18.04
Complex analysis with applications

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L 04: Analytic functions (Part 2)

4 Analytic functions

Last class

4.1 The derivative: preliminaries

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Example 4.1. Find the derivative of $f(z) = z^2$.

Solution: We compute using the definition of the derivative as a limit.

$$\lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} 2z + \Delta z = 2z$$

Last class

Example 4.2. Let $f(z) = \overline{z}$. Show that the limit for $f'(0)$ does not converge.

Solution: Let's try to compute $f'(0)$ using a limit:

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

Here we used $\Delta z = \Delta x + i\Delta y$.

Now, $\Delta z \to 0$ means both Δx and Δy have to go to 0. There are lots of ways to do this. For example, if we let Δz go to 0 along the x-axis then, $\Delta y = 0$ while Δx goes to 0. In this case, we would have

$$f'(0) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

On the other hand, if we let Δz go to 0 along the positive y-axis then

$$f'(0) = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

The limits **don't** agree! The problem is that the limit depends on how Δz approaches 0.

(Unique) limit does not exist !

Last class

4.5 Derivatives

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

If the limit exists we say f is analytic at z_0 or f is differentiable at z_0.

Remember: The limit has to exist and be the same no matter how you approach z_0!

If f is analytic at all the points in an open region A then we say f is analytic on A.

Alternative notations:

$$f'(z_0) = \left.\frac{dw}{dz}\right|_{z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Last class

4.5.1 Derivative rules

assuming f and g are differentiable we have:

- Sum rule: $\frac{d}{dz}(f(z)+g(z)) = f' + g'$
- Product rule: $\frac{d}{dz}(f(z)g(z)) = f'g + fg'$
- Quotient rule: $\frac{d}{dz}(f(z)/g(z)) = \frac{f'g - fg'}{g^2}$
- Chain rule: $\frac{d}{dz}g(f(z)) = g'(f(z))f'(z)$
- Inverse rule: $\frac{df^{-1}(z)}{dz} = \frac{1}{f'(f^{-1}(z))}$

(same as for real function)

4.6 Cauchy-Riemann equations

4.6.1 Partial derivatives as limits

If $u(x,y)$ is a function of two variables then the partial derivatives of u are defined as

$$\frac{\partial u}{\partial x}(x,y) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x}$$

i.e. the derivative of u holding y constant.

$$\frac{\partial u}{\partial y}(x,y) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y}$$

i.e. the derivative of u holding x constant.

4.6.2 The Cauchy-Riemann equations

Theorem. (Cauchy-Riemann equations) If $f(z) = u(x, y) + iv(x, y)$ is analytic (complex differentiable) then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

In particular,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Here is the short form of the Cauchy-Riemann equations:

$$u_x = v_y$$
$$u_y = -v_x$$

Proof. Let's suppose that $f(z)$ is differentiable in some region A and

$$f(z) = f(x+iy) = u(x,y) + iv(x,y).$$

We'll compute $f'(z)$ by approaching z first from the horizontal direction and then from the vertical direction. We'll use the formula

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z},$$

where $\Delta z = \Delta x + i\Delta y$.

Horizontal direction: $\Delta y = 0$, $\Delta z = \Delta x$

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \\ &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x+iy) - f(x+iy)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(u(x+\Delta x,y) + iv(x+\Delta x,y)) - (u(x,y)+iv(x,y))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{u(x+\Delta x,y) - u(x,y)}{\Delta x} + i\frac{v(x+\Delta x,y) - v(x,y)}{\Delta x} \\ &= \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) \end{aligned}$$

Vertical direction: $\Delta x = 0$, $\Delta z = i\Delta y$

$$\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \\
&= \lim_{\Delta y \to 0} \frac{(u(x, y + \Delta y) + iv(x, y + \Delta y)) - (u(x, y) + iv(x, y))}{i\Delta y} \\
&= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i\frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y} \\
&= \frac{1}{i}\frac{\partial u}{\partial y}(x, y) + \frac{\partial v}{\partial y}(x, y) \\
&= \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}(x, y)
\end{aligned}$$

We have found two different representations of $f'(z)$ in terms of the partials of u and v. If put them together we have the Cauchy-Riemann equations:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \qquad \Rightarrow \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \text{and} \quad -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

It turns out that the converse is true and will be very useful to us.

Theorem. Consider the function $f(z) = u(x,y) + iv(x,y)$ defined on a region A. If u and v satisfy the Cauchy-Riemann equations and have continuous partials then $f(z)$ is differentiable on A.

The proof of this is a tricky exercise in analysis. It is somewhat beyond the scope of this class, so we will skip it.

4.6.3 Using the Cauchy-Riemann equations

Example 4.10. Use the Cauchy-Riemann equations to show that e^z is differentiable and its derivative is e^z.

Solution: We write $e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y)$. So

$$u(x,y) = e^x \cos(y) \qquad v(x,y) = e^x \sin(y)$$

$$u_x = e^x \cos(y) \qquad v_x = e^x \sin(y)$$

$$u_y = -e^x \sin(y) \qquad v_y = e^x \cos(y)$$

We see that $u_x = v_y$ and $u_y = -v_x$ $\Rightarrow$ e^z is differentiable and

$$\frac{d}{dz} e^z = u_x + iv_x = e^x \cos(y) + ie^x \sin(y) = e^z$$

Example 4.11. Use the Cauchy-Riemann equations to show that $f(z) = \overline{z}$ is not differentiable.

Solution: $f(x + iy) = x - iy$, so $u(x, y) = x$, $v(x, y) = -y$. Taking partial derivatives

$$u_x = 1, \quad u_y = 0, \quad v_x = 0, \quad v_y = -1$$

Since $u_x \neq v_y$ the Cauchy-Riemann equations are not satisfied and therefore f is not differentiable.

Theorem. If $f(z)$ is differentiable on a disk and $f'(z) = 0$ on the disk then $f(z)$ is constant.
Proof. Since f is differentiable and $f'(z) \equiv 0$, the Cauchy-Riemann equations show that

$$u_x(x, y) = u_y(x, y) = v_x(x, y) = v_y(x, y) = 0$$

We know from multivariable calculus that a function of (x, y) with both partials identically zero is constant. Thus u and v are constant, and therefore so is f.

