18.04
Complex analysis with applications

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L 02: Complex functions

3 Complex functions

3.1 The exponential function

We have Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. We can extend this to the complex exponential function e^z.

Definition. For $z = x + iy$ the complex exponential function is defined as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

all the usual rules of exponents hold:

1. $e^0 = 1$
2. $e^{z_1+z_2} = e^{z_1}e^{z_2}$
3. $(e^z)^n = e^{nz}$ for positive integers n.
4. $(e^z)^{-1} = e^{-z}$
5. $e^z \neq 0$

It will turn out that the property $\frac{de^z}{dz} = e^z$ also holds, but we can't prove this yet because we haven't defined what we mean by the complex derivative $\frac{d}{dz}$.

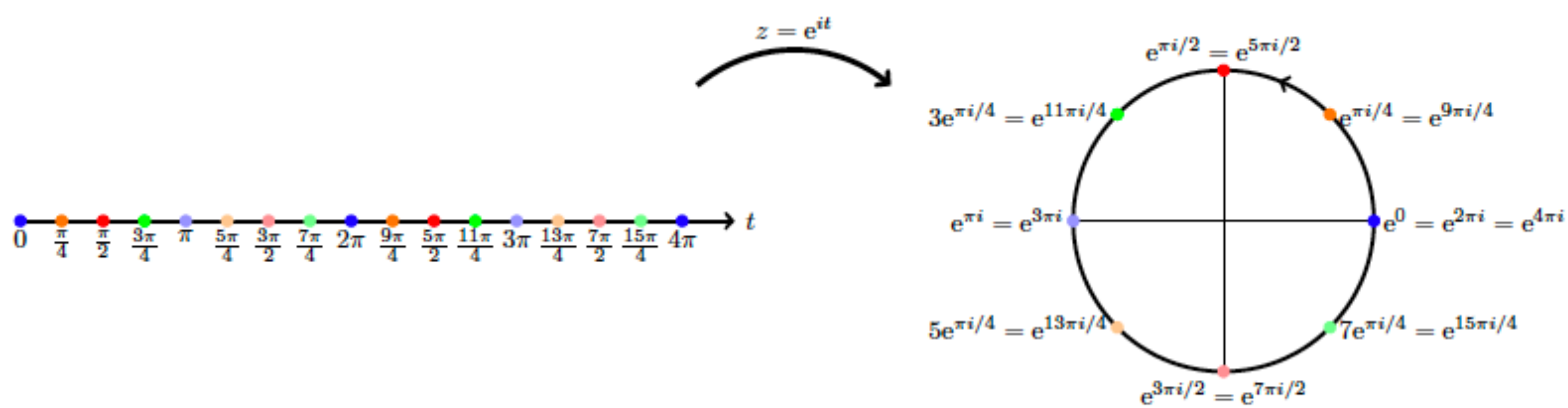
6. $|e^{i\theta}| = 1$

   Proof.
   $$|e^{i\theta}| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

7. $|e^{x+iy}| = e^x$ (as usual $z = x + iy$ and x, y are real).

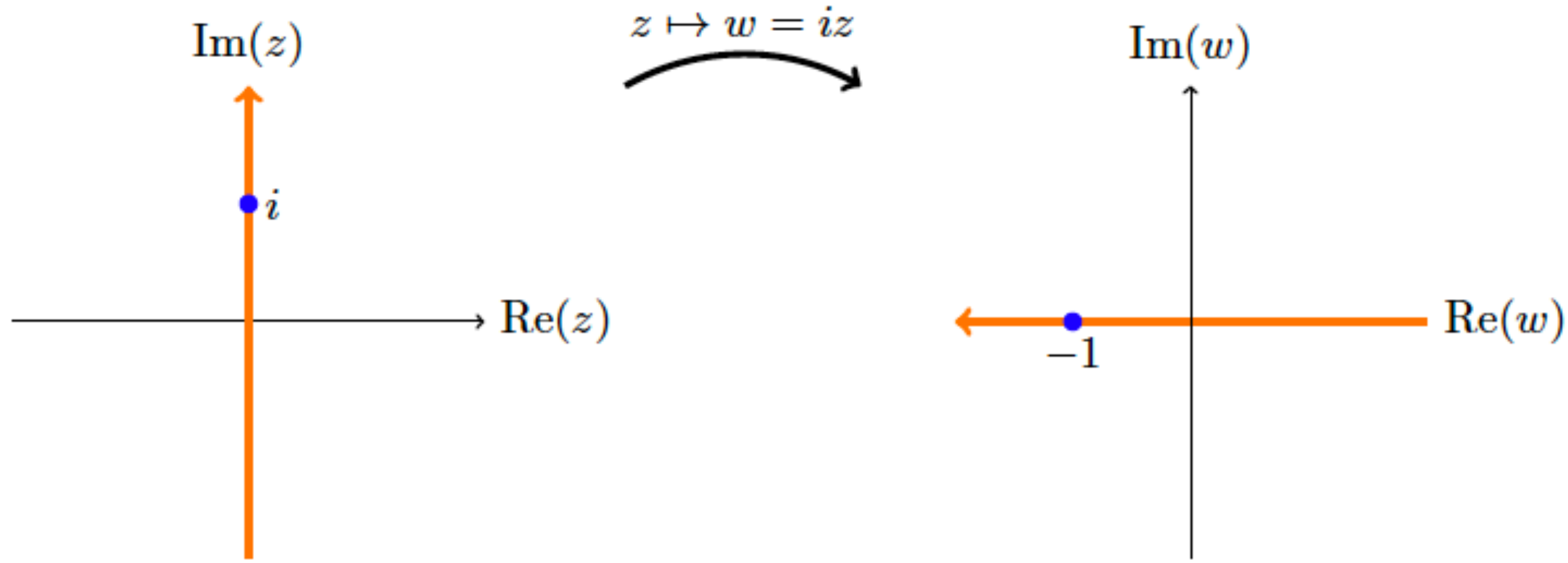
   Proof. You should be able to supply this. If not: ask a teacher or TA.

8. The path e^{it} for $0 < t < \infty$ wraps counterclockwise around the unit circle. It does so infinitely many times. This is illustrated in the following picture.



The map $t \to e^{it}$ wraps the real axis around the unit circle.

3.2 Complex functions as mappings



The image of the imaginary axis under $z \mapsto iz$.

