WEYL'S LEMMA, ONE OF MANY

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Abstract. This note is a brief, and somewhat biased, account of the evolution of what people working in P.D.E.'s call Weyl's Lemma about the regularity of solutions to second order elliptic equations. As distinguished from most modern treatments, which are based on pseudodifferential operator technology, the approach adopted here is, like Weyl's own, potential theoretic.

§1: Where It All Started

Given a bounded, connected open region $\Omega \subseteq \mathbb{R}^N$ with smooth boundary $\partial\Omega$ and a smooth $\Phi : \Omega \longrightarrow \mathbb{R}^N$, consider the problem of smoothly decomposing Φ into a divergence free part Φ_0 and an exact part Φ_1. That is, the problem of writing $\Phi = \Phi_0 + \Phi_1$, where Φ_0 and Φ_1 are smooth, $\mathrm{div}\Phi_0$ vanishes, and $\Phi_1 = \nabla\varphi$ for some φ which vanishes at $\partial\Omega$.

To solve this problem, one should begin by observing that, if one ignores questions of smoothness, then it is reasonably clear how to proceed. Namely, because Φ_0 is divergence free, and Φ_1 is exact, $\mathrm{div}\Phi_0 = 0$ and $\Phi_1 = \nabla\varphi$. Hence, if φ vanishes at $\partial\Omega$, the divergence theorem says that $\nabla\varphi$ is perpendicular to Φ_0 in $L^2(\Omega;\mathbb{R}^N)$. With this in mind, let Φ_1 denote the orthogonal projection in $L^2(\Omega;\mathbb{R}^N)$ of Φ onto the closure in $L^2(\Omega;\mathbb{R}^N)$ of $\{\nabla\psi : \psi \in C_c^\infty(\Omega;\mathbb{R}^N)\}$. Next, choose $\{\varphi_n : n \geq 0\} \subseteq C_c^\infty(\Omega;\mathbb{R})$ so that $\nabla\varphi_n \longrightarrow \Phi_1$ in $L^2(\Omega;\mathbb{R}^N)$. Because the φ_n's vanish at $\partial\Omega$, $\lambda_0\|\varphi_n - \varphi_m\|^2_{L^2(\Omega;\mathbb{R})} \leq \|\nabla\varphi_n - \nabla\varphi_m\|^2_{L^2(\Omega;\mathbb{R}^N)}$, where $-\lambda_0 < 0$ is the largest Dirichlet eigenvalue of Laplacian Δ on $L^2(\Omega;\mathbb{R})$. Hence, there is a $\varphi \in L^2(\Omega;\mathbb{R})$ to which the φ_n's converge. Moreover, if $\psi \in C_c^\infty(\Omega;\mathbb{R})$, then

$$\begin{aligned}\int_\Omega \Delta\psi(x)\varphi(x)\,dx &= \lim_{n\to\infty}\int_\Omega \Delta\psi(x)\varphi_n(x)\,dx \\ &= -\lim_{n\to\infty}\int_\Omega \nabla\psi(x)\cdot\nabla\varphi_n(x)\,dx = -\int_\Omega \nabla\psi(x)\cdot\Phi_1(x)\,dx \\ &= -\int_\Omega \nabla\psi(x)\cdot\Phi(x)\,dx = \int_\Omega \psi(x)\mathrm{div}\Phi(x)\,dx.\end{aligned}$$

That is, $\Delta\varphi = \mathrm{div}\Phi$ is the sense of (Schwartz) distributions.

In view of the preceding, we will be done as soon as we show that φ is smooth. Indeed, if φ is smooth, then $\Delta\varphi = \mathrm{div}(\Phi)$ in the classical sense, and so, when $\Phi_1 = \nabla\varphi$, $\Phi_0 \equiv \Phi - \nabla\varphi$ satisfies $\mathrm{div}(\Phi_0) = \mathrm{div}(\Phi) - \Delta\varphi = 0$.

§2: Weyl's Lemma

As we saw in §1, the problem posed there will be solved as soon as we show that φ is smooth, and it is at this point that Weyl made a crucial contribution. Namely, he proved (cf. [6]) the following statement.