4.6.4 $f'(z)$ as a 2×2 matrix

Recall that we could represent a complex number $a + ib$ as a 2×2 matrix

$$a + ib \quad \leftrightarrow \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

$$f'(z) = u_x + iv_x \qquad \longleftrightarrow \qquad \begin{bmatrix} u_x & -v_x \\ v_x & u_x \end{bmatrix}$$

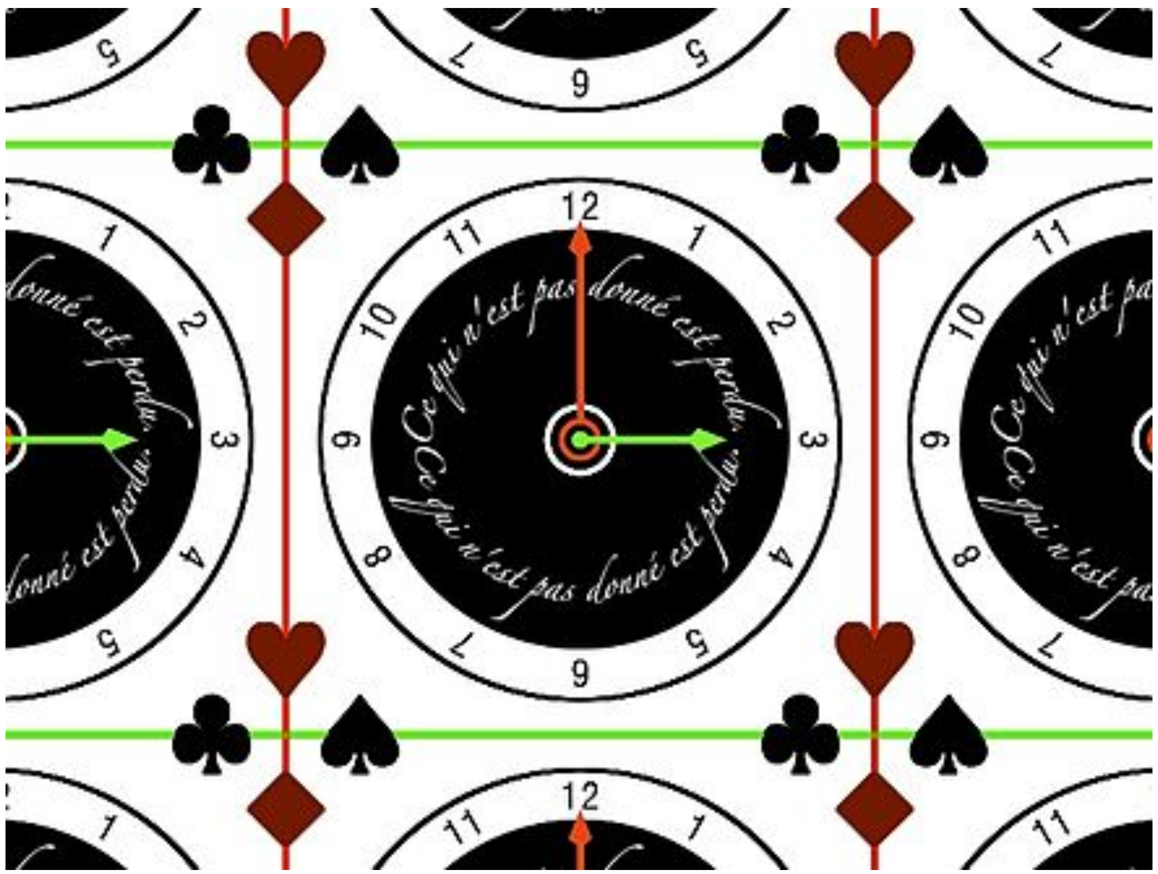
$$u_x = v_y \text{ and } u_y = -v_x$$

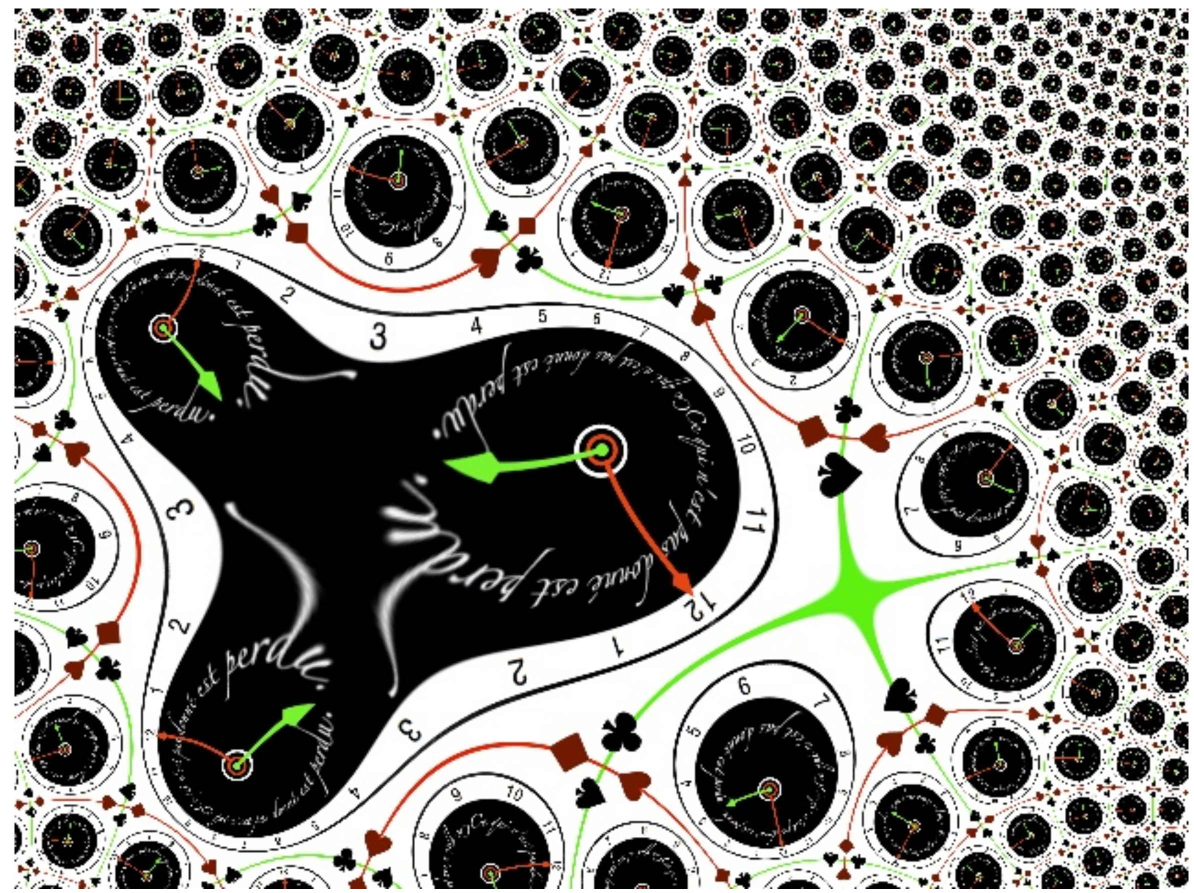
$$f'(z) \quad \leftrightarrow \quad \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

"Jacobian"