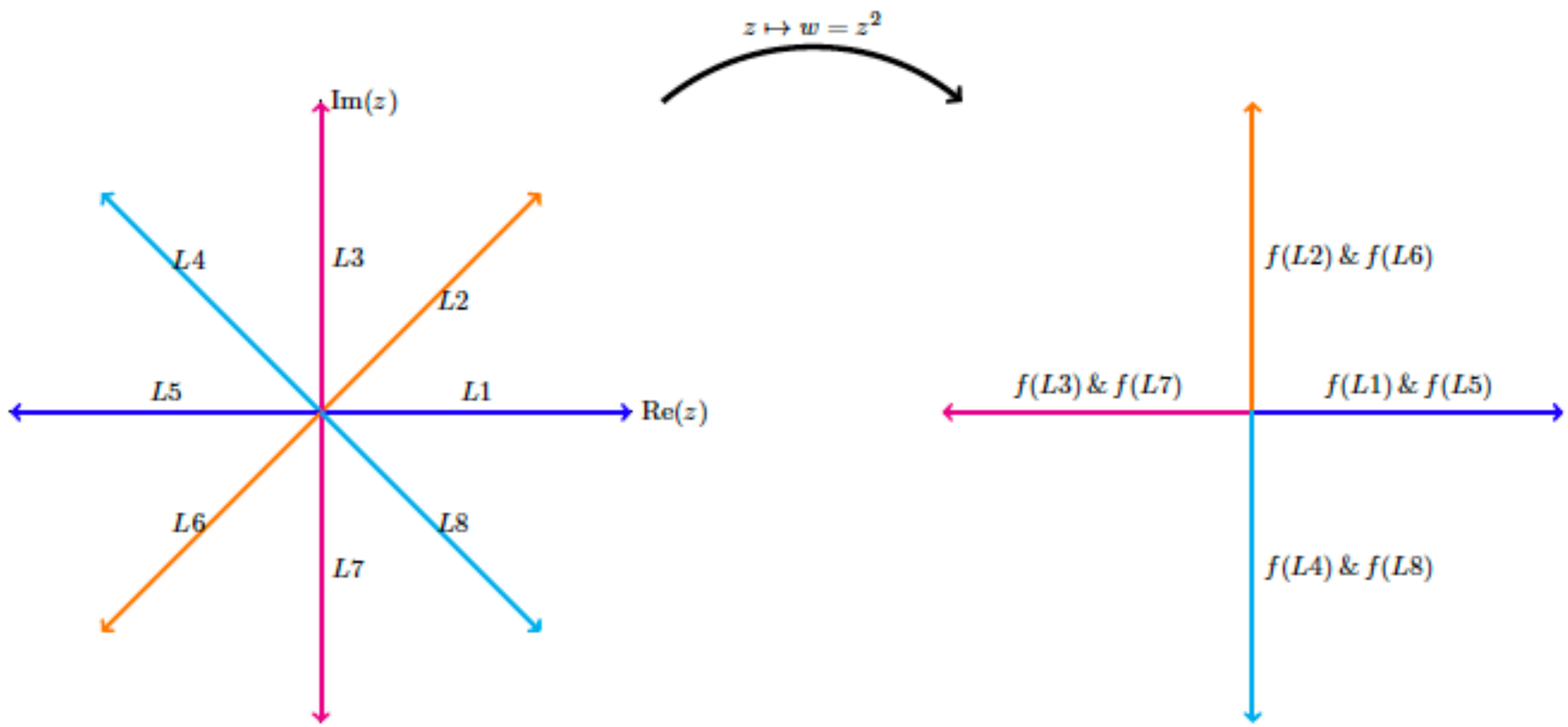
- A function $w = f(z)$ will also be called a mapping of z to w.
- Alternatively we will write $z \mapsto w$ or $z \mapsto f(z)$. This is read as "z maps to w".
- We will say that "w is the image of z under the mapping" or more simply "w is the image of z".
- If we have a set of points in the z-plane we will talk of the image of that set under the mapping. For example, under the mapping $z \mapsto iz$ the image of the imaginary z-axis is the real w-axis.

Example. The mapping $w = z^2$.

1. The ray L_2 at $\pi/4$ radians is mapped to the ray $f(L_2)$ at $\pi/2$ radians.

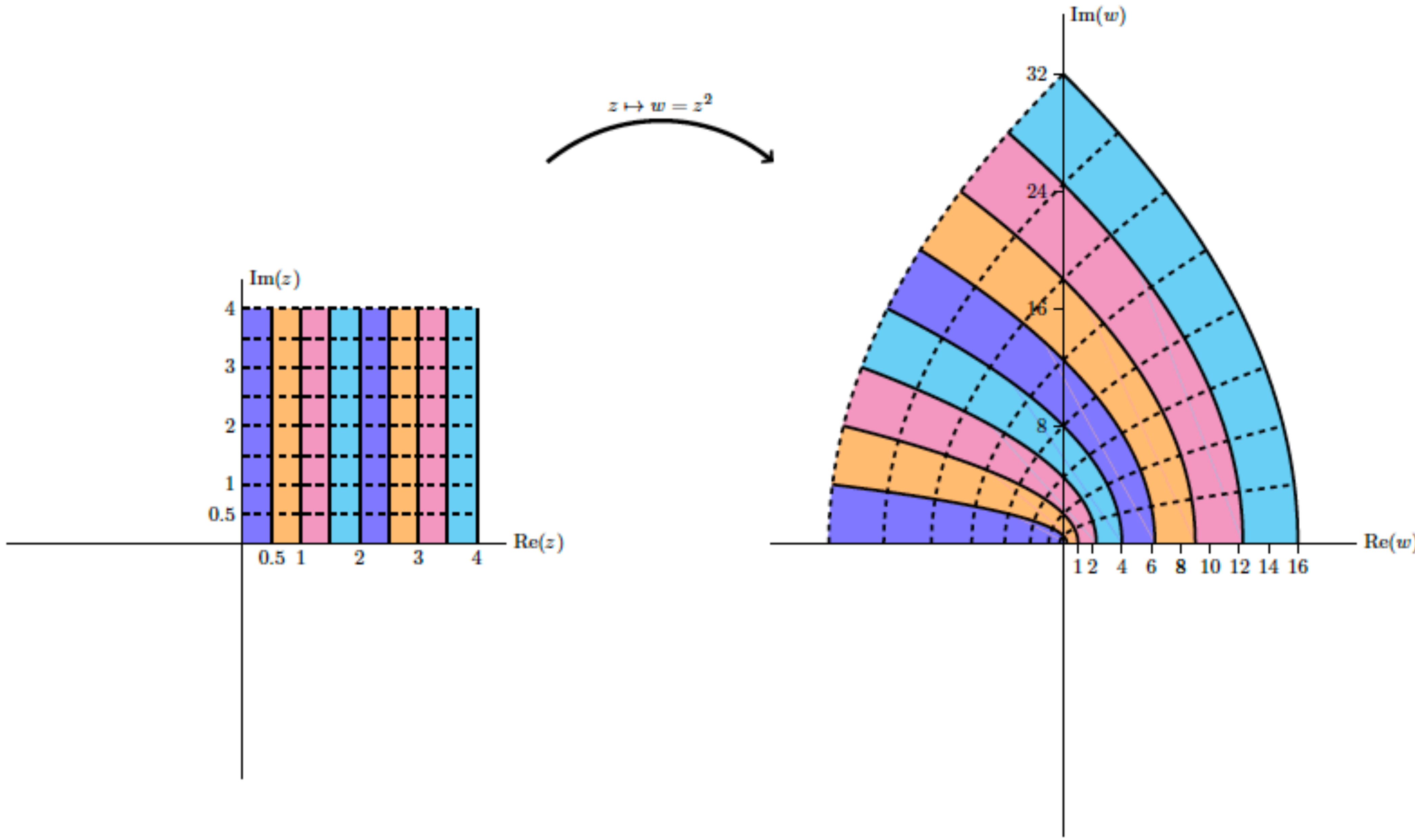
2. The rays L_2 and L_6 are both mapped to the same ray. This is true for each pair of diametrically opposed rays.

