18.04 Complex analysis with applications

Jörn Dunkel

L 03: Analytic functions

4 Analytic functions

4.1 The derivative: preliminaries

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Example 4.1. Find the derivative of $f(z) = z^2$.

Solution: We compute using the definition of the derivative as a limit.

$$\lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + (\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} 2z + \Delta z = 2z$$

Example 4.2. Let $f(z) = \overline{z}$. Show that the limit for $f'(0)$ does not converge.

Solution: Let's try to compute $f'(0)$ using a limit:

$$f'(0) = \lim_{\Delta z \to 0} \frac{f(\Delta z) - f(0)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}.$$

Here we used $\Delta z = \Delta x + i\Delta y$.

Now, $\Delta z \to 0$ means both Δx and Δy have to go to 0. There are lots of ways to do this. For example, if we let Δz go to 0 along the x-axis then, $\Delta y = 0$ while Δx goes to 0. In this case, we would have

$$f'(0) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1.$$

On the other hand, if we let Δz go to 0 along the positive y-axis then

$$f'(0) = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1.$$

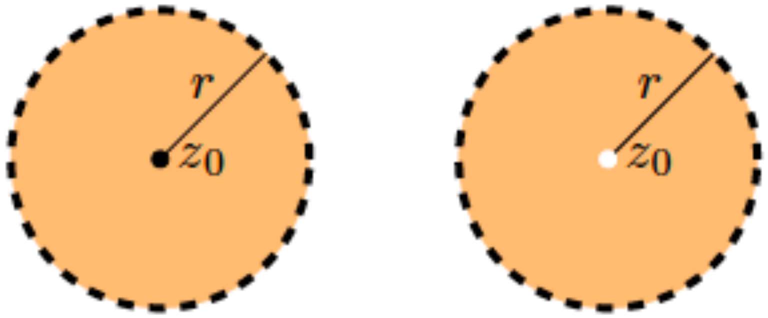
The limits **don't** agree! The problem is that the limit depends on how Δz approaches 0.

(Unique) limit does not exist !

4.2 Open disks, open deleted disks, open regions

Definition. The open disk of radius r around z_0 is the set of points z with $|z - z_0| < r$, i.e. all points within distance r of z_0.

The open deleted disk of radius r around z_0 is the set of points z with $0 < |z - z_0| < r$. That is, we remove the center z_0 from the open disk. A deleted disk is also called a punctured disk.



Left: an open disk around z_0; right: a deleted open disk around z_0

Definition. An open region in the complex plane is a set A with the property that every point in A can be be surrounded by an open disk that lies entirely in A. We will often drop the word open and simply call A a region.