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Weyl's Lemma. *Let $\Omega \subseteq \mathbb{R}^N$ be open. If $u \in \mathfrak{D}'(\Omega;\mathbb{R})$ (the space of Schwartz distributions on Ω) satisfies $\Delta u = f \in C^\infty(\Omega;\mathbb{R})$ in the sense that*

$$\langle \Delta\psi, u\rangle = \langle \psi, f\rangle, \quad \psi \in C_c^\infty(\Omega;\mathbb{R}),$$

then $u \in C^\infty(\Omega;\mathbb{R})$.

Proof. : Set

$$\gamma_t(x) = (4\pi t)^{-\frac{N}{2}} \exp\left[-\frac{|x|^2}{4t}\right].$$

Given $x_0 \in \Omega$, choose $r > 0$ so that $\bar{B}(x_0, 3r) \subset\subset \Omega$ and $\eta \in C_c^\infty\big(B(x_0,3r);[0,1]\big)$ so that $\eta = 1$ on $\bar{B}(x_0, 2r)$.

Set $v = \eta u$ and $w = \Delta v - \eta f$. Then w is supported in $B(x_0,3r)\setminus \bar{B}(x_0,2r)$. Now take

$$v_t(x) = \gamma_t \star v(x) \equiv \langle \gamma_t(\,\cdot - x), v\rangle \text{ and } w_t(x) = \gamma_t \star w \equiv \langle \gamma_t(\,\cdot - x), w\rangle.$$

For each $t > 0$, v_t is smooth. Moreover,

$$\dot{v}_t(x) = \langle \gamma_t(\,\cdot - x), \eta f\rangle + \langle \gamma_t(\,\cdot - x), w\rangle,$$

and so

$$v_t(x) = v_1(x) - \int_t^1 \gamma_\tau \star (\eta f)(x)\, d\tau - \int_t^1 w_\tau(x)\, d\tau.$$

The first term on the right causes no problems, since $\eta f \in C_c^\infty\big(B(x_0,3);\mathbb{R}\big)$. Finally,

$$\sup_{(\tau,x)\in(0,1)\times B(x_0,r)} \big|\partial^\alpha w_\tau(x)\big| < \infty$$

for all $\alpha \in \mathbb{N}^N$. Hence, we have now shown that every derivative of $v_t \restriction B(x_0,1)$ is uniformly bounded by a bound which is independent of $t \in (0,1]$. Since, as $t \searrow 0$, $v_t \longrightarrow v$ on $B(x_0,1)$ in the sense of distributions, this means that $v \restriction B(x_0,1)$ is smooth. □

So far as I know, Weyl's Lemma is the first definitive statement of what are now known as *elliptic regularity* results. More precisely, it is the statement that Δ is *hypoelliptic* in the sense that the singular support of a distribution u is contained in that of Δu.

The spirit of Weyl's own proof is very much like that of the one just given. Namely, it is based on an analysis of the singularity in the Green's function. The only difference is that he dealt with the Green's function directly, whereas we have used the mollification of the Green's function provided by the heat flow. Most modern proofs of hypoellipticity prove a more quantitative statement. Namely, they prove hypoellipticity as a consequence of a *subelliptic* estimate which says that the s-order Sobolev norm $(I-\Delta)u$ can be used to dominate the $(s+2)$-order Sobolev norm of u.

§3: Weyl's Lemma for Heat Equation

As we will see, there are various directions in which Weyl's Lemma has been extended. The following is an important example of such an extension.

Weyl's Lemma for the Heat Equation. *Let $\Omega \subseteq \mathbb{R}^1 \times \mathbb{R}^N$ be open. If $u \in \mathfrak{D}'(\Omega;\mathbb{R})$ satisfies $(\partial_\xi + \Delta)u = f \in C^\infty(\Omega;\mathbb{R})$ in the sense that*

$$\big\langle(-\partial_\xi + \Delta)\psi, u\big\rangle = \langle \psi, f\rangle, \quad \psi \in C_c^\infty(\Omega;\mathbb{R}),$$

then $u \in C^\infty(\Omega;\mathbb{R})$.