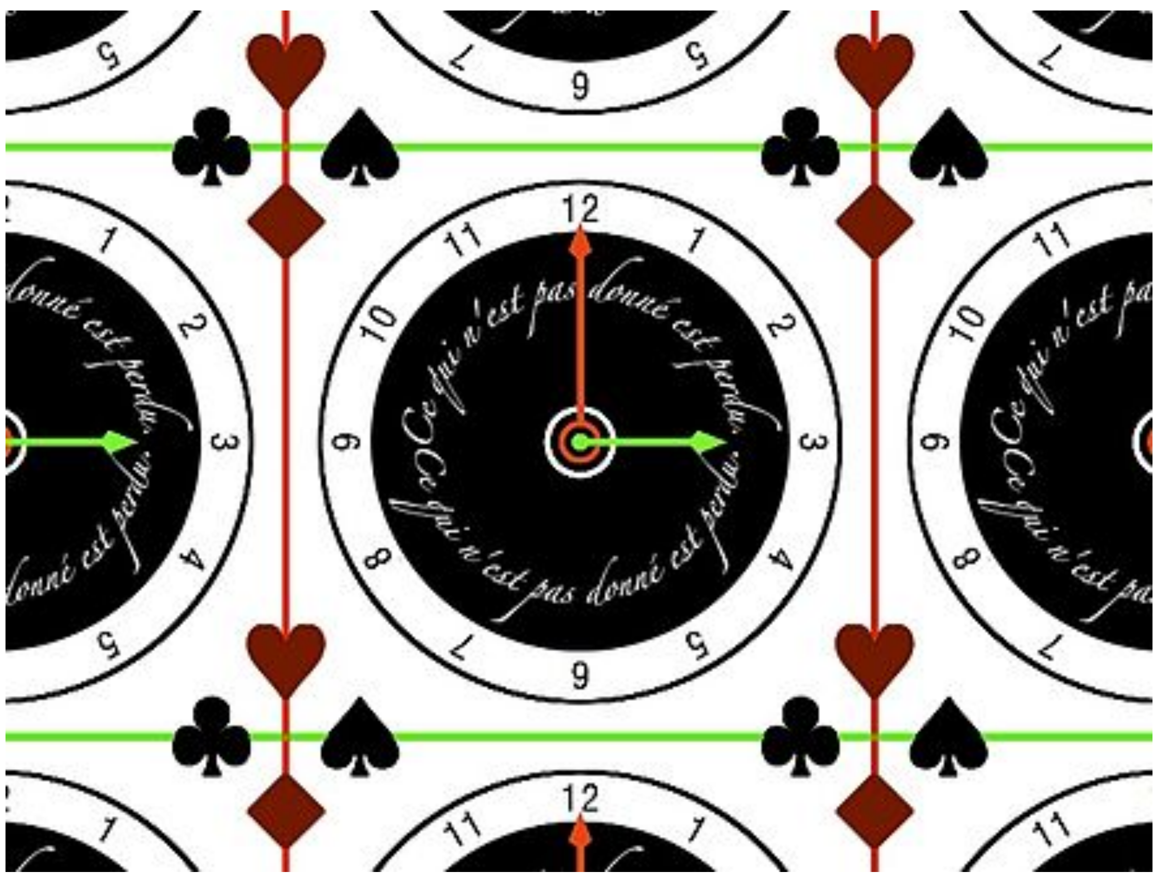
Examples

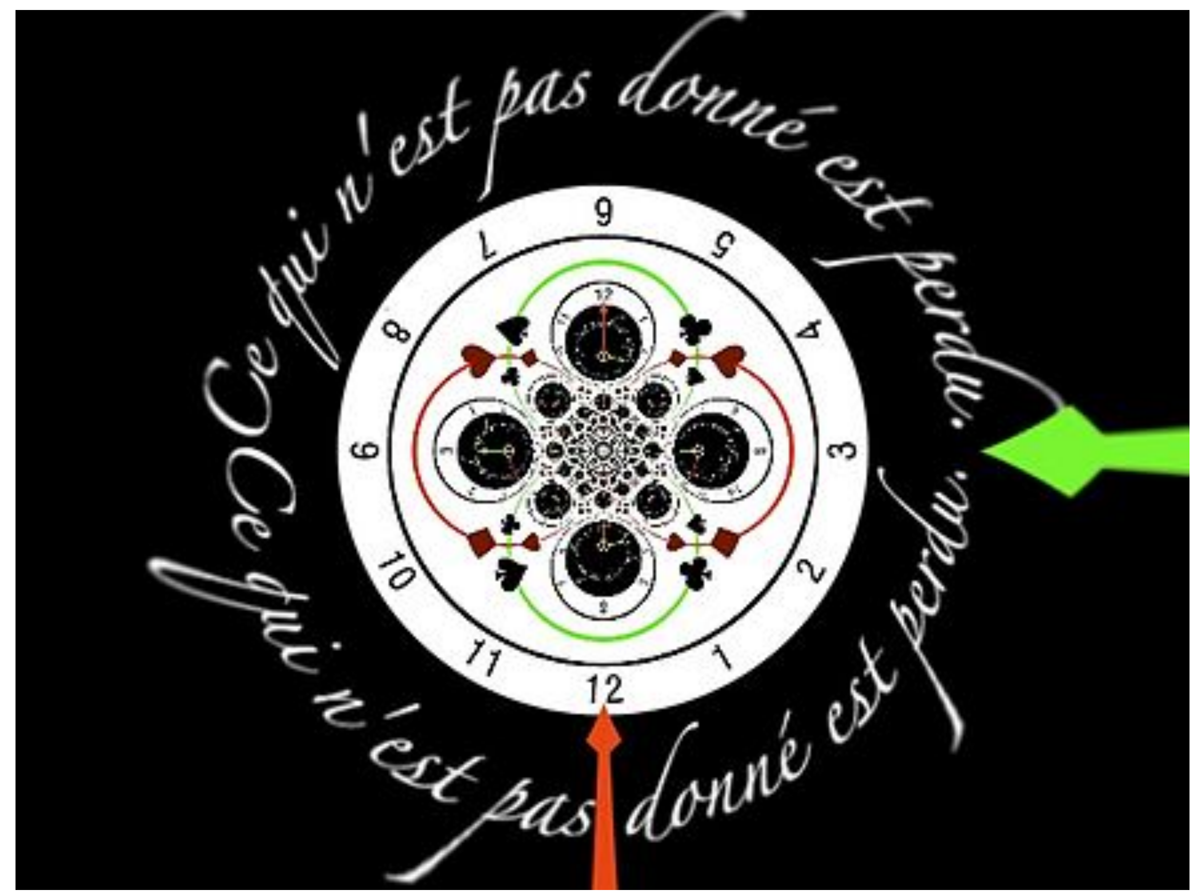
polynomial of degree 4





Inversion $z \mapsto 1/z$