3. A ray at angle θ is mapped to the ray at angle 2θ.



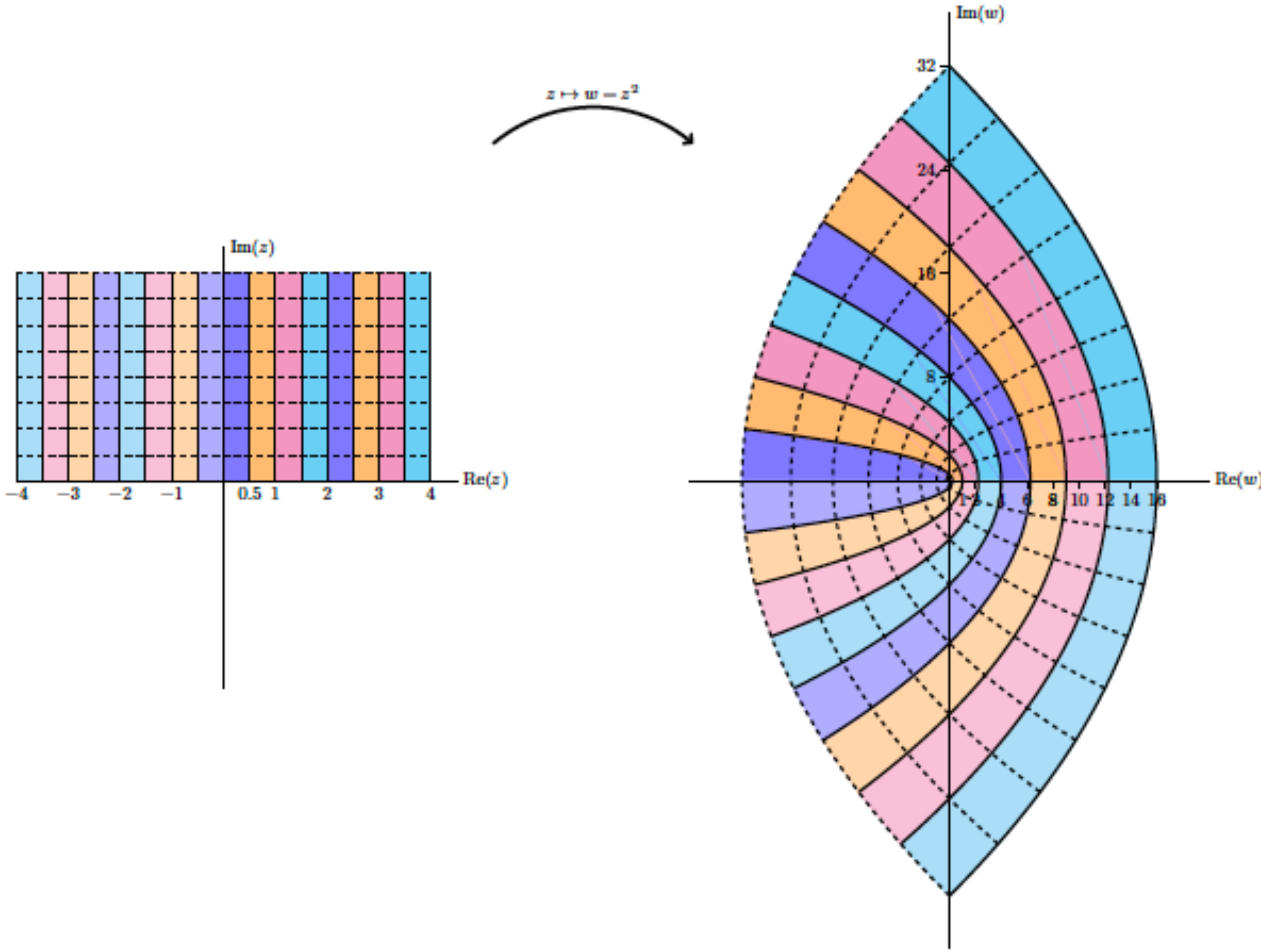
$f(z) = z^2$ maps rays from the origin to rays from the origin.

The next figure gives another view of the mapping. Here we see vertical stripes in the first quadrant are mapped to parabolic stripes that live in the first and second quadrants.



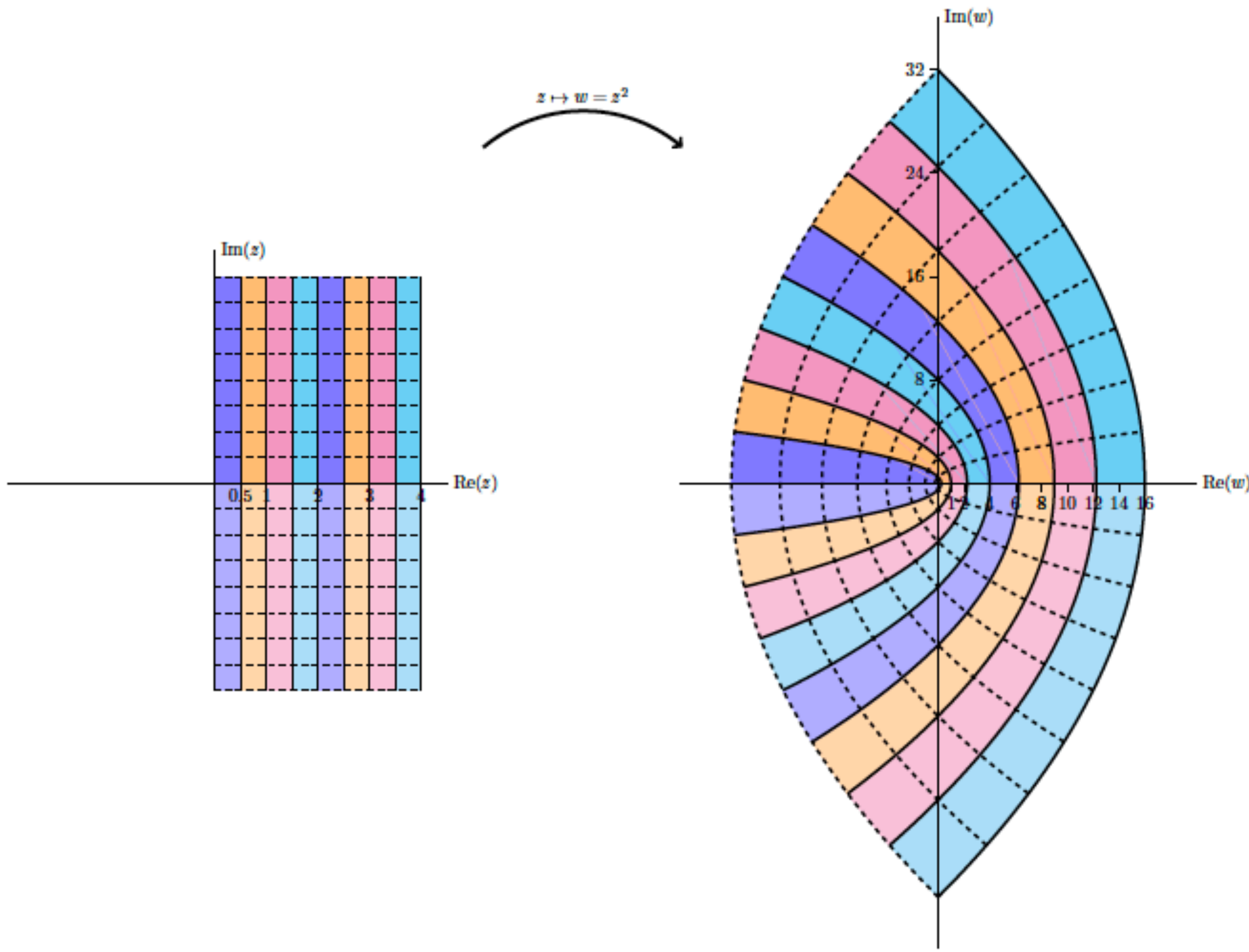
$z^2 = (x^2 - y^2) + i2xy$ maps vertical lines to left facing parabolas.

The next figure is similar to the previous one, except in this figure we look at vertical stripes in both the first and second quadrants. We see that they map to parabolic stripes that live in all four quadrants.