Proof. Given $(\xi_0, x_0) \in \Omega$, set $P(r) = (\xi_0 - r, \xi_0 + r) \times B(x_0, r)$, and choose $r > 0$ so that $\bar{P}(3r) \subset\subset \Omega$. Next, choose $\eta \in C_c^\infty\big(P(3r); [0,1]\big)$ so that $\eta = 1$ on $P(2r)$, and set $v = \eta u$ and $w = (\partial_\xi + \Delta)v - \eta f$. Also, choose $\rho \in C_c^\infty\big((2,3); [0,\infty)\big)$ with total integral 1, and set $\rho_t(x) = t^{-1}\rho(t^{-1}x)$. Finally, for $t \in (0,1]$, set

$$v_t(\xi, x) = \big\langle \rho_t(\,\cdot - \xi)\gamma_{\cdot\,-\xi}(* - x), v\big\rangle \quad \text{and} \quad w_t(\xi, x) = \big\langle \tilde\rho_t(\,\cdot - \xi)\gamma_{\cdot\,-\xi}(* - x), w\big\rangle,$$

where $\tilde\rho(\xi) = \xi\rho(\xi)$ and $\tilde\rho_t(\xi) = t^{-1}\tilde\rho(t^{-1}\xi)$. Because $\frac{d}{dt}\rho_t(\xi) = -\tilde\rho_t'(\xi)$,

$$\frac{d}{dt}v_t(\xi, x) = -\big\langle \tilde\rho_t'(\,\cdot - \xi)\gamma_{\cdot\,-\xi}(* - x), v\big\rangle = \langle \tilde\rho_t(\,\cdot - \xi)\gamma_{\cdot\,-\xi}(* - x), \eta f\rangle + w_t(\xi, x).$$

The first term causes no problem as $t \searrow 0$ because $\eta f \in C_c^\infty(\Omega; \mathbb{R})$ and $\tilde\rho \in L^1(\mathbb{R})$. As for the second term, so long as $(\xi, x) \in P(r)$, its derivatives of this term are controlled independent of $t \in (0,1]$ because

(1) $\operatorname{supp}(w) \subseteq P(3r) \setminus P(2r)$.
(2) Derivatives of $\tilde\rho_t$ are bounded by powers of t^{-1}.
(3) For $(\xi, x) \in P(r)$ and $(\xi', x') \notin P(2r)$ with $0 < \xi' - \xi < 3t$, all derivatives of $\gamma_{\xi'-\xi}(x' - x)$ are bounded uniformly by any power of t.

Thus, just as before, we can conclude that $v \in C^\infty\big(P(r); \mathbb{R}\big)$. □

§4: A General Result

If one examines the proofs given in §§ 2 & 3, one see that they turn on two properties of the classic heat flow. The first of these is that the heat flow does "no damage" to initial data. That is, if one starts with smooth data, then it evolves smoothly. The second property is that so long as one stays away from the diagonal, the heat kernel $\gamma_t(y - x)$ remains smooth as $t \searrow 0$. When dealing with Δ, these two suffice. When dealing with $\Delta + \partial_\xi$, one needs a more quantitative statement of the latter property. Namely, one needs to know that away from the diagonal, the heat kernel goes to 0 faster than any power of t. Based on this discussion, we formulate the following general principle.

Suppose that L is a linear partial differential operator from $C^\infty(\mathbb{R}^N; \mathbb{R})$ to itself, and assume that associated with L there is a kernel

$$(t, x, y) \in (0, 2) \times \mathbb{R}^N \times \mathbb{R}^N \longmapsto q(t, x, y) \in \mathbb{R}$$

and the operators $t \rightsquigarrow Q_t$ given by

$$Q_t\varphi(x) = \int \varphi(y) q(t, x, y)\, dy, \quad \varphi \in C_c^\infty(\mathbb{R}^N; \mathbb{R})$$

with the properties that

(1) For each $x \in \mathbb{R}^N$, $(t, y) \rightsquigarrow q(t, x, y)$ satisfies the adjoint equation with initial value δ_x. That is, $\partial_t q(t, x, y) = [L^* q(t, x, \cdot\,)](y)$ and $q(t, x, \cdot\,) \longrightarrow \delta_x$ as $t \searrow 0$.
(2) For each $n \geq 0$, there exists a $C_n < \infty$ such that

$$\sup_{t \in (0,2]} \|Q_t\varphi\|_{C_b^n} \vee \|Q_t^*\varphi\|_{C_b^n} \leq C_n \|\varphi\|_{C_b^n}.$$

Using the same ideas on which we based the proofs in §§ 2 & 3, one can prove the following.

