatorname{Var}(\bar{X}_\gamma)=1.$$

Next, by the first calculation in (4), we see that $g(x)=(2\pi)^{-\frac12}e^{-\frac{x^2}{2}}$ is a probability density. In fact, g may be the most important probability density function, an assertion which is reflected in the fact that a random variable with density g is called a *standard normal* random variable. The origin of its importance is a general result, known as the *Central Limit Theorem*, which says that, with remarkable frequency, a standard normal random variable will be a good approximation for any random variables which has expectation value 0 and variance 1. (In Exercise 3 below, you will be asked to verify, among other things, that 0 and 1 are the expectation value and variance of a standard normal random variable.) In the present setting, the central limit theorem says that, for any $-\infty<a<b<\infty$,

$$\begin{aligned}\lim_{\gamma\to\infty}\frac{1}{\Gamma(\gamma)}\int_{\gamma+\gamma^{\frac12}a}^{\gamma+\gamma^{\frac12}b} t^{\gamma-1}e^{-t}\,dt &= \lim_{\gamma\to\infty}\mathbb{P}\big(a\leq \bar{X}_\gamma\leq b\big)\\ &= \mathbb{P}\big(a\leq Y\leq b\big) = (2\pi)^{-\frac12}\int_a^b e^{-\frac{x^2}{2}}\,dx,\end{aligned} \tag{9}$$

where Y is a standard normal random variable. The rest of this lecture will be devoted to proving (9), and, along the way, we will give a proof of Stirling's formula (cf. (11) below).

Stirling's Formula: We begin by noting that, by (7), for $R > 0$ and $\gamma > R^2$,

$$1 - \frac{1}{R^2} \le \mathbb{P}\big(-R \le \bar{X}_\gamma \le R\big) \le 1$$

and

$$\begin{aligned}\mathbb{P}\big(-R \le \bar{X}_\gamma \le R\big) &= \mathbb{P}\big(\gamma - \gamma^{\frac{1}{2}}R \le X_\gamma \le \gamma + \gamma^{\frac{1}{2}}R\big) = \frac{1}{\Gamma(\gamma)} \int_{\gamma-\gamma^{\frac{1}{2}}R}^{\gamma+\gamma^{\frac{1}{2}}R} t^{\gamma-1}e^{-t}\,dt \\ &= \frac{e^{-\gamma}}{\Gamma(\gamma)} \int_{-\gamma^{\frac{1}{2}}R}^{\gamma^{\frac{1}{2}}R} (\gamma+t)^{\gamma-1}e^{-t}\,dt = \frac{\gamma^{\gamma-1}e^{-\gamma}}{\Gamma(\gamma)} \int_{-\gamma^{\frac{1}{2}}R}^{\gamma^{\frac{1}{2}}R} (1+\gamma^{-1}t)^{\gamma-1}e^{-t}\,dt \\ &= \frac{\gamma^{\gamma-\frac{1}{2}}e^{-\gamma}}{\Gamma(\gamma)} \int_{-R}^{R} (1+\gamma^{-\frac{1}{2}}t)^{\gamma-1}e^{-\gamma^{\frac{1}{2}}t}\,dt.\end{aligned}$$

Hence, if

$$\mathcal{S}(\gamma) \equiv \frac{\gamma^{\gamma-\frac{1}{2}}e^{-\gamma}}{\Gamma(\gamma)},$$

then, for any $0 < R \le \gamma^{\frac{1}{2}}$,

$$1 - \frac{1}{R^2} \le \mathcal{S}(\gamma) \int_{-R}^{R} \big(1+\gamma^{-\frac{1}{2}}t\big)^{\gamma-1}e^{-\gamma^{\frac{1}{2}}t}\,dt \le 1. \tag{10}$$

To complete the derivation of Stirling's formula starting from (10), use Taylor's theorem to see that

$$\log(1+x) = x - \frac{x^2}{2} + E(x) \quad \text{where } 0 \le \frac{E(x)}{x^3} \le 1 \text{ for } 0 < |x| \le \frac{2}{3}. \tag{11}$$