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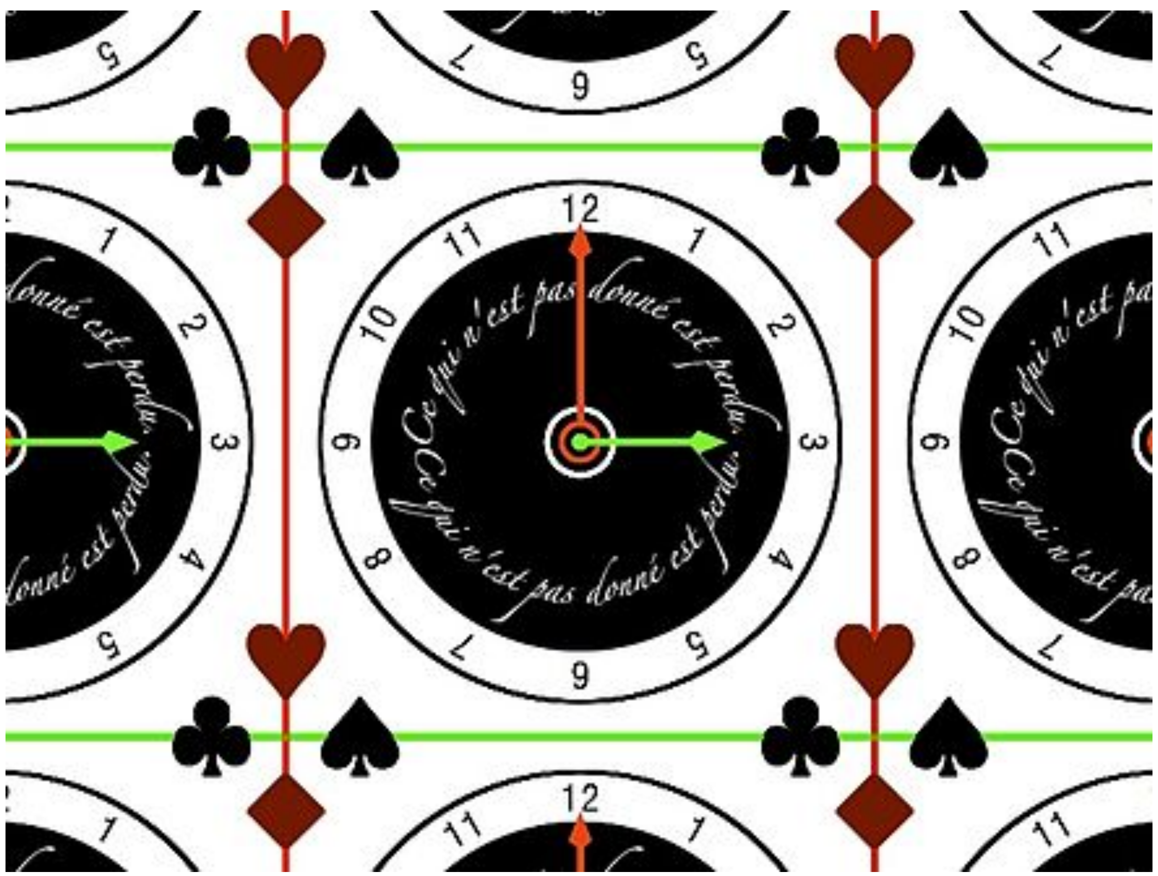
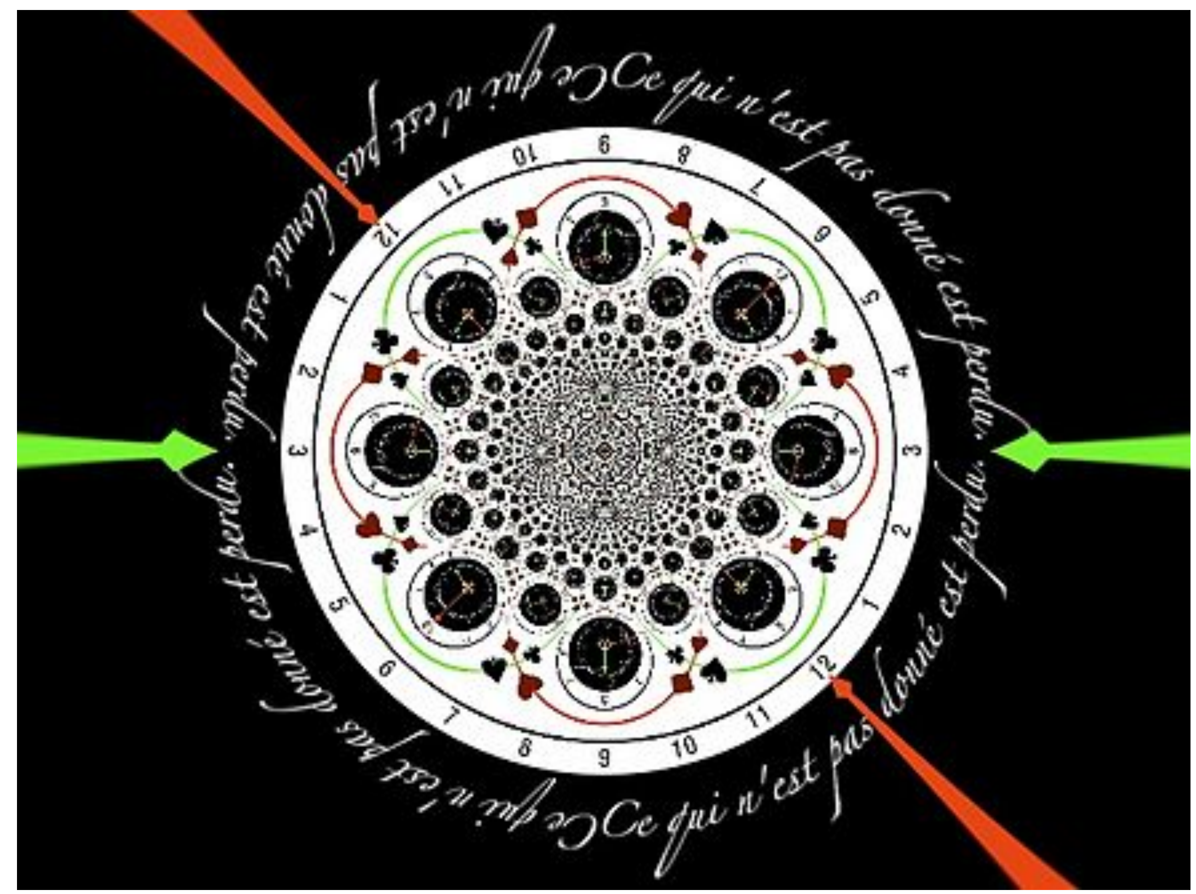
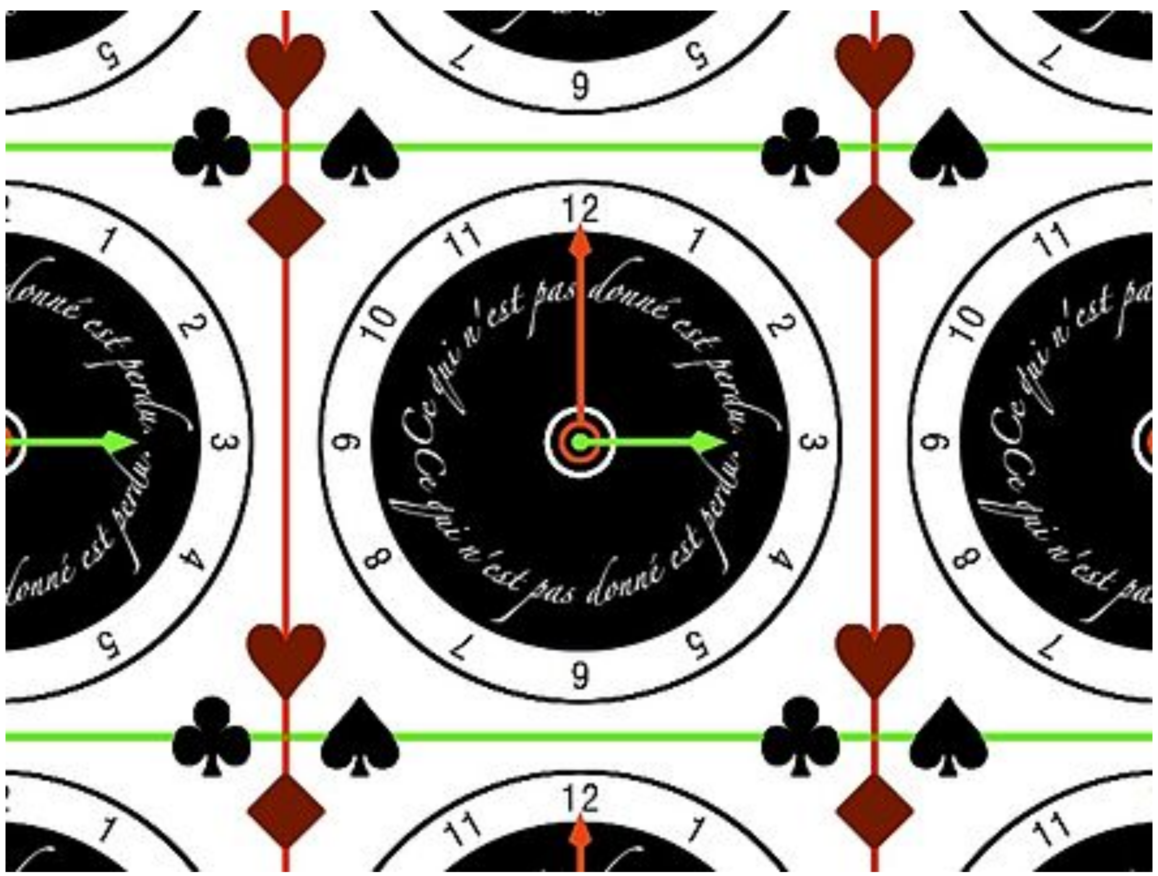
Examples