Theorem. *If, in addition to (1) and (2), for each $n \geq 0$ and $\epsilon > 0$,*

$$\max_{\|\alpha\| + \|\beta\| \leq n} \sup_{\substack{t \in (0,1] \\ |y - x| \geq \epsilon}} \big|\partial_x^\alpha \partial_y^\beta q(t, x, y)\big| < \infty,$$

then L is hypoelliptic. If, for each $n \geq 0$, $\epsilon > 0$, and $\nu > 0$,

$$\max_{m + \|\alpha\| + \|\beta\| \leq n} \sup_{\substack{t \in (0,1] \\ |y - x| \geq \epsilon}} t^{-\nu}\big|\partial_t^m \partial_x^\alpha \partial_y^\beta q(t, x, y)\big| < \infty,$$

then $L + \partial_\xi$ is hypoelliptic.

§5: Elliptic Operators

The original generalization of Weyl's Lemma was to replace the Laplace operator by a variable coefficient, second order, elliptic partial differential operator. That is, let $a:\mathbb{R}^N\longrightarrow\mathbb{R}^N\otimes\mathbb{R}^N$, $b:\mathbb{R}^N\longrightarrow\mathbb{R}^N$, and $c:\mathbb{R}^N\longrightarrow\mathbb{R}$ be smooth, bounded functions with bounded derivatives of all orders, and set

$$L=\sum_{i,j=1}^N a(x)_{ij}\partial_{x_i}\partial_{x_j}+\sum_{i=1}^N b(x)_i\partial_{x_i}+c(x).$$

Without loss in generality, we assume that $a(x)_{ij}=a(x)_{ji}$. The operator L is said to be *uniformly elliptic* if $a(x)\ge\delta I$ for some $\delta>0$.

Theorem. *If L is uniformly elliptic, then there is a $q\in C^\infty\big((0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}\big)$ such that $\partial_t q(t,x,y)=[L^*q(t,x,\cdot\,)](y)$, $q(t,x,\cdot\,)\longrightarrow\delta_x$ as $t\searrow 0$ for each $x\in\mathbb{R}^N$, and, for each $m\ge 0$ and $(\alpha,\beta)\in(\mathbb{N}^N)^2$, there is a $K_{m,\alpha,\beta}<\infty$ such that*

$$\big|\partial_t^m\partial_x^\alpha\partial_y^\beta q(t,x,y)\big|\le K_{m,\alpha,\beta}t^{-\frac{N+2m+\|\alpha\|+\|\beta\|}{2}}\exp\left[-\frac{|y-x|^2}{K_{m.\alpha,\beta}t}\right].$$

In particular, $L+\partial_\xi$ is hypoelliptic.

To see how such a result gets applied, consider the following construction. Take a and b as in the preceding and $c\equiv 0$. Let $\Gamma(t,x)$ denote the Gaussian probability measure on $\mathbb{R}^N$ with mean $x+tb(x)$ and covariance $2ta(x)$ That is, $\Gamma(t,x)$ has density given by

$$\big[(4\pi t)^N\det(a(x))\big]^{-\frac12}\exp\left[-\frac{(y-x-tb(x))\cdot a(x)^{-1}(y-x-tb(x))}{4t}\right].$$

Then

$$\frac{d}{dt}\int\varphi(y)\,\Gamma(t,x,dy)=\int L_x\varphi(y)\,\Gamma(t,x,dy),$$

where L_x is the constant coefficient operator obtained by freezing the coefficients of L at x.

