

## 7 The Navier-Stokes Equations

In the previous section, we have seen how one can deduce the general structure of hydrodynamic equations from purely macroscopic considerations and we also showed how one can derive macroscopic continuum equations from an underlying microscopic model. For the remainder of this course, we will return to the macroscopic viewpoint developed in Sec. 6.1.

### 7.1 Viscosity

A main insight from the discussion in the previous section is that the Euler equations, as given in Sec. 6.1.2, do not account for one final element needed to complete the macroscopic fluid equations: viscosity. Viscous stresses try to stop relative motion between nearby parts of the fluid. Another way of saying this is that wherever there is a rate of strain in the fluid, a stress acts to reduce the strain. As with pressure, viscosity has its origins in intermolecular forces and momentum transfer across a surface.

To understand more about viscosity, let's first have a general discussion of the stress acting on an infinitesimal fluid element. There are two kinds of stresses: normal stresses and tangential stresses. We anticipate that the viscous stresses act tangentially to the fluid element (as their role is to get rid of relative motion.) Our goal is to figure out the form of these tangential stresses. All of the information related to normal and tangential stresses within a fluid can be collected in the *stress tensor*. The stress tensor is a three by three matrix  $\sigma$  with components  $\sigma_{ij}$ , which has the property that the stress acting on a surface  $S$  with unit normal  $\mathbf{n}$  is just  $\sigma_{ij}n_j$ . For an arbitrary fluid element, the net force arising from surface stresses is

$$\int_S \sigma \cdot \mathbf{n} dS = \int_V (\nabla \cdot \sigma) dV. \quad (1)$$

We must therefore determine the form of  $\sigma$  in order to derive our equations of motion.

Firstly, suppose there were no tangential stresses on a fluid element. The normal stresses are just pressures. In this case the stress tensor would just be

$$\sigma = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix}. \quad (2)$$

The momentum equation for the component  $u_i$  of the velocity is then

$$\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i}, \quad (3)$$

and this is Euler's equation, which we derived in the previous lecture.

Now let's think a little more about the tangential stresses acting on a fluid element. We argue that the net *torque* on an element must vanish. This means that  $\sigma_{12}$ , the stress in the

$x$ -direction on the face with normal in the  $y$ -direction must be equal to  $\sigma_{21}$ , the stress in the  $y$ -direction on the face with normal in the  $x$ -direction. If these do not exactly cancel, the forces will make the fluid element spin. The fact that the tangential stresses on the fluid element balance means that the stress tensor must be *symmetric*. We therefore deduce that the stress tensor is of the general form

$$\sigma = -pI + \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{pmatrix}, \quad (4)$$

where  $I$  is the identity matrix.

In determining the form of the viscous tangential stresses, we reason that these must arise from relative motion between fluid elements. Thus the stress should somehow depend on  $\nabla \mathbf{u}$ , which will only be nonzero if there are velocity gradients. Note that  $\nabla \mathbf{u}$  is also a tensor, and can be written explicitly as

$$\nabla \mathbf{u} = \begin{pmatrix} \partial_x u_x & \partial_y u_x & \partial_z u_x \\ \partial_x u_y & \partial_y u_y & \partial_z u_y \\ \partial_x u_z & \partial_y u_z & \partial_z u_z \end{pmatrix}. \quad (5)$$

We immediately have a problem because the tensor is not symmetric, whilst we know that the stress tensor is necessarily symmetric. There is, however, a mathematical fact that says a general tensor can be expressed as the sum of a symmetric tensor and an antisymmetric tensor, i.e., if  $A$  is a tensor then

$$A_{ij} = A_{ij}^s + A_{ij}^a = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji}). \quad (6)$$

The first part of the formula corresponds to a symmetric tensor and the second part to an antisymmetric tensor. Using this construction, the velocity gradient tensor can thus be divided into symmetric part

$$\nabla \mathbf{u}^s = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \quad (7a)$$

and antisymmetric part

$$\nabla \mathbf{u}^a = \frac{1}{2}(\partial_i u_j - \partial_j u_i). \quad (7b)$$