Left: an open region A; right: B is not an open region

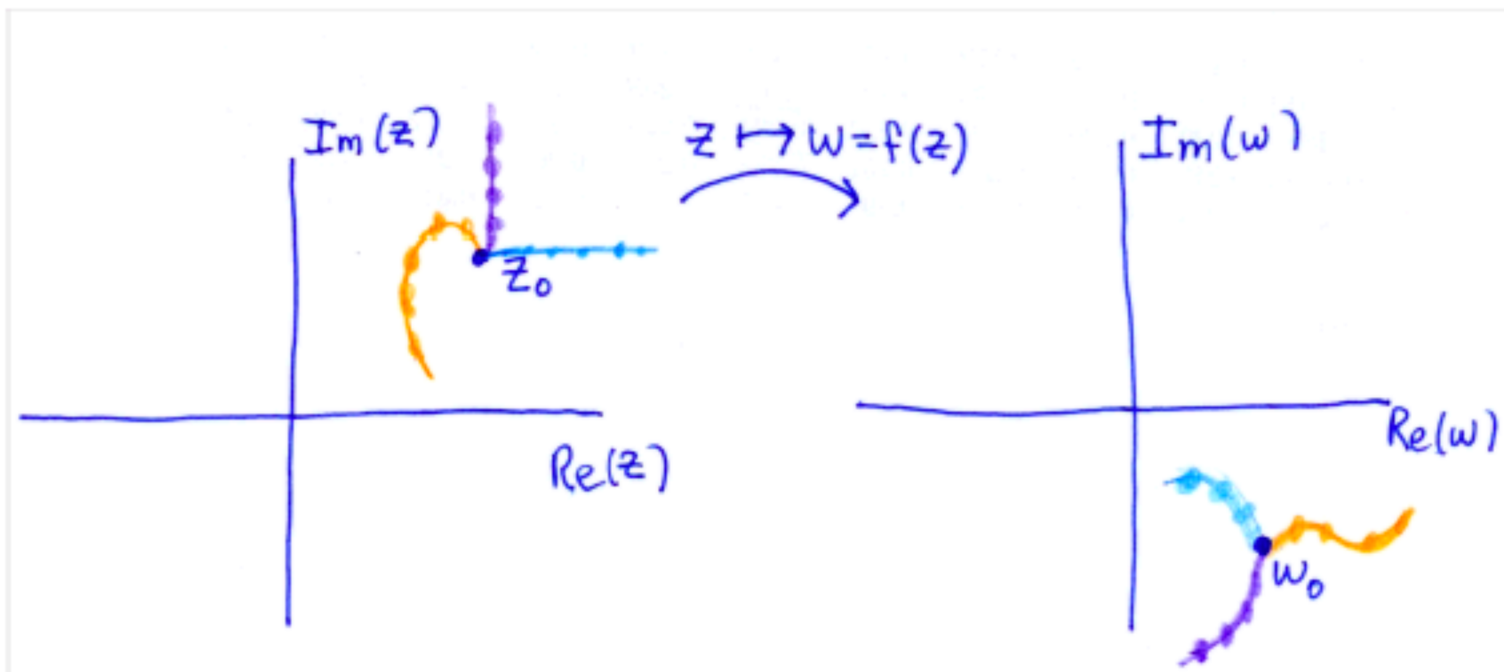
4.3 Limits and continuous functions

Definition. If $f(z)$ is defined on a punctured disk around z_0 then we say

$$\lim_{z \to z_0} f(z) = w_0$$

if $f(z)$ goes to w_0 no matter what direction z approaches z_0.

The figure below shows several sequences of points that approach z_0. If $\lim_{z \to z_0} f(z) = w_0$ then $f(z)$ must go to w_0 along each of these sequences.



Sequences going to z_0 are mapped to sequences going to w_0.

Example 4.3. Many functions have obvious limits

$$\lim_{z\to 2} z^2 = 4$$

and

$$\lim_{z\to 2}(z^2+2)/(z^3+1) = 6/9$$

Example 4.4. (No limit) Show that

$$\lim_{z\to 0}\frac{z}{\bar{z}} = \lim_{z\to 0}\frac{x+iy}{x-iy}$$

does not exist.

Solution: On the real axis we have

$$\frac{z}{\bar{z}} = \frac{x}{x} = 1$$

so the limit as $z \to 0$ along the real axis is 1. By contrast, on the imaginary axis we have

$$\frac{z}{\bar{z}} = \frac{iy}{-iy} = -1,$$

so the limit as $z \to 0$ along the imaginary axis is -1. Since the two limits do not agree the limit as $z \to 0$ does not exist!

4.3.1 Properties of limits

We have the usual properties of limits. Suppose

$$\lim_{z\to z_0} f(z) = w_1 \text{ and } \lim_{z\to z_0} g(z) = w_2$$

then

- $\lim_{z\to z_0} f(z) + g(z) = w_1 + w_2$.
- $\lim_{z\to z_0} f(z)g(z) = w_1 \cdot w_2$.
- If $w_2 \neq 0$ then $\lim_{z\to z_0} f(z)/g(z) = w_1/w_2$
- If $h(z)$ is continuous and defined on a neighborhood of w_1 then $\lim_{z\to z_0} h(f(z)) = h(w_1)$
  (Note: we will give the official definition of continuity in the next section.)