For $n\ge 0$, define $P_n(t,x)=\delta_x$ and

$$P_n(t,x)=\int\Gamma\big(t-[t]_n,x'\big)\,P_n\big([t]_n,x,dx'),$$

where $[t]_n=2^{-n}[2^nt]$ is the largest dyadic $m2^{-n}$ dominated by t. Then,

$$\langle\varphi,P_n(t,x)\rangle-\varphi(x)=\int_0^t\left(\int\big\langle L_y\varphi,\Gamma\big(\{\tau\}_n,y)\big\rangle\,P_n\big([\tau_n],x,dy)\right)d\tau,$$

where $\{\tau\}_n=\tau-[\tau]_n$. Using elementary facts about weak convergence of probability measures, one can show that there is a continuous map $(t,x)\rightsquigarrow P(t,x)$ such that $P_n(t,x)\longrightarrow P(t,x)$ uniformly on compacts. Moreover, because, uniformly on compacts,

$$\big\langle L_y\varphi,\Gamma\big(\{\tau\}_n,y)\big\rangle\longrightarrow L\varphi(y)\quad\text{as } n\to\infty,$$

$$\langle\varphi,P(t,x)\rangle=\varphi(x)+\int_0^t\langle L\varphi,P(\tau,x)\rangle\,d\tau.$$

Thus, $\partial_tP(t,x)=L^*P(t,x)$ and $P(t,x)\longrightarrow\delta_x$. Finally, given $T>0$, define the distribution u on $(0,T)\times\mathbb{R}^N$ by

$$\langle\varphi,u\rangle=\int_0^T\left(\int\varphi(y)\,P(T-t,x,dy)\right)dt.$$

Then $(L+\partial_\xi)u=0$, and so u is smooth on $\{(t,y):\ a(y)>0\}$. That is, $P(t,x,dy)=p(t,x,y)dy$ where $(t,y)\rightsquigarrow p(t,x,y)$ is a smooth function there. Similar reasoning shows that $(t,x,y)\rightsquigarrow p(t,x,y)$ is smooth on $\{(t,x,y):\ a(x)>0\ \&\ a(y)>0\}$, and more delicate considerations show that it is smooth on $\{(t,x,y):\ a(x)>0\text{ or }a(y)>0\}$.

§6: Kolmogorov's Example

Although ellipticity guarantees hypoellipticity, hypoellipticity holds in for many operators which are not elliptic. The following example due to Kolmogorov is seminal.

Take $N = 2$, and consider $L = \partial_{x_1}^2 + x_1\partial_{x_2}$, which is severely non-elliptic. As Kolmogorov realized, the corresponding diffusion has coordinates

$$X_1(t) = x_1 + \sqrt{2}\,B(t) \;\&\; X_2(t) = x_2 + \int_0^t X_1(\tau)\,d\tau,$$

where $t\rightsquigarrow B(t)$ is a standard, 1-dimensional Brownian motion. In particular, this means that the distribution of $\big(X_1(t), X_2(t)\big)$ is the Gaussian measure on $\mathbb{R}^2$ with mean

$$\mathbf{m}(t,x) = \begin{pmatrix} x_1 \\ x_2 + tx_2 \end{pmatrix}$$

and covariance

$$C(t) = \begin{pmatrix} 2t & t^2 \\ t^2 & \frac{2t^3}{3}. \end{pmatrix}$$

Thus, the fundamental solution to the heat equation $\partial_t u = Lu$ is

$$q(t,x,y) = \frac{1}{2\pi\sqrt{\det C(t)}} \times \exp\left[-\frac{\big(y-\mathbf{m}(t,x)\big)\cdot C(t)^{-1}\big(y-\mathbf{m}(t,x)\big)}{2}\right].$$

In particular, $L + \partial_\xi$ is hypoelliptic. One can use the same reasoning to draw the same conclusion about when

$$L = \partial_{x_1}^2 + \sum_{i=2}^N x_{i-1}\partial_{x_i}.$$

§7: Results of Hörmander Type

Kolmogorov's example was put into context by a remarkable result proved by Hörmander. To state his result, let $\{X_0, \dots, X_r\}$ be a set of smooth vector fields on $\mathbb{R}^N$ and set

$$L = X_0 + \sum_{i=1}^r X_i^2,$$

where the X_k's are interpreted as directional derivative operators and X_k^2 is the composition of X_k with itself. Equivalently, if

$$X_i(x) = \sum_{j=1}^N \sigma(x)_{ij}\partial_{x_j}, \quad 1 \le i \le r, \quad \text{and} \quad X_0(x) = \sum_{j=1}^N \beta_j(x)\partial_{x_j},$$

then the matrix a of second order coefficients equals $\sigma\sigma^\top$ and the vector b of first order coefficient part equals

$$\begin{pmatrix} \beta_1(x) \\ \vdots \\ \beta_N(x) \end{pmatrix} + \sum_{i=1}^r \begin{pmatrix} X_i\sigma_{i1}(x) \\ \vdots \\ X_i\sigma_{iN}(x) \end{pmatrix}.$$