Physically, the symmetric part  $\nabla \mathbf{u}^s$  corresponds to the deformation of the fluid element and is called the *rate of strain tensor*. The antisymmetric part corresponds to rotation of the fluid element and is called the *vorticity tensor*. To see this, let's consider a flow that is rotating, but not deforming, and also a flow that is deforming, but not rotating. In two dimensions a rotating flow is  $\mathbf{u} \propto (-y, x)$  and a deforming flow is  $\mathbf{u} \propto (x, y)$ . For the rotating flow it can be shown that the antisymmetric part  $\nabla \mathbf{u}^a$  is non-zero, and for the deforming flow the symmetric part is non-zero  $\nabla \mathbf{u}^s$ .

The grand conclusion of this is that we expect the strain tensor  $\sigma$  to be a function of the rate of strain tensor i.e.,  $\sigma = \sigma(\nabla \mathbf{u}^s)$ . The question now is, what function is it? This depends on the fluid and the situation is usually divided into two categories.

(i) *Newtonian fluids*: In Book II of the *Principia* Newton writes

‘The resistance arising from the want of lubricity in the parts of a fluid is, other things being equal, proportional to the velocity with which the parts of the fluid are separated from one another.’

Thus Newton’s guess, which corresponds to the simplest situation, was that the stress is a linear function of the strain,

$$\sigma = 2\mu\nabla\mathbf{u}^s. \quad (8)$$

(ii) *Non-Newtonian fluids*: This encompasses all other cases. That is, whenever the stress depends on the strain in a more complicated way, the fluid is called non-Newtonian.

Which of these two possibilities happens can only be determined experimentally for a particular fluid. In general, whether a fluid is non-Newtonian or not depends on how hard you are shearing it. Fortunately, it happens that most simple fluids are Newtonian under ordinary conditions. So for water, oil, air etc. it is often possible to approximate fluids as being Newtonian. Non-Newtonian also happens frequently in nature (e.g. liquid crystals) and gives rise to fascinating flow phenomena, but this is more specialised.

Now let’s put everything together and write down the equations for Newtonian viscous flow. If we consider the equation for  $u_i$ , the  $i^{\text{th}}$  component of the velocity, this is

$$\begin{aligned} \rho \frac{Du_i}{Dt} &= -\frac{\partial p}{\partial x_i} + 2\mu \sum_j \frac{\partial}{\partial x_j} \frac{1}{2} \left( \frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \\ &= -\nabla_i p + \mu \nabla_i (\nabla \cdot \mathbf{u}) + \mu \nabla^2 u_i. \end{aligned} \quad (9)$$

When the fluid density doesn’t change very much we have seen that  $\nabla \cdot \mathbf{u} = 0$ , and under these conditions the *Navier-Stokes equations* for fluid motion are

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \mu \nabla^2 \mathbf{u}. \quad (10)$$

These form the basis for much of our studies, and it should be noted that the derivation relies crucially on the incompressibility of the flow. The equations are essentially a macroscopic description of microscopic laws, and were written down even before the notion of a molecule was fully understood.

The parameter  $\mu$  is called the *coefficient of viscosity*, and since our derivation of the viscous force is phenomenological, it is both important and useful to make sure that all of the assumptions have been clearly stated. Is it true in general that only one number is sufficient to completely characterise the viscosity? Stated another way, the viscous effects in a fluid capture the macroscopic consequences of dissipative collisions between fluid particles. Is it obvious that only one number is sufficient to characterise this (enormously complicated!) process? Interestingly, one can show that if the fluid is both assumed to be both incompressible and isotropic (i.e., whichever way you look at the fluid it’s macroscopic properties are the same) then the parameter  $\mu$  is all that is needed.