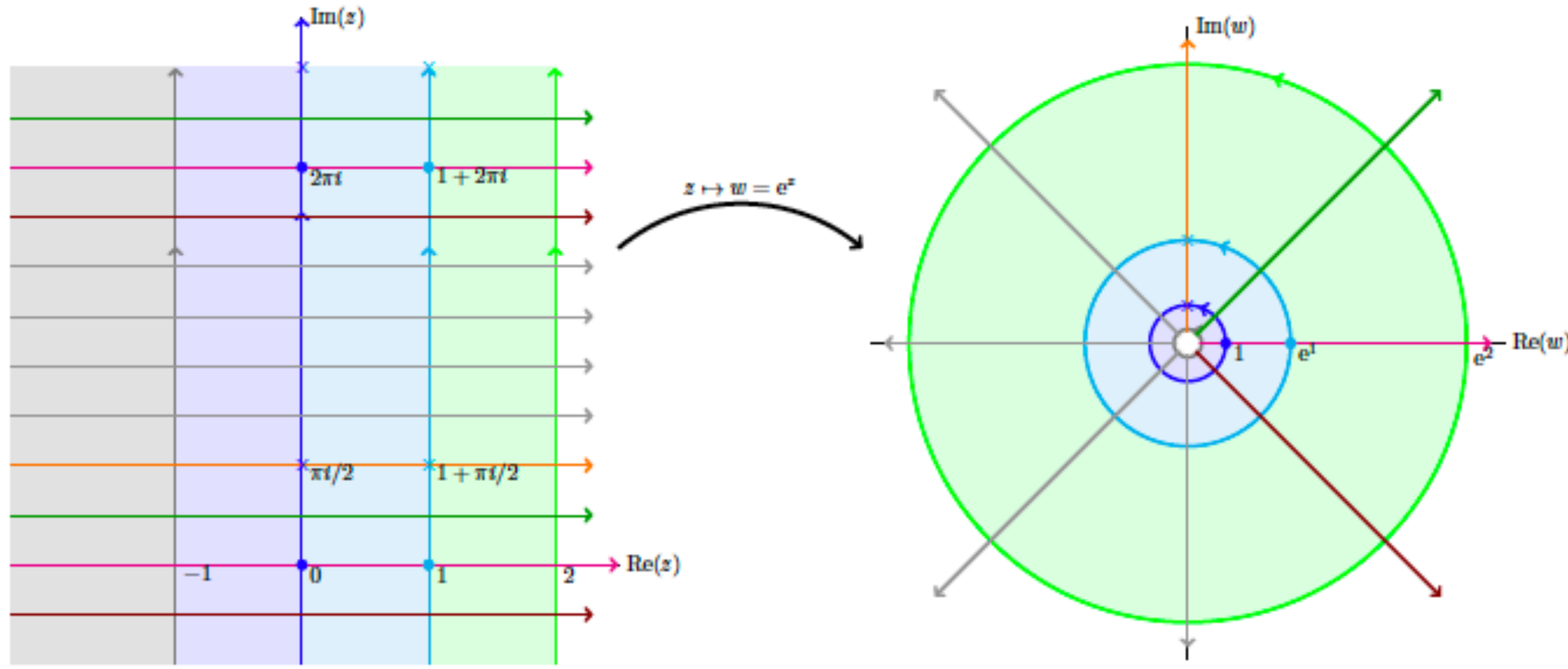
$f(z) = z^2$ maps the first two quadrants to the entire plane.

The next figure shows the mapping of stripes in the first and fourth quadrants. The image map is identical to the previous figure. This is because the fourth quadrant is minus the second quadrant, but $z^2 = (-z)^2$.



Vertical stripes in quadrant 4 are mapped identically to vertical stripes in quadrant 2.

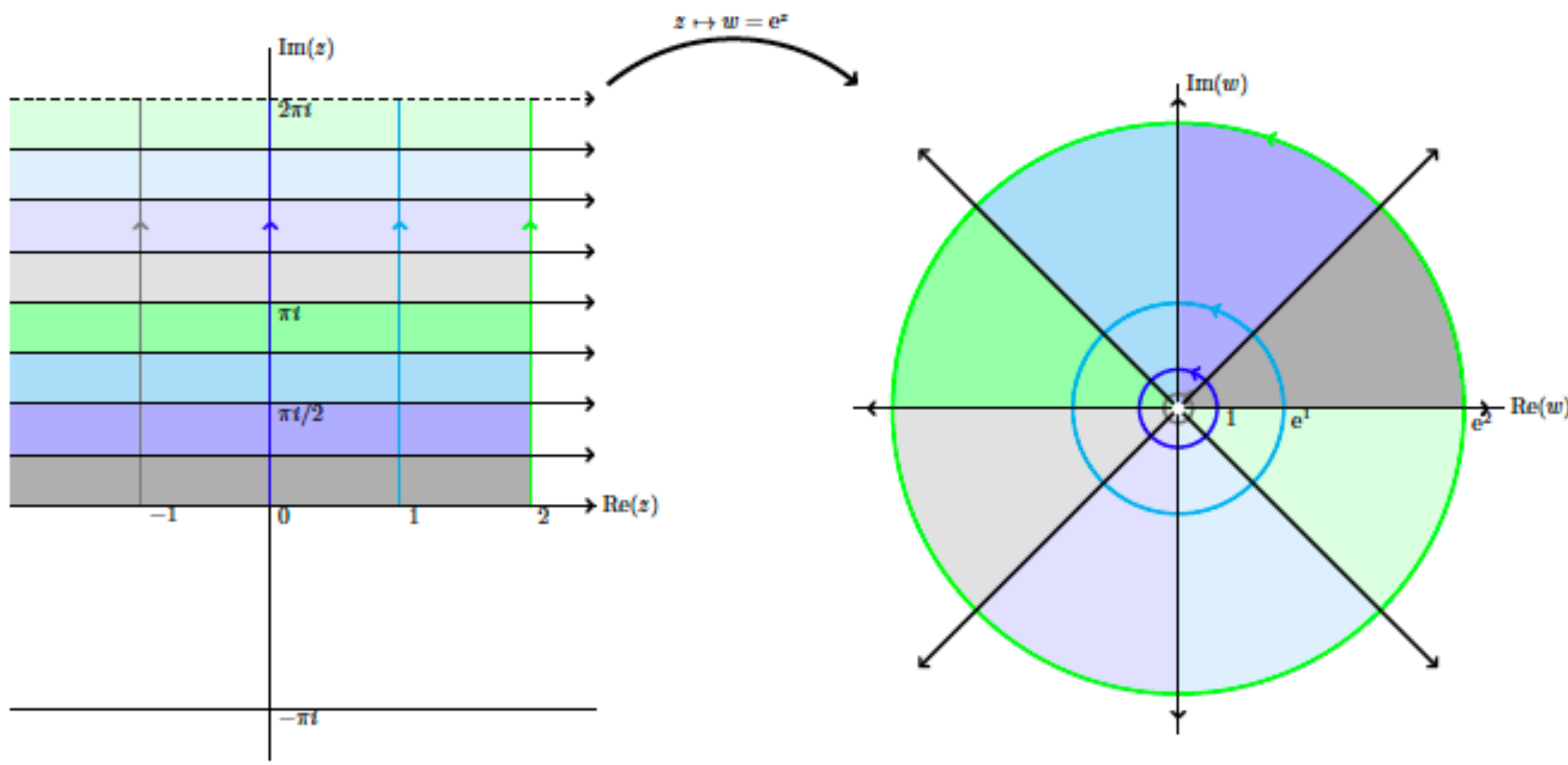
Example. The mapping $w = e^z$.