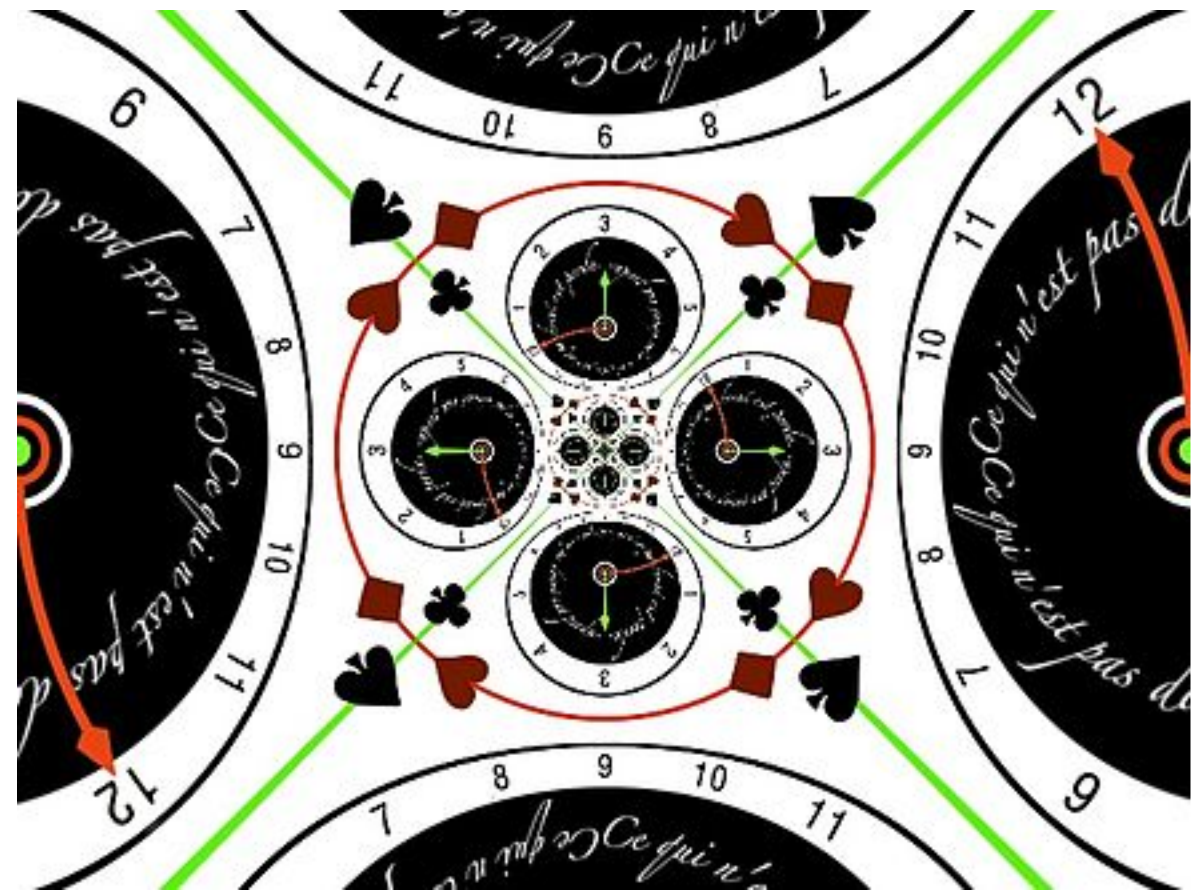
"squared" inversion $z \mapsto 1/z^2$





Exponential map $z \mapsto exp(z)$