We can restate the definition of limit in terms of functions of (x, y). To this end, let's write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

and abbreviate

$$P = (x, y)\ , \qquad P_0 = (x_0, y_0)\ , \qquad w_0 = u_0 + iv_0$$

Then

$$\lim_{z \to z_0} f(z) = w_0 \qquad \text{iff} \qquad \begin{cases} \lim_{P \to P_0} u(x, y) = u_0 \\ \lim_{P \to P_0} v(x, y) = v_0. \end{cases}$$

4.3.2 Continuous functions

Definition. If the function $f(z)$ is defined on an open disk around z_0 and $\lim_{z \to z_0} f(z) = f(z_0)$ then we say f is continuous at z_0. If f is defined on an open region A then the phrase 'f is continuous on A' means that f is continuous at every point in A.

Fact. $f(z) = u(x, y) + iv(x, y)$ is continuous iff $u(x, y)$ and $v(x, y)$ are continuous as functions of two variables.

Example 4.5. (Some continuous functions)
(i) A polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$$

is continuous on the entire plane. Reason: it is clear that each power $(x + iy)^k$ is continuous as a function of (x, y).
(ii) The exponential function is continuous on the entire plane. Reason:

$$e^z = e^{x+iy} = e^x \cos(y) + ie^x \sin(y),$$

so the both the real and imaginary parts are clearly continuous as a function of (x, y).

(iii) The principal branch $\mathrm{Arg}(z)$ is continuous on the plane minus the non-positive real axis.

(iv) The principal branch of the function $\log(z)$ is continuous on the plane minus the non-positive real axis. Reason: the principal branch of log has

$$\log(z) = \log(r) + i\,\mathrm{Arg}(z),$$

so the continuity of $\log(z)$ follows from the continuity of $\mathrm{Arg}(z)$.

4.3.3 Properties of continuous functions

Suppose $f(z)$ and $g(z)$ are continuous on a region A. Then

- $f(z) + g(z)$ is continuous on A.
- $f(z)g(z)$ is continuous on A.
- $f(z)/g(z)$ is continuous on A except (possibly) at points where $g(z) = 0$.
- If h is continuous on $f(A)$ then $h(f(z))$ is continuous on A.

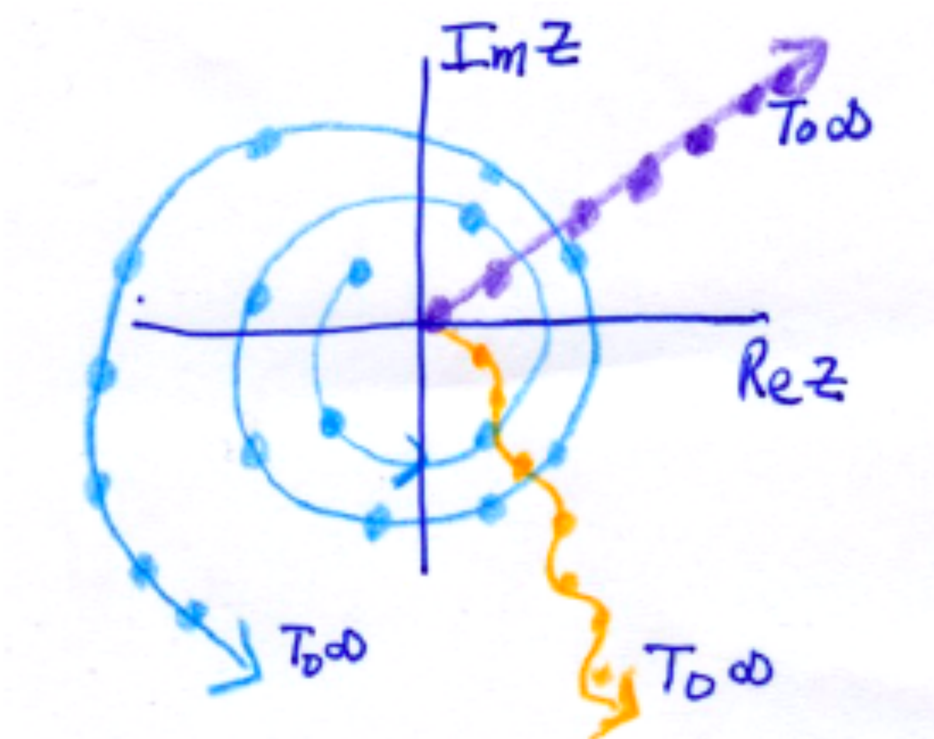
Using these properties we can claim continuity for each of the following functions:

- e^{z^2}
- $\cos(z) = (e^{iz} + e^{-iz})/2$
- If $P(z)$ and $Q(z)$ are polynomials then $P(z)/Q(z)$ is continuous except at roots of $Q(z)$.

