Homework #5

(3.2.1) Let $(E, B, \mu)$ be a finite measure space, and show that $f_n \to f$ in $\mu$-measure if and only if $\int |f_n - f| \land 1 \, d\mu \to 0$.

(3.2.3) Let $J$ be a compact rectangle in $\mathbb{R}^N$ and $f : J \to \mathbb{R}$ a continuous function.

(i) Show that the Riemann integral $\int_J f(x) \, dx$ of $f$ over $J$ is equal to the Lebesgue integral $\int_J f(x) \, \lambda_N(dx)$. Next, suppose that $f \in L^1(\lambda_N; \mathbb{R})$ is continuous, and use the preceding to show that

$$\int f(x) \, \lambda_N(dx) = \lim_{J \nearrow \mathbb{R}^N} (R) \int_J f(x) \, dx,$$

where the limit means that, for any $\epsilon > 0$, there exists a rectangle $J_\epsilon$ such that

$$\left| \int f(x) \, \lambda_N(dx) - (R) \int_J f(x) \, dx \right| < \epsilon$$

whenever $J$ is a rectangle containing $J_\epsilon$. For this reason, even when $f$ is not continuous, it is conventional to use $\int f(x) \, dx$ instead of $\int f \, d\lambda_N$ to denote the Lebesgue integral of $f$ when greater precision is not required.

(ii) Now assume that $N = 1$ and $J = [a, b]$. Given a right-continuous, non-decreasing function $\psi : J \to \mathbb{R}$, let $\mu_\psi$ be the Borel measure on $\mathbb{R}$ for which $x \mapsto \psi((a \land x) \land b) - \psi(a)$ is the distribution function. Show that for every $\varphi \in C(J; \mathbb{R})$,

$$(R) \int_J \varphi(x) \, d\psi(x) = \int \varphi \, d\mu_\psi.$$  

(3.2.4) Let $f$ be a non-negative, measurable function on $(E, B, \mu)$. Show that

$$f \in L^1(\mu; \mathbb{R}) \implies \lim_{R \to \infty} R\mu(f \geq R) = 0.$$  

Next produce an example that shows that the preceding implication does not go in the opposite direction. Finally, show that

$$\sum_{n=0}^{\infty} \mu(f > n) < \infty \implies f \in L^1(\mu; \mathbb{R}).$$

See Exercise 5.1.2 for further information.

(3.2.6) Let $(E, B, \mu)$ be a measure space and $\{f_n : n \geq 1\}$ a sequence of measurable functions on $(E, B)$. Suppose that $\{g_n : n \geq 1\} \subseteq L^1(\mu; \mathbb{R})$ and that $g_n \to g \in L^1(\mu; \mathbb{R})$ in $L^1(\mu; \mathbb{R})$. The following variants of Fatou’s lemma and Lebesgue’s dominated convergence theorem are often useful.

(i) If $f_n \leq g_n$ (a.e., $\mu$) for each $n \geq 1$, show that
Although almost everywhere convergence does not follow from convergence in measure, it nearly does. Indeed, suppose \( \{f_n : n \geq 1\} \) is a sequence of measurable, \( \mathbb{R} \)-valued functions on \( (E, B, \mu) \). Given an \( \mathbb{R} \)-valued, measurable function \( f \), show that (3.2.2) holds, and therefore that \( f_n \to f \) both (a.e., \( \mu \)) and in \( \mu \)-measure if

\[
\sum_{i=1}^{\infty} \mu(|f_n - f| \geq \epsilon) < \infty \quad \text{for every } \epsilon > 0.
\]

In particular, if \( \{f_n : n \geq 1\} \cup \{f\} \subseteq L^1(\mu; \mathbb{R}) \) and \( \sum_{i=1}^{\infty} \|f_n - f\|_{L^1(\mu; \mathbb{R})} < \infty \), conclude that \( f_n \to f \) (a.e., \( \mu \)) and in \( L^1(\mu; \mathbb{R}) \).

(3.2.9) Let \( (E, B, \mu) \) be a measure space. A family \( \mathcal{K} \) of measurable functions \( f \) on \( (E, B, \mu) \) is said to be uniformly \( \mu \)-absolutely continuous if, for each \( \epsilon > 0 \), there is a \( \delta > 0 \) such that \( \int_E |f| \, d\mu \leq \epsilon \) for all \( f \in \mathcal{K} \) whenever \( \Gamma \in B \) and \( \mu(\Gamma) < \delta \), and it is said to be uniformly \( \mu \)-integrable if for each \( \epsilon > 0 \) there is an \( R < \infty \) such that \( \int_{|f| \geq R} |f| \, d\mu \leq \epsilon \) for all \( f \in \mathcal{K} \).

(i) Show that \( \mathcal{K} \) is uniformly \( \mu \)-integrable if it is uniformly \( \mu \)-absolutely continuous and \( \sup_{f \in \mathcal{K}} \|f\|_{L^1(\mu; \mathbb{R})} < \infty \). Conversely, suppose that \( \mathcal{K} \) is uniformly \( \mu \)-integrable and show that it is then necessarily uniformly \( \mu \)-absolutely continuous and, when \( \mu(E) < \infty \), that \( \sup_{f \in \mathcal{K}} \|f\|_{L^1(\mu; \mathbb{R})} < \infty \).

(ii) If \( \sup_{f \in \mathcal{K}} \int |f|^{1+\delta} \, d\mu < \infty \) for some \( \delta > 0 \), show that \( \mathcal{K} \) is uniformly \( \mu \)-integrable.

(iii) Let \( \{f_n : n \geq 1\} \subseteq L^1(\mu; \mathbb{R}) \) be given. If \( f_n \to f \) in \( L^1(\mu; \mathbb{R}) \), show that \( \{f_n : n \geq 1\} \cup \{f\} \) is uniformly \( \mu \)-absolutely continuous and uniformly \( \mu \)-integrable. Conversely, assuming that \( \mu(E) < \infty \), show that \( f_n \to f \) in \( L^1(\mu; \mathbb{R}) \) if \( f_n \to f \) in \( \mu \)-measure and \( \{f_n : n \geq 1\} \) is uniformly \( \mu \)-integrable.

(iv) Assume that \( \mu(E) = \infty \). One says that a family \( \mathcal{K} \) of measurable functions \( f \) on \( (E, B, \mu) \) is tight if, for each \( \epsilon > 0 \), there is a \( \Gamma \in B \) for which \( \mu(\Gamma) < \infty \) and \( \sup_{f \in \mathcal{K}} \int_{\Gamma} |f| \, d\mu \leq \epsilon \). Assuming that \( \mathcal{K} \) is tight, show that \( \mathcal{K} \) is uniformly \( \mu \)-integrable if and only if it is uniformly \( \mu \)-absolutely continuous and \( \sup_{f \in \mathcal{K}} \|f\|_{L^1(\mu; \mathbb{R})} < \infty \). Finally, suppose that \( \{f_n : n \geq 1\} \subseteq L^1(\mu; \mathbb{R}) \) is tight and that \( f_n \to f \) in \( \mu \)-measure. Show that \( \|f_n - f\|_{L^1(\mu; \mathbb{R})} \to 0 \) if and only if \( \{f_n : n \geq 1\} \) is uniformly \( \mu \)-integrable.