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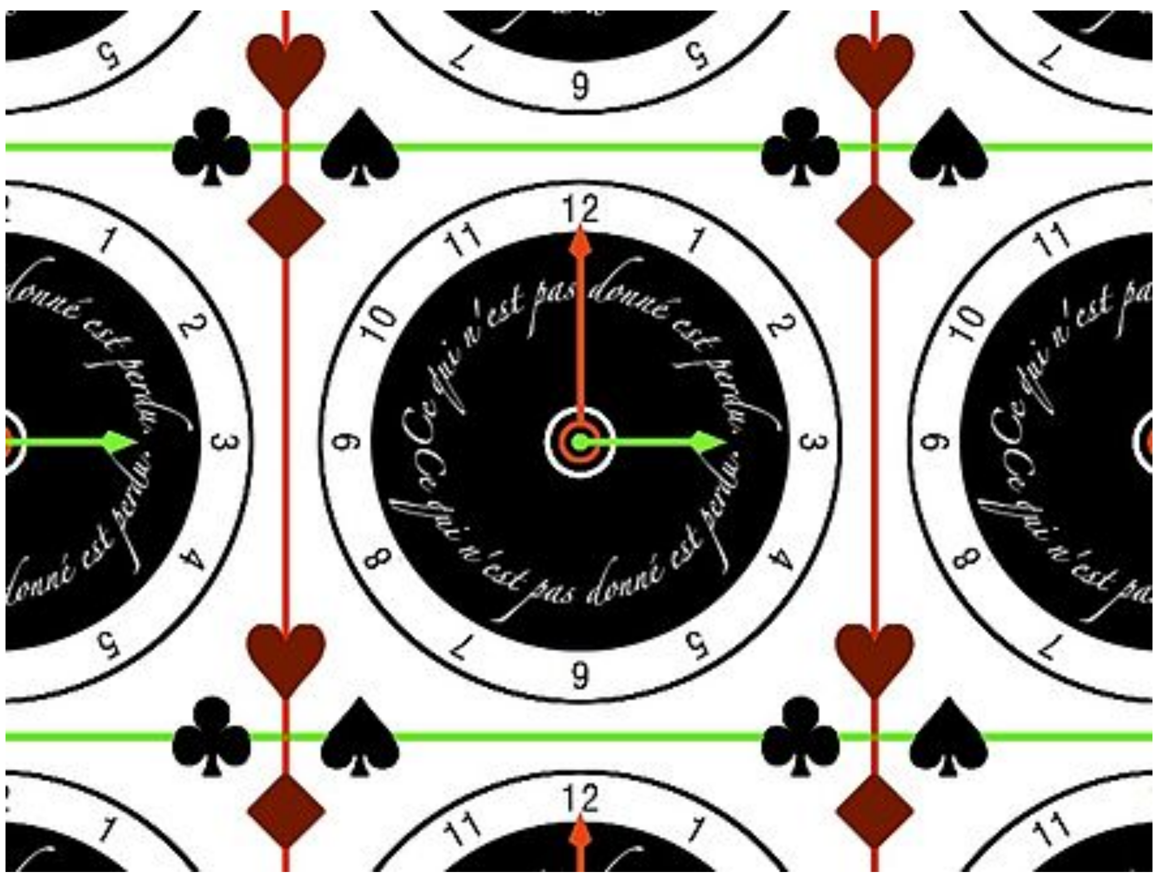
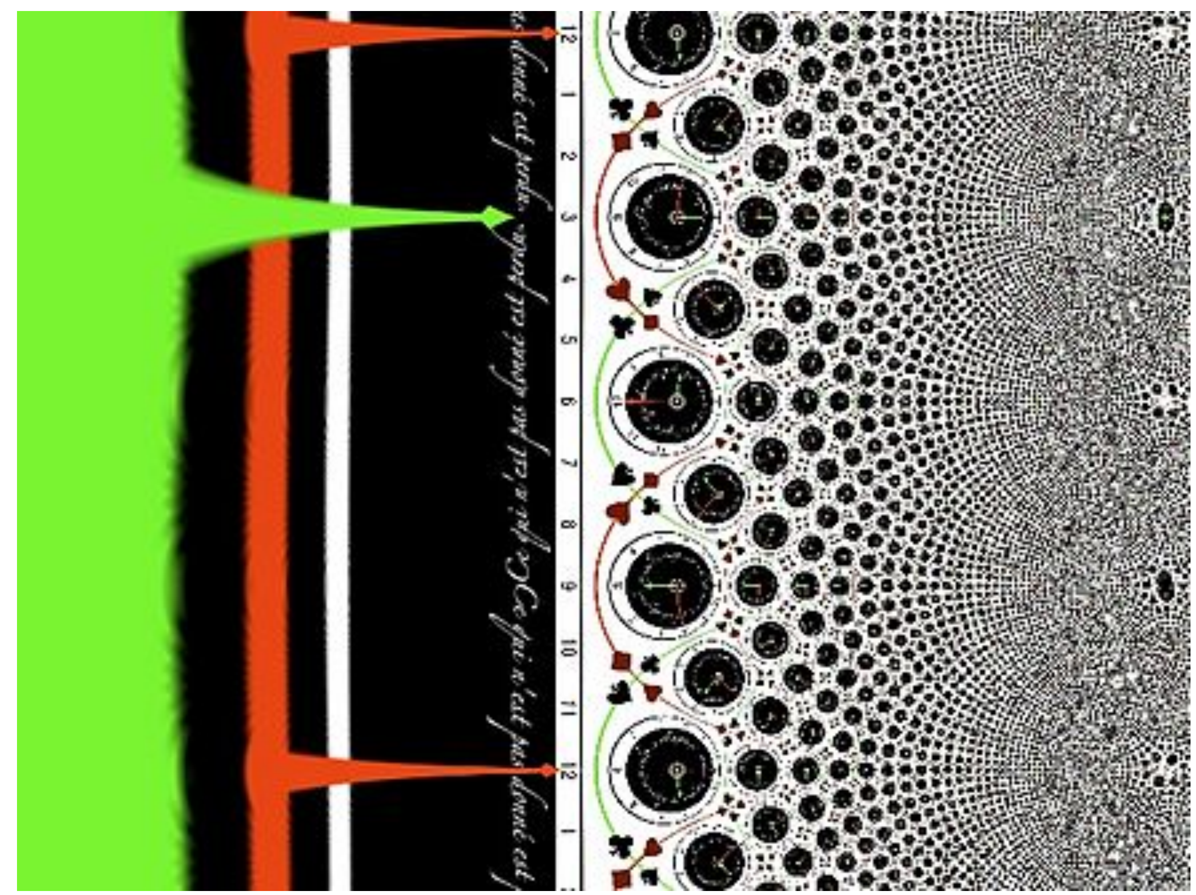
Logarithm $z \mapsto log(z)$