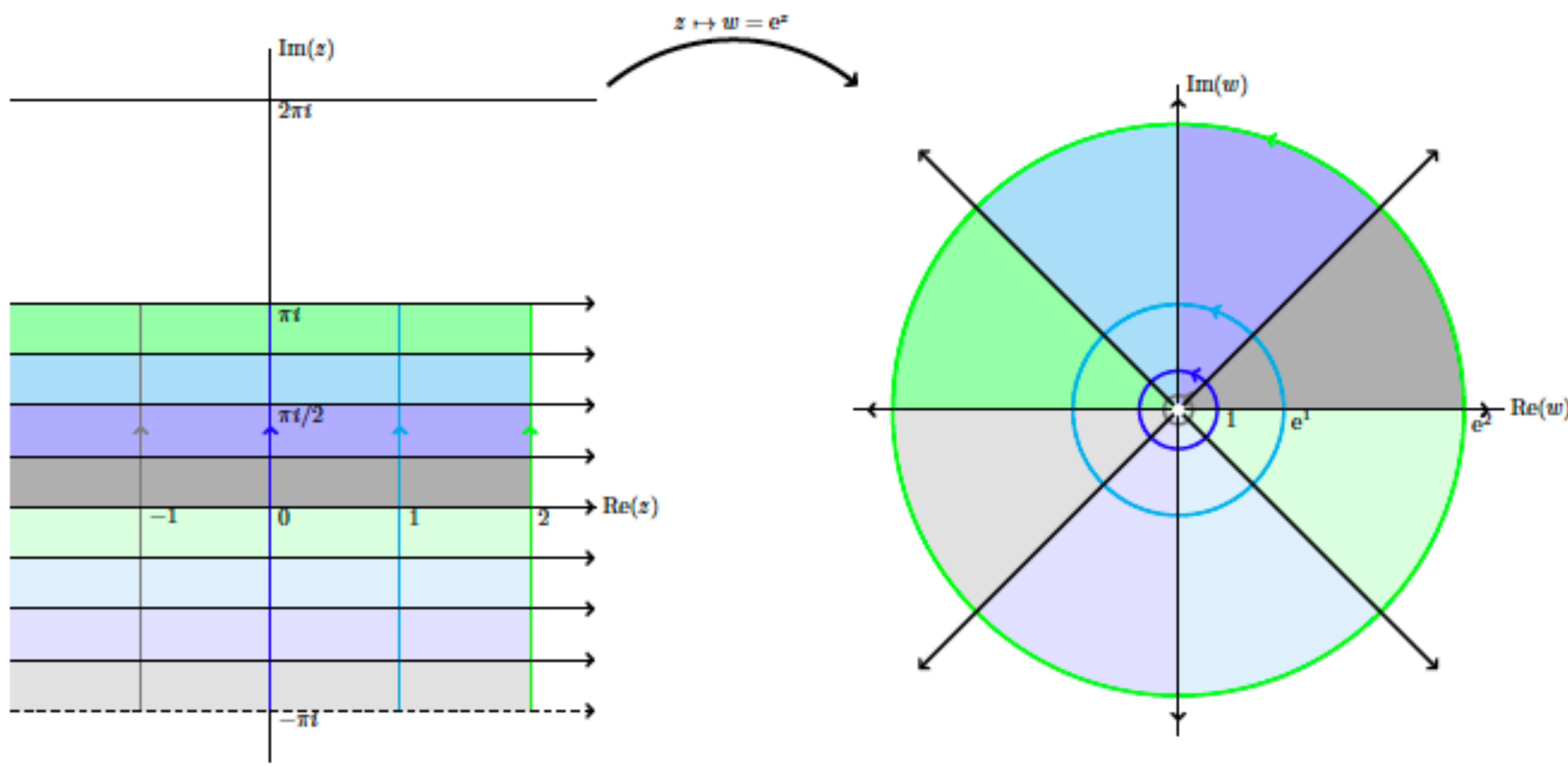
Notice that vertical lines are mapped to circles and horizontal lines to rays from the origin.

Because the plane minus the origin comes up frequently we give it a name:

Definition. The punctured plane is the complex plane minus the origin. In symbols we can write it as $\mathbf{C} - \{0\}$ or $\mathbf{C}/\{0\}$.



The horizontal strip $0 \le y < 2\pi$ is mapped to the punctured plane



The horizontal strip $-\pi < y \leq \pi$ is mapped to the punctured plane

3.3 The function $\arg(z)$

3.3.1 Many-to-one functions

The function $f(z) = z^2$ maps $\pm z$ to the same value, e.g. $f(2) = f(-2) = 4$.

Example 1. $w = z^3$ is a 3-to-1 function. For example, 3 different z values get mapped to $w = 1$:

$$1^3 = \left(\frac{-1+\sqrt{3}\,i}{2}\right)^3 = \left(\frac{-1-\sqrt{3}\,i}{2}\right)^3 = 1$$

Example 2. The function $w = \mathrm{e}^z$ maps infinitely many points to each value. For example

$$\mathrm{e}^0 = \mathrm{e}^{2\pi i} = \mathrm{e}^{4\pi i} = \ldots = \mathrm{e}^{n 2\pi i} = \ldots = 1$$
$$\mathrm{e}^{i\pi/2} = \mathrm{e}^{i\pi/2+2\pi i} = \mathrm{e}^{i\pi/2+4\pi i} = \ldots = \mathrm{e}^{i\pi/2+n2\pi i} = \ldots = i$$

In general, $\mathrm{e}^{z+n\,2\pi i}$ has the same value for every integer n.

3.3.2 Branches of $\arg(z)$

The key point is that the argument is only defined up to multiples of $2\pi i$ so every z produces infinitely many values for $\arg(z)$. Because of this we will say that $\arg(z)$ is a multiple-valued function.

Definition. By a branch of the argument function we mean a choice of range so that it becomes single-valued. By specifying a branch we are saying that we will take the single value of $\arg(z)$ that lies in the branch.

(i) If we specify the branch as $0 \le \arg(z) < 2\pi$ then we have the following arguments.

$$\arg(1) = 0; \quad \arg(i) = \pi/2; \quad \arg(-1) = \pi; \quad \arg(-i) = 3\pi/2$$