## 7.2 The Reynolds number

For an incompressible flow, we have established that the equations of motion are

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}_{ext}, \quad (11)$$

as well as incompressibility  $\nabla \cdot \mathbf{u} = 0$ . Now note that the equation has five terms in it. The first two have to do with inertia and the third is pressure gradient, the fourth is viscosity and the fifth is an external force. In many situations, all of these terms are not equally important. The most trivial situation is a *static* situation. Here all of the terms involving the velocity are zero, and the only nonzero terms are the pressure gradient and the external forces. There are many other possibilities. The most difficult part is to figure out in any particular situation which of the terms in the equation are large, and which are small. In different limits the Navier-Stokes equations contain all of the important classes of partial differential equations (i.e., diffusion equation, Laplace's equation, wave equations) which are usually considered. In the next lecture we shall find an example which has within it a diffusion equation.

An important parameter that indicates the relative importance of viscous and inertial forces in a given situation is the *Reynolds number*. Suppose we are looking at a problem where the characteristic velocity scale is  $U_0$ , and the characteristic length scale for variation of the velocity is  $L$ . Then the size of the terms in the equation are

$$\frac{\partial \mathbf{u}}{\partial t} \sim \frac{U_0^2}{L}, \quad \mathbf{u} \cdot \nabla \mathbf{u} \sim \frac{U_0^2}{L}, \quad \mu \nabla^2 \mathbf{u} \sim \frac{\mu U_0}{L^2}. \quad (12)$$

The ratio of the inertial terms to the viscous term is

$$\frac{\rho U_0^2 / L}{\mu U_0 / L^2} = \frac{\rho U_0 L}{\mu} = Re, \quad (13)$$

and this is called the Reynolds number,  $Re$ . When the Reynolds number is very high the flow is rather inviscid, and when the Reynolds number is low the flow is very viscous. Honey is at low Reynolds number and turbulence is at high Reynolds number. For low Reynolds number it may be possible to ignore the inertial terms in the Navier-Stokes equations and obtain the so-called *slow (or creeping) flow equations* for very viscous flow. At high Reynolds number one ends up with the Euler equations.

## 8 Solving the Navier-Stokes equations

### 8.1 Boundary conditions

Now we have the equations of motion governing a fluid, the basic claim is that all the phenomena of normal fluid motion are contained in the equations. Unfortunately, there is no general theory of obtaining solutions to the Navier-Stokes equations. In fact, so difficult can it be, that the challenge of proving the existence and smoothness of solutions has been named as one of the seven Millennium Prize Problems by the Clay Mathematics Institute ([www.claymath.org/prizeproblems/statement.htm](http://www.claymath.org/prizeproblems/statement.htm)). All is not lost however, as one can turn to experiments to find new phenomena and, on the basis of this mathematicians can go looking for solutions. To become familiar with the equations we shall find some simple solutions to well known problems, but to directly compare solutions to experiments we first need to think a little about boundary conditions.

There are several types of boundaries that occur in practice, though the most common is simply the solid wall. The equations of motion for the fluid specify what happens everywhere except for the fluid element right next to the wall. At the wall, the forces between the wall and the fluid determine the dynamics.

To derive boundary conditions from first principles, it is necessary to take these interactions into account. Suffice to say that (a) in the nineteenth century there were vigorous debates between Maxwell, Stokes, etc. about what the correct conditions were, and (b) though today there is agreement for most situations, no decent derivation beginning with the microscopics has been given. From a purely mathematical point of view, it is necessary to check how many boundary conditions are allowed to still have unique solutions to the equations. If too many conditions are specified there might be no solutions of the equations which are consistent with them.

What are the boundary conditions? First, fluid cannot penetrate the boundary. That is, the component of the velocity normal to the solid boundary vanishes. What about the tangential component of the velocity? If the fluid has no viscosity, it is inconsistent to demand any relation on the tangential component of the velocity at the boundary. If there is viscosity then it is possible to demand another condition. The condition which is mostly accepted to be true is called the *no slip boundary condition* (there is important ongoing research which aims at determining if and when such a condition breaks down). The *no-slip* condition says that dissipative processes are so strong that the tangential component of the velocity actually vanishes there! This is known to be true from experimental studies of the motion of fluid near walls. It is extremely important, however, to understand that this is a phenomenological observation, not derived from first principles.