Let $\mathcal{L}$ and $\mathcal{L}'$ denote the Lie algebras generated by, respectively, $\{X_0, \dots, X_r\}$ and

$$\{[X_0, X_1], \dots, [X_0, X_r], X_1, \dots, X_r\}.$$

Hörmander proved the following result in [1].

Hörmander's Theorem. *If $\mathcal{L}(x)$ has dimension N at each $x \in \Omega$, then L is hypoelliptic on Ω. If $\mathcal{L}'(x)$ has dimension N at each $x \in \Omega$, then $\partial_\xi + L$ is hypoelliptic in $\mathbb{R}^1 \times \Omega$.*

First Oleinik and Radekevich [5] and later (see [2]) Fefferman, Phong, and others extended and sharpened this theorem to cover cases when L cannot be represented in terms of vector fields. That is, when there is no smooth square root of a. There are situations in which more detailed information is available. For instance, suppose that $X_0 = \sum_1^r c_k X_k$ for some smooth $c_1, \dots, c_k$ with bounded derivatives of all orders and that the vector fields X_k have bounded derivatives of all orders. Further, assume that there is an $n \in \mathbb{N}$ and $\epsilon > 0$ such that

$$\sum_{m=1}^{n} \sum_{\alpha \in \{1,\dots,r\}^m} \big(V_\alpha(x), \xi\big)^2_{\mathbb{R}^N} \geq \epsilon |\xi|^2, \quad (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where, for $\alpha \in \{1,\dots,r\}^m$, $V_\alpha(x) = X_\alpha x$ and, for $m \geq 2$,

$$X_\alpha = \big[X_{\alpha_m}, X_{\alpha'}\big] \quad \text{when } \alpha' = (\alpha_1, \dots, \alpha_{m-1}).$$

Then Rothschild and Stein showed that the operator L can be interpreted in terms of a degenerate Riemannian geometry in which the model space is a nilpotent Lie group instead of Euclidean space. Variations on and extensions of their ideas can be found in [2] and [4].

§ 8: Concluding Remarks

One can show that if there is smooth differentiable manifold M for which $\mathcal{L}(x)$ is the tangent space at each $x \in M$, then a diffusion process generated by L which starts on at an $x \in M$ will stay on M for a positive length of time. As a consequence, one can see that when such a manifold exists, L cannot be hypoelliptic in a neighborhood of M. Thus, when one combines (cf. § 7 in part II of [4]) this with Nagumo's Theorem about integral manifolds for real analytic vector fields, one realizes that the criterion in Hörmander's Theorem is necessary and sufficient when the X_k's are real analytic.

When the X_k's are not real analytic, Hörmander's criterion is necessary and sufficient for subellipticiy but not for hypoellipticity, For example, take $N = 3$ and consider

$$L = \partial^2_{x_1} + \big(\alpha(x_1)\partial_{x_2}\big)^2 + \partial^2_{x_3},$$

where α is a smooth function which vanishes only at 0 but vanishes to all orders there. Further, assume that α^2 is an even function on $\mathbb{R}$ which is non-decreasing on $[0,\infty)$. Then (cf. the last part of § 8 in part II of [4]) L is hypoelliptic in a neighborhood of 0 if and only if

$$\lim_{\xi \searrow 0} \xi \log\big(|\alpha(\xi)|\big) = 0.$$

When dealing with elliptic operators, hypoellipticity extends easily to systems. However, the validity of a Hörmander type theorem for systems remains an open question. Indeed, there is considerable doubt about what criterion replaces Hörmander's for systems. Recently, J.J. Kohn [3] has made some progress in this direction. Namely, he has found examples of complex vector fields for which hypoellipticity holds in the absence of ellipticity. Perhaps the most intriguing aspect of Kohn's example is that, from a subelliptic standpoint, his operators "lose derivatives."

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