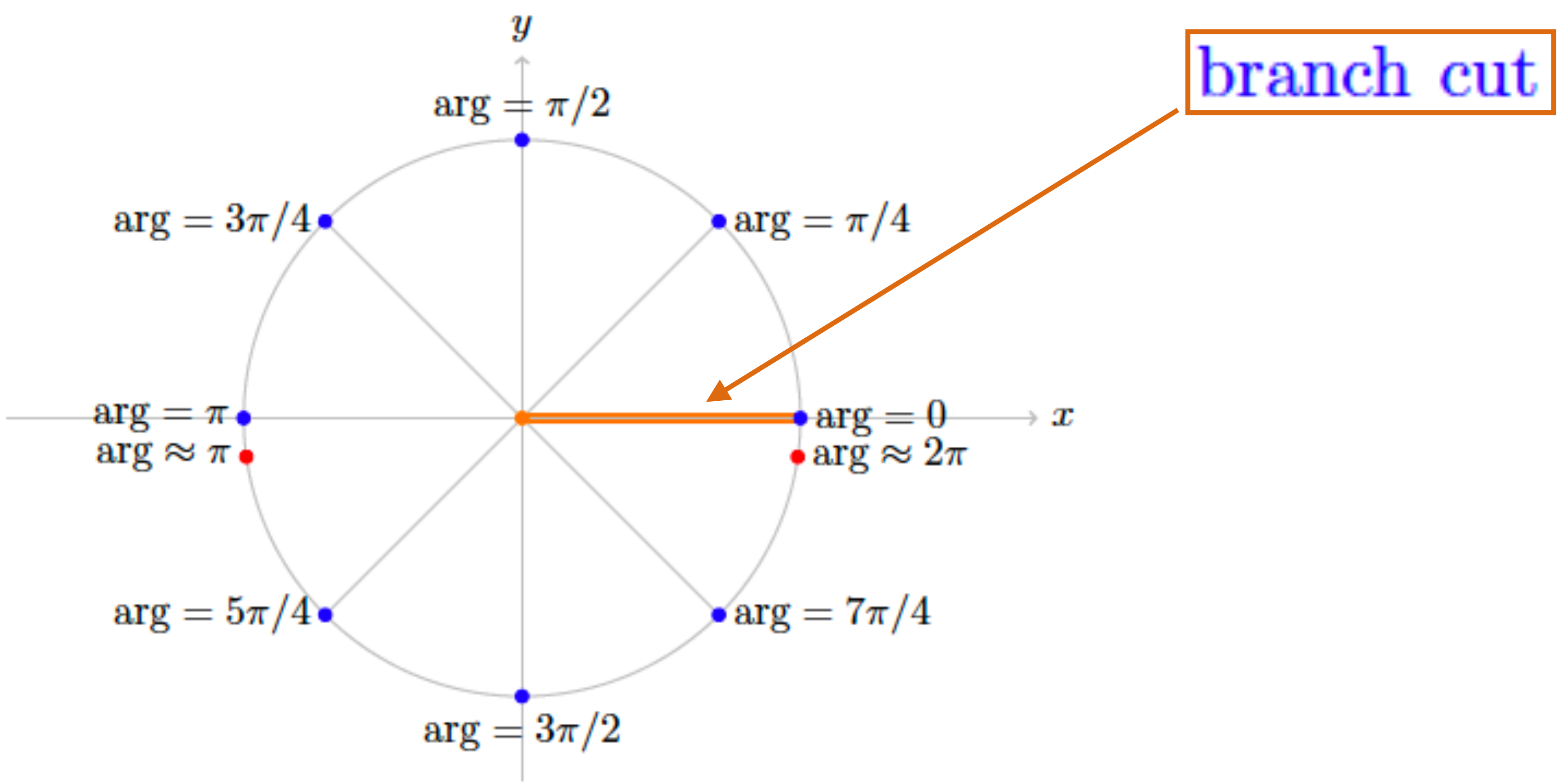


Figure (i): The branch $0 \le \arg(z) < 2\pi$ of $\arg(z)$.

For future reference you should note that, on this branch, $\arg(z)$ is continuous near the negative real axis, i.e. the arguments of nearby points are close to each other.

(ii) If we specify the branch as $-\pi < \arg(z) \leq \pi$ then we have the following arguments:

$$\arg(1) = 0; \quad \arg(i) = \pi/2; \quad \arg(-1) = \pi; \quad \arg(-i) = -\pi/2$$

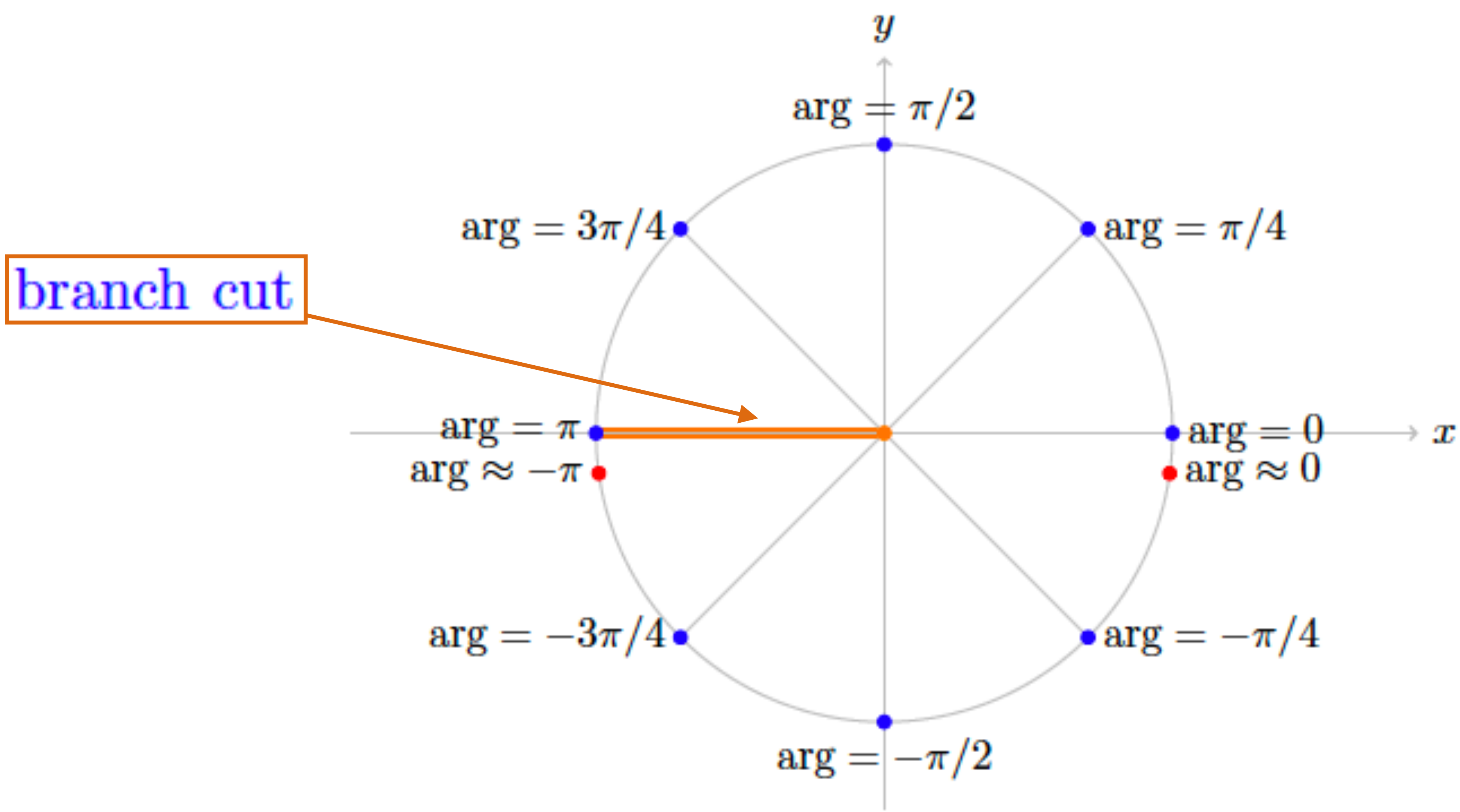


Figure (ii): The branch $-\pi < \arg(z) \leq \pi$ of $\arg(z)$.

(iii) Figure (iii) shows the branch of $\arg(z)$ with $\pi/4 \le \arg(z) < 9\pi/4$.