4.4 The point at infinity

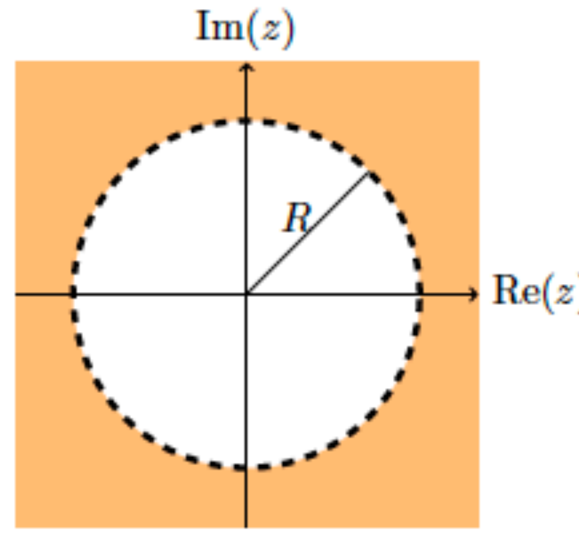
By definition the extended complex plane $= \mathbf{C} \cup \{\infty\}$. That is, we have **one** point at infinity to be thought of in a limiting sense described as follows.

A sequence of points $\{z_n\}$ goes to infinity if $|z_n|$ goes to infinity.



Various sequences all going to infinity.

If we draw a large circle around 0 in the plane, then we call the region **outside** this circle a neighborhood of infinity.



The shaded region outside the circle of radius R is a neighborhood of infinity.

4.4.1 Limits involving infinity

The key idea is $1/\infty = 0$. By this we mean

$$\lim_{z \to \infty} \frac{1}{z} = 0$$

We then have the following facts:

- $\lim_{z \to z_0} f(z) = \infty \Leftrightarrow \lim_{z \to z_0} 1/f(z) = 0$
- $\lim_{z \to \infty} f(z) = w_0 \Leftrightarrow \lim_{z \to 0} f(1/z) = w_0$
- $\lim_{z \to \infty} f(z) = \infty \Leftrightarrow \lim_{z \to 0} \dfrac{1}{f(1/z)} = 0$

Example 4.6. $\lim_{z\to\infty} e^z$ is not defined because it has different values if we go to infinity in different directions, e.g. we have $e^z = e^x e^{iy}$ and

$\lim_{x\to-\infty} e^x e^{iy} = 0$

$\lim_{x\to+\infty} e^x e^{iy} = \infty$

$\lim_{y\to+\infty} e^x e^{iy}$ is not defined, since x is constant, so $e^x e^{iy}$ loops in a circle indefinitely.