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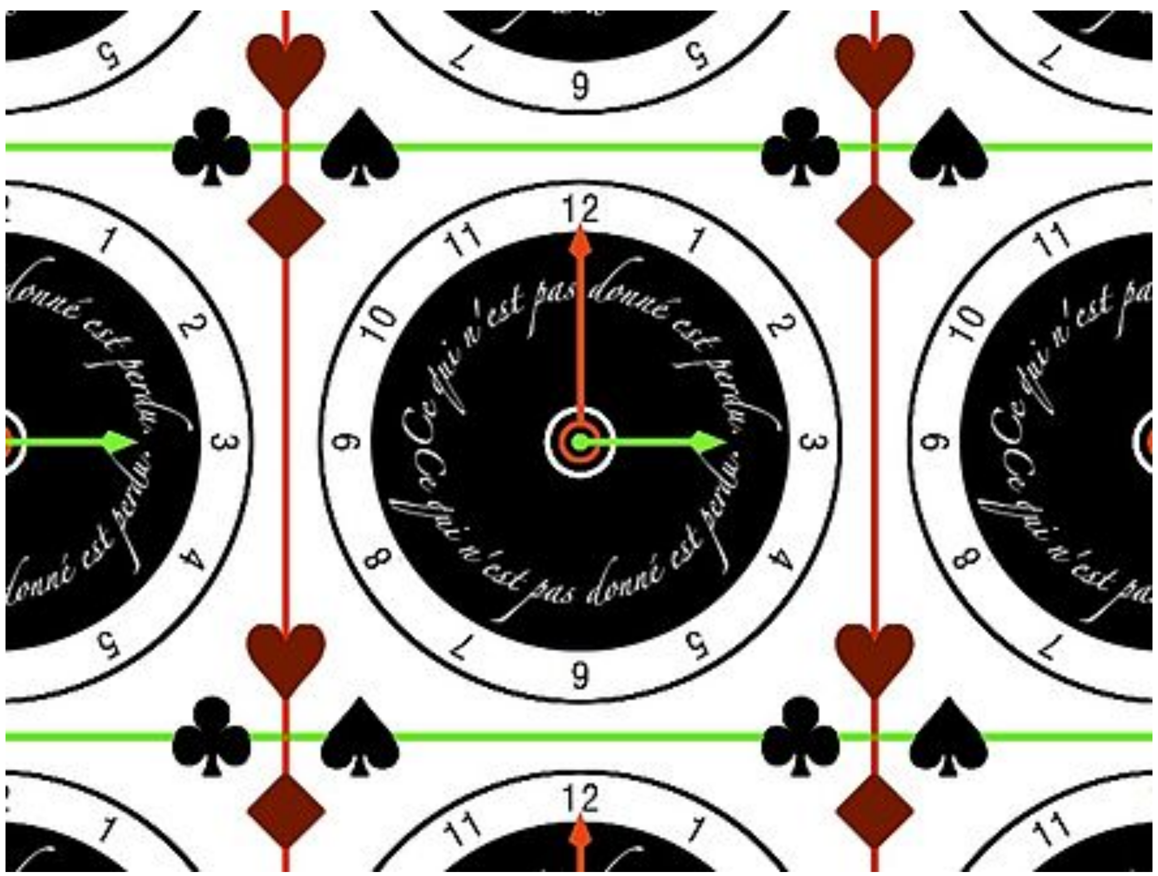
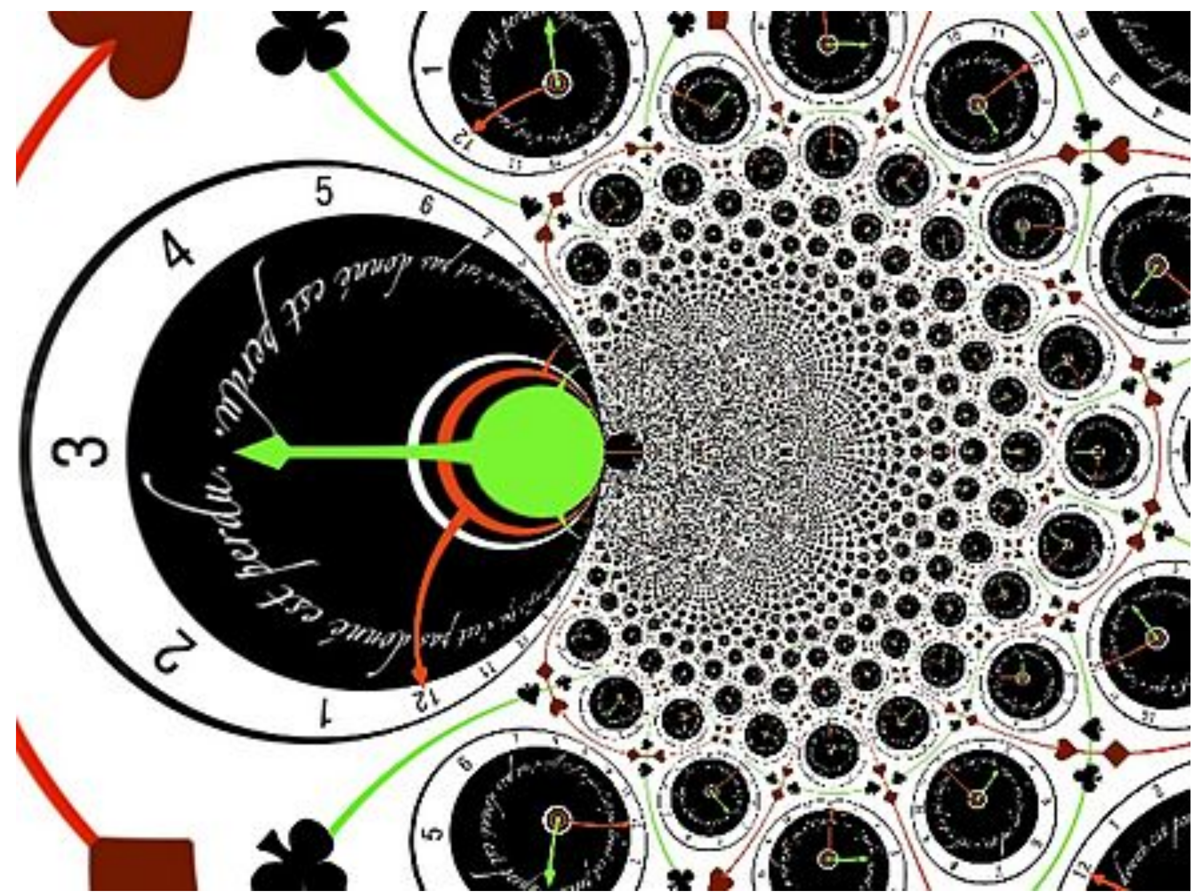
$$z \mapsto exp(1/z)$$