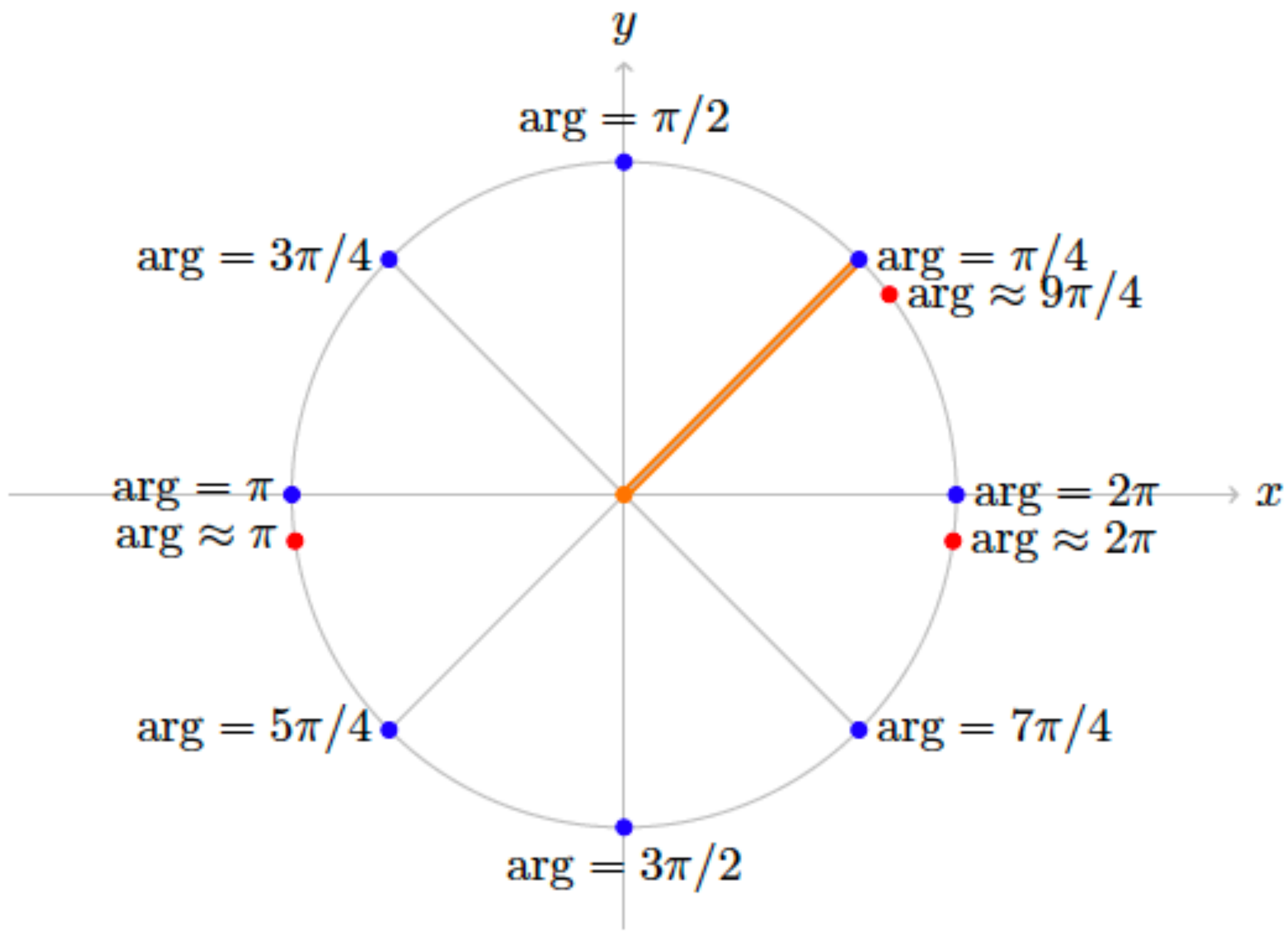


Figure (iii): The branch $\pi/4 \le \arg(z) < 9\pi/4$ of $\arg(z)$.

(iv) Obviously, there are many many possible branches. For example,

$$42 < \arg(z) \leq 42 + 2\pi.$$

(v) We won't make use of this in 18.04, but, in fact, the branch cut doesn't have to be a straight line. Any curve that goes from the origin to infinity will do. The argument will be continuous except for a jump by 2π when z crosses the branch cut.

3.3.3 The principal branch of $\arg(z)$

Definition. The branch $-\pi < \arg(z) \leq \pi$ is called the principal branch of $\arg(z)$. We will use the notation $\text{Arg}(z)$ (capital A) to indicate that we are using the principal branch.

Need Arg to be continuous ⇒ restrict to plane minus branch cut

3.4 Concise summary of branches and branch cuts

Consider the function $w = f(z)$. Suppose that $z = x + iy$ and $w = u + iv$.

Domain. The domain of f is the set of z where we are allowed to compute $f(z)$.

Range. The range (image) of f is the set of all $f(z)$ for z in the domain, i.e. the set of all w reached by f.

Branch. For a multiple-valued function, a branch is a choice of range for the function. We choose the range to exclude all but one possible value for each element of the domain.

Branch cut. A branch cut removes (cuts) points out of the domain. This is done to remove points where the function is discontinuous.

3.5 The function $\log(z)$

$$\boxed{\text{want } e^{\log(z)} = z}$$

Example. Find $\log(1)$

$$\log(1) = n\,2\pi i, \text{ where } n \text{ is any integer}$$

"multi-valued"

If $z = re^{i\theta}$

$$\log(z) = \log(re^{i\theta}) = \log(r) + i\theta,$$

Definition. The function $\log(z)$ is defined as

$$\log(z) = \log(|z|) + i\arg(z),$$

where $\log(|z|)$ is the usual natural logarithm of a positive real number.

Remarks.

1. Since $\arg(z)$ has infinitely many possible values, so does $\log(z)$.

2. $\log(0)$ is not defined. (Both because $\arg(0)$ is not defined and $\log(|0|)$ is not defined.)

3. Choosing a branch for $\arg(z)$ makes $\log(z)$ single valued. The usual terminology is to say we have chosen a branch of the log function.

4. The principal branch of log comes from the principal branch of arg. That is,

$$\log(z) = \log(|z|) + i \arg(z), \quad \text{where} \quad -\pi < \arg(z) \leq \pi \quad \text{(principal branch)}.$$

Example. Compute all the values of $\log(i)$.

Solution: We have that $|i| = 1$ and $\arg(i) = \dfrac{\pi}{2} + 2\pi n$, so

$$\log(i) = \log(1) + i\frac{\pi}{2} + i2\pi n = i\frac{\pi}{2} + i2\pi n, \text{ where } n \text{ is any integer.}$$

The principal branch of $\arg(z)$ is between $-\pi$ and π, so $\mathrm{Arg}(i) = \pi/2$. The value of $\log(i)$ from the principal branch is therefore $i\pi/2$.

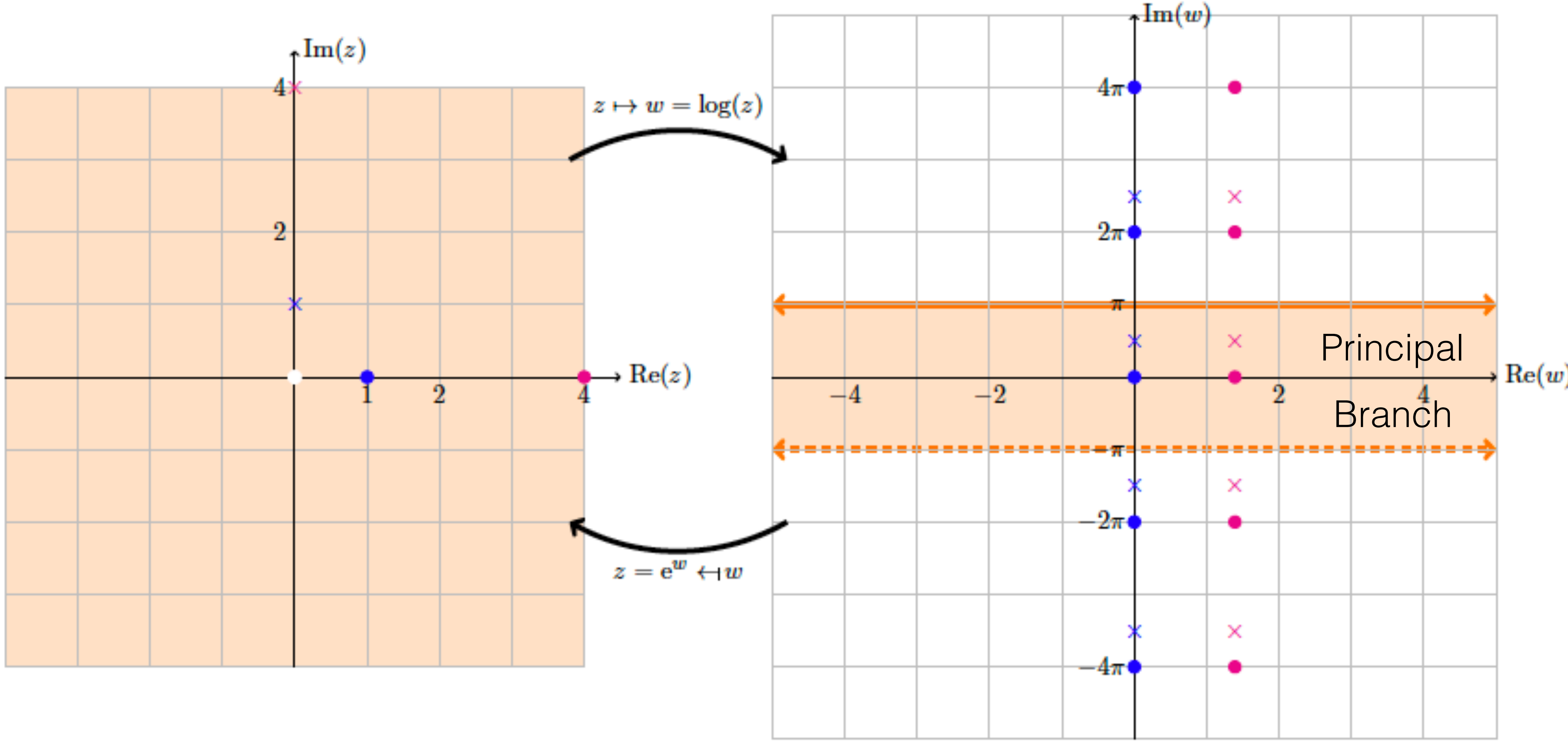
Example. Compute all the values of $\log(-1 - \sqrt{3}\,i)$. Specify which one comes from the principal branch.