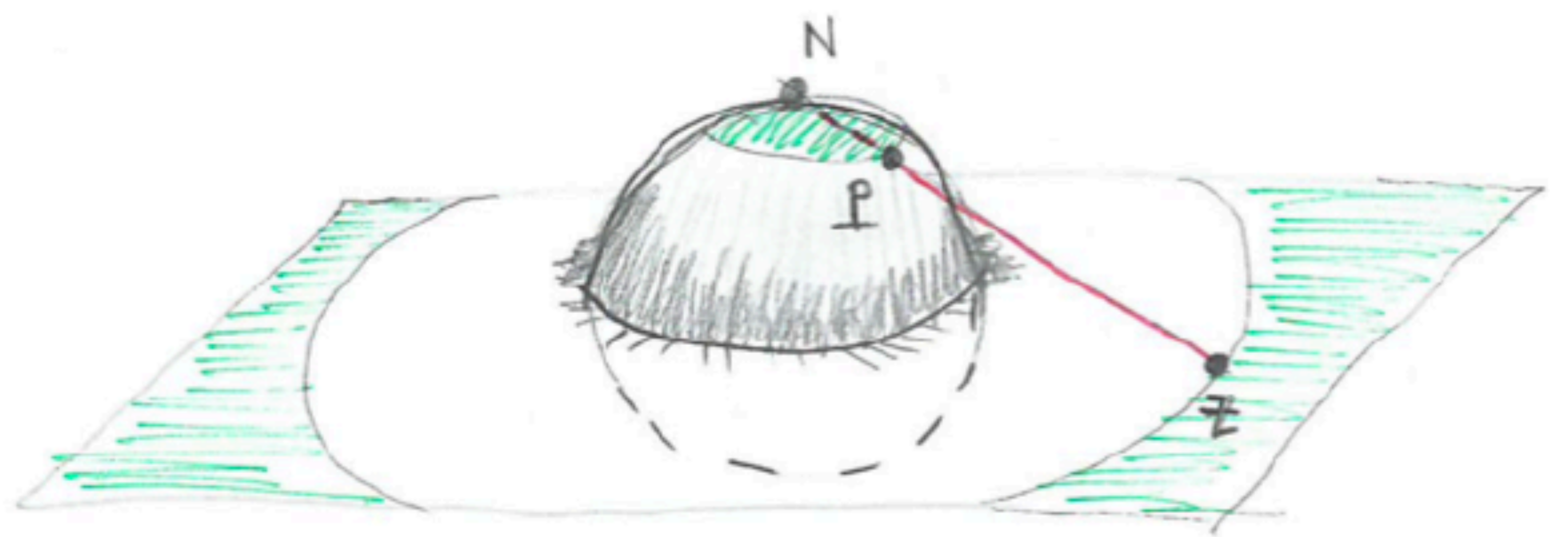
Example 4.7. Show $\lim_{z\to\infty} z^n = \infty$ (for n a positive integer).

Solution: We need to show that $|z^n|$ gets large as $|z|$ gets large. Write $z = Re^{i\theta}$, then

$$|z^n| = |R^n e^{in\theta}| = R^n = |z|^n$$

Clearly, as $|z| = R \to \infty$ also $|z|^n = R^n \to \infty$.

4.4.2 Stereographic projection from the Riemann sphere



Stereographic projection from the sphere to the plane.

$$P = (a, b, c) \mapsto z = \frac{a}{1-c} + i\frac{b}{1-c}$$

The point $N = (0, 0, 1)$ is special, the secant lines from N through P become tangent lines to the sphere at N which never intersect the plane. We consider N the point at infinity.

4.5 Derivatives

$$\boxed{f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}}$$

If the limit exists we say f is analytic at z_0 or f is differentiable at z_0.

Remember: The limit has to exist and be the same no matter how you approach z_0!

If f is analytic at all the points in an open region A then we say f is analytic on A.

Alternative notations:

$$f'(z_0) = \left.\frac{dw}{dz}\right|_{z_0} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

4.5.1 Derivative rules

assuming f and g are differentiable we have:

- Sum rule: $\frac{d}{dz}(f(z)+g(z)) = f' + g'$
- Product rule: $\frac{d}{dz}(f(z)g(z)) = f'g + fg'$
- Quotient rule: $\frac{d}{dz}(f(z)/g(z)) = \frac{f'g - fg'}{g^2}$
- Chain rule: $\frac{d}{dz}g(f(z)) = g'(f(z))f'(z)$
- Inverse rule: $\frac{df^{-1}(z)}{dz} = \frac{1}{f'(f^{-1}(z))}$

(same as for real function)

To give you the flavor of these arguments we'll prove the product rule.

$$
\begin{aligned}
\frac{d}{dz}(f(z)g(z)) &= \lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} \\
&= \lim_{z\to z_0} \frac{(f(z) - f(z_0))g(z) + f(z_0)(g(z) - g(z_0))}{z - z_0} \\
&= \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} g(z) + f(z_0)\frac{(g(z) - g(z_0))}{z - z_0} \\
&= f'(z_0)g(z_0) + f(z_0)g'(z_0)
\end{aligned}
$$