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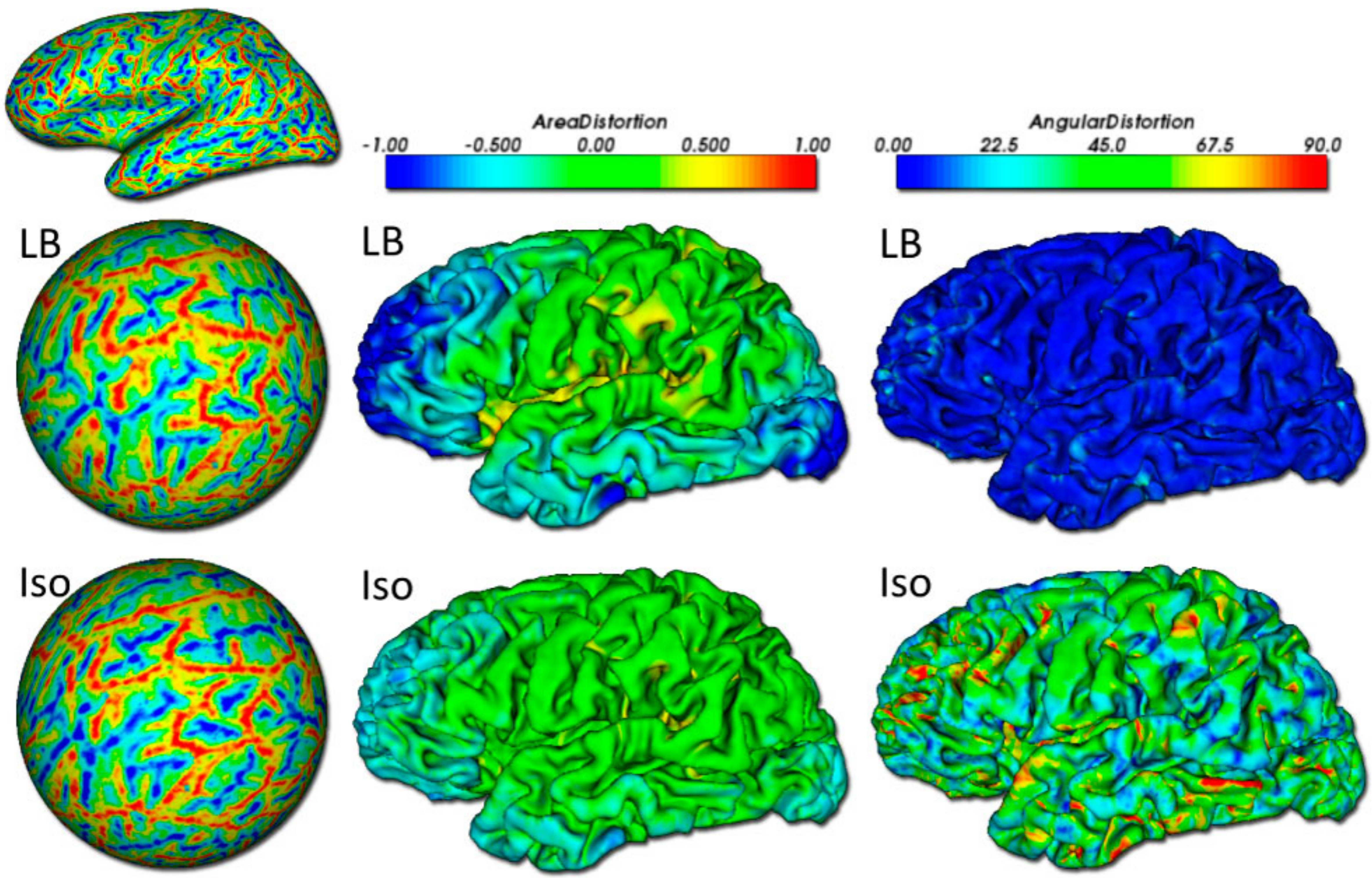
Structural brain imaging



www.neuro.uni-jena.de/research/methods/

4.7 Geometric interpretation & linear elasticity theory

Consider a mapping $\mathbb{R}^2 \to \mathbb{R}^2$

$$z = (x, y) \quad \to \quad w = f(z) = (u(x,y), v(x,y))$$

Interpret as deformation map (field) of 2D flat elastic sheet

associated displacement field $$d(z) := f(z) - z$$

Linear elasticity ⇔ small deformation limit

$$|d(z)| = |f(z) - z| \ll 1$$

In terms of the components d_1 and d_2 of d, this means that

$$d_1(x,y) = u(x,y) - x \qquad \text{and} \qquad d_2(x,y) = v(x,y) - y$$

are small everywhere.

Taylor-expand $d(z)$ at $z+\epsilon$, where $\epsilon = (\delta x, \delta y)$

Define

$$x_1 = x \;, \qquad x_2 = y \;, \qquad d_{i,j} = \frac{\partial}{\partial x_j} d_i(x_1, x_2)$$

Using Einstein's summation convention, $a_i b_i = \sum_{i=1}^{2} a_i b_i$

Taylor-expansion in component form

$$d_i(z+\epsilon) = d_i(z) + d_{i,j}(z)\epsilon_j + \mathcal{O}(\epsilon^2)$$

Jacobian matrix

$$(d_{i,j}) = \begin{pmatrix} u_x - 1 & u_y \\ v_x & v_y - 1 \end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} - \boldsymbol{I}$$

Rewrite

$$\begin{aligned} d_{i,j} &= \frac{1}{2}(d_{i,j} + d_{j,i}) + \frac{1}{2}(d_{i,j} - d_{j,i}) - \delta_{ij} \\ &= \frac{1}{2}d_{i,i}\delta_{ij} + \frac{1}{2}\left[(d_{i,j} + d_{j,i}) - d_{i,i}\delta_{ij}\right] + \frac{1}{2}(d_{i,j} - d_{j,i}) - \delta_{ij} \end{aligned}$$

In terms of $u(x,y)$ and $v(x,y)$

$$(d_{i,j}) = \frac{1}{2}(u_x + v_y)\boldsymbol{I} + \frac{1}{2}\begin{pmatrix} u_x - v_y & u_y + v_x \\ v_x + u_y & v_y - u_x \end{pmatrix} - \left\{\boldsymbol{I} + \frac{1}{2}(u_y - v_x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$$

Now recall that an infinitesimal rotation is

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \approx \boldsymbol{I} + \phi\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

we can interpret the contributions in terms of intuitive fundamental deformations:

- The first term

$$\frac{1}{2}(u_x + v_y)\boldsymbol{I} = \frac{1}{2}(\nabla \cdot d)\boldsymbol{I}$$

  represents stretching or compression.

- The last term term

$$\left\{\boldsymbol{I} + \frac{1}{2}(u_y - v_x)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\}$$

  represents an infinitesimal rotation by an angle $\phi = \frac{1}{2}(u_y - v_x) = \frac{1}{2}\nabla \wedge d$.

- The middle term

$$\frac{1}{2}\begin{pmatrix} u_x - v_y & v_x + u_y \\ v_x + u_y & -(u_x - v_y) \end{pmatrix} = \frac{1}{2}(u_x - v_y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2}(v_x + u_y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is the sum of a scaled *reflection* (via the diagonal components) and *shear* strain (via the off-diagonal components).

Thus, deformations that preserve orientation <u>and</u> angles locally must satisfy

$$u_x = v_y \ , \qquad u_y = -v_x$$

But these are just the Cauchy-Riemann conditions !