Solution: Let $z = -1 - \sqrt{3}\,i$. Then $|z| = 2$ and in the principal branch $\mathrm{Arg}(z) = -2\pi/3$. So all the values of $\log(z)$ are

$$\log(z) = \log(2) - i\frac{2\pi}{3} + i2\pi n.$$

The value from the principal branch is $\log(z) = \log(2) - i2\pi/3$.

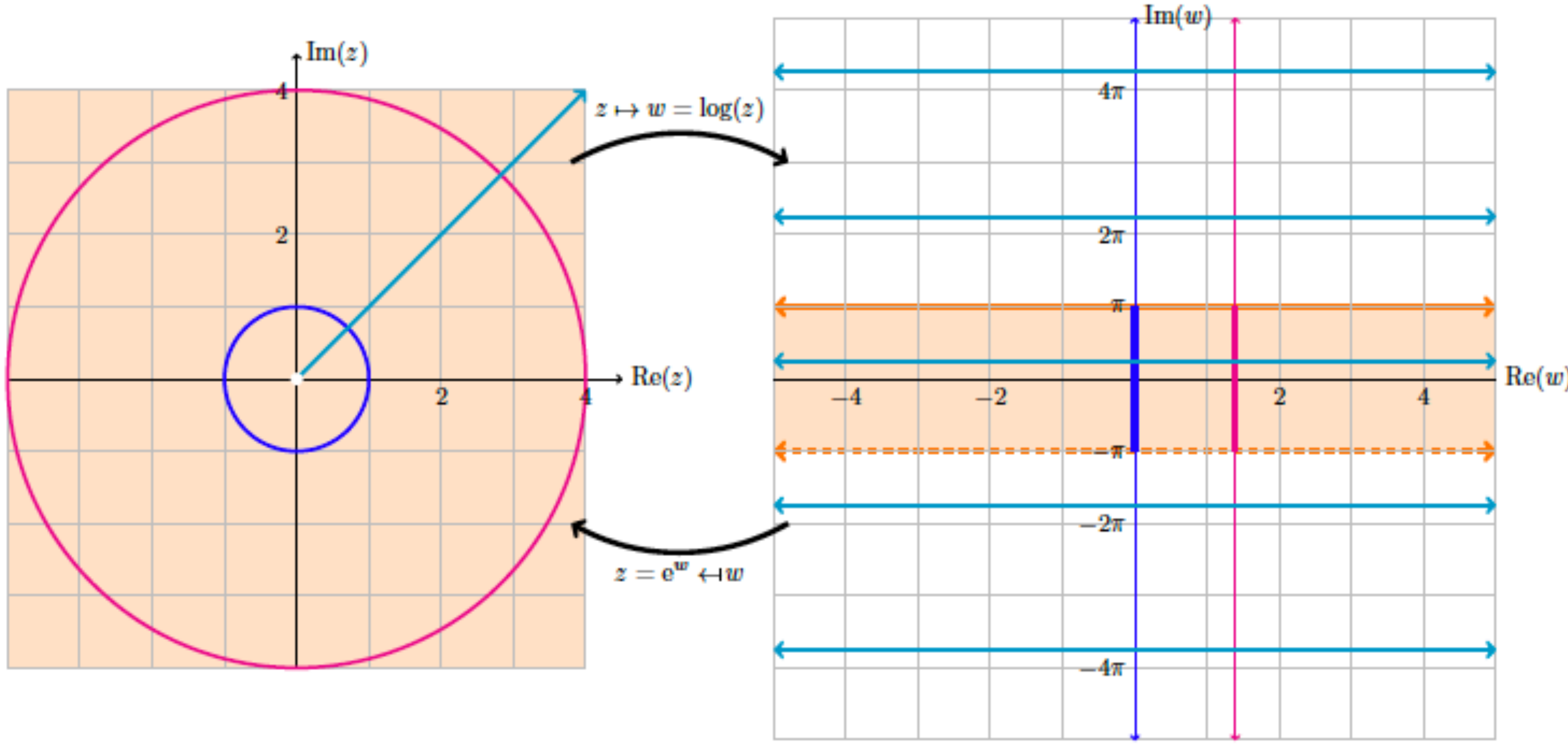
3.5.1 Figures showing $w = \log(z)$ as a mapping



Mapping $\log(z)$: $\log(1)$, $\log(4)$, $\log(i)$, $\log(4i)$

circles centered on 0 are mapped to vertical lines,

rays from the origin are mapped to horizontal lines.



Mapping $\log(z)$: mapping circles and rays

3.5.2 Complex powers

Definition. Let z and a be complex numbers then the power z^a is defined as

$$z^a = \mathrm{e}^{a\log(z)}.$$

"multi-valued"

Example. Compute all the values of $\sqrt{2i}$. Give the value associated to the principal branch of $\log(z)$.

Solution: We have

$$\log(2i) = \log(2\mathrm{e}^{\frac{i\pi}{2}}) = \log(2) + i\frac{\pi}{2} + i2n\pi$$

So,

$$\sqrt{2i} = (2i)^{1/2} = \mathrm{e}^{\frac{\log(2i)}{2}} = \mathrm{e}^{\frac{\log(2)}{2}+\frac{i\pi}{4}+in\pi} = \sqrt{2}\mathrm{e}^{\frac{i\pi}{4}+in\pi}.$$

principal branch has $\mathrm{Arg}(2i) = \pi/2$, so

$$\sqrt{2i} = \sqrt{2}\mathrm{e}^{\left(\frac{i\pi}{4}\right)} = \sqrt{2}\frac{(1+i)}{\sqrt{2}} = 1+i.$$

The other distinct value is when $n = 1$ and gives minus the value just above.

Example. Cube roots: Compute all the cube roots of i. Give the value which comes from the principal branch of $\log(z)$.

Solution: We have $\log(i) = i\dfrac{\pi}{2} + i2\pi n$, where n is any integer. So,

$$i^{1/3} = e^{\frac{\log(i)}{3}} = e^{i\frac{\pi}{6} + i\frac{2n\pi}{3}}$$

This gives only three distinct values

$$e^{i\pi/6}, \quad e^{i5\pi/6}, \quad e^{i9\pi/6}$$

On the principal branch $\log(i) = i\dfrac{\pi}{2}$, so the value of $i^{1/3}$ which comes from this is

$$e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}.$$

Example. Compute all the values of 1^i. What is the value from the principal branch?

Solution: This is similar to the problems above. $\log(1) = 2n\pi i$, so

$$1^i = \mathrm{e}^{i\log(1)} = \mathrm{e}^{i2n\pi i} = \mathrm{e}^{-2n\pi}, \text{ where } n \text{ is an integer.}$$

The principal branch has $\log(1) = 0$ so $1^i = 1$.