Hence, for $R \le \frac{2\gamma^{\frac{1}{2}}}{3}$,

$$\big(1+\gamma^{-\frac{1}{2}}t\big)^{\gamma}e^{-\gamma^{\frac{1}{2}}t} = \exp\Big[\gamma\log\big(1+\gamma^{-\frac{1}{2}}t\big) - \gamma^{\frac{1}{2}}t\Big] = \exp\Big[-\frac{t^2}{2} + \gamma E\big(\gamma^{-\frac{1}{2}}t\big)\Big],$$

and so (cf. (4))

$$\int_{-R}^{R} \big(1+\gamma^{-\frac{1}{2}}t\big)^{\gamma-1}e^{-\gamma^{\frac{1}{2}}t}\,dt \le \big(1-\gamma^{-\frac{1}{2}}R\big)^{-1}e^{\gamma^{-\frac{1}{2}}R^3} \int_{-R}^{R} e^{-\frac{t^2}{2}}\,dt \le \sqrt{2\pi}\big(1-\gamma^{-\frac{1}{2}}R\big)^{-1}e^{\gamma^{-\frac{1}{2}}R^3}$$

and

$$\int_{-R}^{R} \big(1+\gamma^{-\frac{1}{2}}t\big)^{\gamma-1}e^{-\gamma^{\frac{1}{2}}t}\,dt \ge \big(1+\gamma^{-\frac{1}{2}}R\big)^{-1}e^{-\gamma^{-\frac{1}{2}}R^3} \int_{-R}^{R} e^{-\frac{t^2}{2}}\,dt \ge \sqrt{2\pi}\frac{1-R^{-2}}{1+\gamma^{-\frac{1}{2}}R}e^{-\gamma^{-\frac{1}{2}}R^3},$$

since, by (7),

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-\frac{t^2}{2}}\,dt = \mathbb{P}\big(|Y| \le R\big) = 1 - \mathbb{P}\big(|Y| \ge R\big) \ge 1 - R^{-2}.$$

After plugging these into (10), we arrive at

$$\big(1-R^{-2}\big)\big(1-\gamma^{-\frac{1}{2}}R\big)e^{-\gamma^{-\frac{1}{2}}R^3} \le \sqrt{2\pi}\mathcal{S}(\gamma) \le \big(1-R^{-2}\big)^{-1}\big(1+\gamma^{-\frac{1}{2}}R\big)e^{\gamma^{-\frac{1}{2}}R^3}$$

for all $R \le \frac{2\gamma^{\frac{1}{2}}}{3}$. In particular, by taking $R = \gamma^{\frac{1}{9}}$ and letting $\gamma \to \infty$, we come to the conclusion that

$$\lim_{\gamma\to\infty} \frac{\sqrt{\frac{2\pi}{\gamma}}\,\big(\frac{\gamma}{e}\big)^{\gamma}}{\Gamma(\gamma)} = 1, \tag{12}$$

which is *Stirling's formula*, although his formula is usually expressed as the *asymptotic formula*

$$\Gamma(\gamma) \sim \sqrt{\frac{2\pi}{\gamma}}\,\Big(\frac{\gamma}{e}\Big)^{\gamma}, \tag{11'}$$

whose meaning is that ratio limit (12) holds.

The Central Limit Theorem: Given (12), it is now an easy matter to verify the Central Limit Theorem for the random variables $\{\bar{X}_\gamma : \gamma > 0\}$. Namely, given $-\infty < a < b < \infty$ and $\gamma > (a^-)^2$, where $a^- \equiv \max\{-a, 0\}$ is the negative part of a, we can proceed in precisely the same way as we did in the first step of the preceding section to get

$$\mathbb{P}\big(a \le \bar{X}_\gamma \le b\big) = \frac{\sqrt{\frac{2\pi}{\gamma}}\left(\frac{\gamma}{e}\right)^\gamma}{\Gamma(\gamma)} \frac{1}{\sqrt{2\pi}} \int_a^b \big(1 + \gamma^{-\frac{1}{2}} t\big)^{\gamma-1} e^{-\frac{t^2}{2} + \gamma E(\gamma^{-\frac{1}{2}} t)}\, dt.$$

Hence, by (11) and (12),

$$\mathbb{P}\big(a \le \bar{X}_\gamma \le b\big) \longrightarrow \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{t^2}{2}}\, dt,$$

which is a special case of the Central Limit Theorem alluded to earlier.

Exercise 1: Most applications of Stirling's formula are to cases in which $\gamma = n$ is a positive integer. As an application of (3) and (11'), we know that

$$n! \sim \sqrt{\frac{2\pi}{n+1}} \left(\frac{n+1}{e}\right)^{n+1}.$$

Starting from this, show that

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n,$$

which is the most familiar form in which Stirling's formula appears.

Exercise 2: Starting from the second part of (4), show that

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{\sqrt{\pi} \prod_{m=0}^{n-1}(2m+1)}{2^n} \quad \text{for } n \ge 1.$$

Exercise 3: Show that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n+1} e^{-\frac{x^2}{2}}\, dx = 0 \quad \text{for all } n \ge 0$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-\frac{x^2}{2}}\, dx = \begin{cases} 1 & \text{if } n = 0 \\ \prod_{m=0}^{n-1}(2m+1) & \text{if } n \ge 1. \end{cases}$$