### 8.2 Some simple solutions

As mentioned before, in different limits the Navier-Stokes equations contain all of the important classes of partial differential equations. Let's proceed to find an example which has

within it a diffusion equation. We consider the following problem, at low Reynolds numbers (taken from Acheson, p.35). Consider a viscous fluid that is at rest in the region  $0 < y < \infty$  and suppose that at  $t = 0$  the rigid boundary at  $y = 0$  is moved at constant speed  $U$  in the  $x$ -direction. What is the motion of the fluid?

Since the jerking motion is uniform in the  $x$ -direction, we expect the velocity to be of the form  $(u, v, w) = (u(y, t), 0, 0)$ . Moreover, there is no pressure variation across the system in the  $x$ -direction, so the pressure is uniform. Thus the equation of motion is just

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}. \quad (1)$$

Note that the assumed form of the velocity automatically satisfies the incompressibility condition. Also, the nonlinear term has vanished because of the form of the velocity field. The equation is a diffusion equation for the velocity in which  $\nu$  is the diffusion coefficient. The initial condition is that  $u = 0$  in the upper half plane. The boundary conditions are that  $u(0, t) = U$  for  $t > 0$  (no-slip), and we expect that  $u$  will vanish as  $y \rightarrow \infty$  (since it vanishes initially).

We look for a *similarity solution* of the form  $u(y, t) = f(y/\sqrt{\nu t}) = f(\eta)$ . The logic is the same as we employed in examining the diffusive spreading of a point cloud. That is, initially there are no characteristic scales in the solution. After time  $t$  the only scale is of order  $\sqrt{t}$  because of the structure of the equations. The factor of  $\nu$  is for convenience. Plugging this into the diffusion equation gives

$$f'' + \frac{1}{2}\eta f' = 0. \quad (2)$$

Integrating twice,

$$f = A + B \int_0^\eta e^{-s^2/4} ds, \quad (3)$$

where  $A$  and  $B$  are integration constants. The boundary conditions imply that  $A + B \int_0^\infty e^{-s^2/4} ds = 0$  and  $A = U$ . Using the fact that

$$\int_0^\infty e^{-s^2/4} ds = \sqrt{\pi}, \quad (4)$$

we have the solution

$$u(y, t) = U \left( 1 - \frac{1}{\sqrt{\pi}} \int_0^{y/\sqrt{\nu t}} e^{-s^2/4} ds \right). \quad (5)$$

The simple form of the initial and boundary conditions was essential to our finding this solution. We see that at different times the velocity profiles are all geometrically similar i.e., the velocity is always the same function of  $(y/\sqrt{\nu t})$ . As time progresses, the velocity profile becomes stretched out and the effects of motion are largely confined to within a distance  $\sqrt{\nu t}$  from the boundary.

The solution above would not have been so simple if, for instance, an upper boundary were present. In this case, a lengthscale is imposed on the problem, this lengthscale being the distance  $h$  between the plates. We can then no longer look for a similarity solution, and instead the solution is found by the method of separation of variables. The problem is described in detail in Acheson (p. 40-41). Without solving the problem in detail however,

we can get a very good idea of what happens. We argue that after a large amount of time we expect the system to reach a steady state. This problem is easy to solve, as all we need to do is drop the time dependent term from equation (1) and solve subject to the boundary conditions  $u(0, t) = U$  and  $u(h, t) = 0$ . The steady state solution is the linear profile

$$u(y) = U \left(1 - \frac{y}{h}\right). \quad (6)$$

The outstanding issue is to then ask, how long does it take for this profile to be realised. Well, we know that viscous diffusion is responsible for setting up the profile, and the dimensions of  $\nu$  are  $L^2/T$ . Since the separation of the plates is  $h$ , then we obtain a timescale by forming the combination  $h^2/\nu$ . This is roughly the time taken for viscous diffusion to act over the gap between the plates, and gives an order of magnitude estimate for the time taken to set up a steady profile. And we have obtained all this information, without having to do any difficult mathematics!