18.04
Complex analysis with applications

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L 01: Exponentials & representations

2.6 Euler's Formula

Def: $$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

2.6.1 $e^{i\theta}$ behaves like a true exponential

P1. e^{it} differentiates as expected:

$$\frac{de^{it}}{dt} = ie^{it}$$

Proof.

$$\frac{de^{it}}{dt} = \frac{d}{dt}(\cos(t) + i\sin(t)) = -\sin(t) + i\cos(t) = i(\cos(t) + i\sin(t)) = ie^{it}$$

P2. $e^{i\cdot 0} = 1$.

Proof.

$$e^{i\cdot 0} = \cos(0) + i\sin(0) = 1$$

P4. The definition of $e^{i\theta}$ is consistent with the power series for e^x

Proof.

$$
\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots
\end{aligned}
$$

$$
\begin{aligned}
e^{i\theta} &= \sum_0^\infty \frac{(i\theta)^n}{n!} \\
&= \sum_0^\infty (-1)^k \frac{\theta^{2k}}{(2k)!} + i \sum_0^\infty (-1)^k \frac{\theta^{2k+1}}{(2k+1)!} \\
&= \cos(\theta) + i\sin(\theta).
\end{aligned}
$$

Properties **P1-P4** should convince you that $e^{i\theta}$ behaves like an exponential.

2.6.2 Complex exponentials and polar form

$$z = x + iy = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta)) = r\mathrm{e}^{i\theta}.$$

the polar form of z.

Magnitude, argument, conjugate, multiplication and division are easy in polar form.

Magnitude. $|\mathrm{e}^{i\theta}| = 1$.

Proof.

$$|\mathrm{e}^{i\theta}| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1$$

Argument. If $z = r\mathrm{e}^{i\theta}$ then $\arg(z) = \theta$.

Proof. This is again the definition: the argument is the polar angle θ.

Conjugate. $\overline{(re^{i\theta})} = re^{-i\theta}$.

Proof.

$$\overline{(re^{i\theta})} = \overline{r(\cos(\theta) + i\sin(\theta))} = r(\cos(\theta) - i\sin(\theta)) = r(\cos(-\theta) + i\sin(-\theta)) = re^{-i\theta}$$

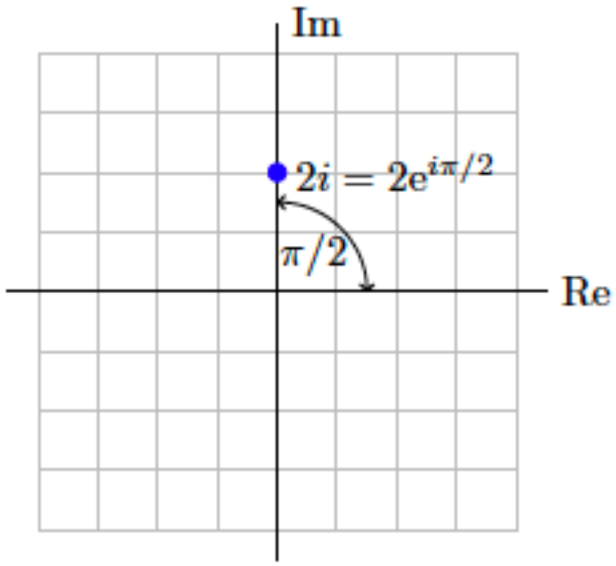
Multiplication. If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

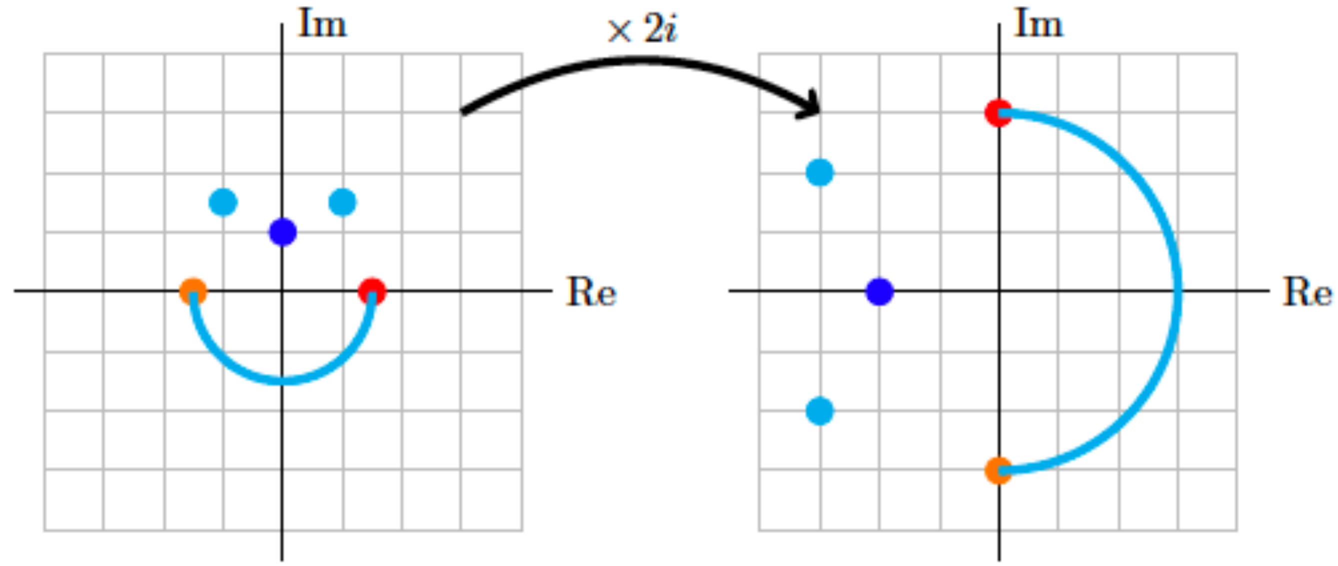
Division. Again it's trivial that

$$\frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Example. (Multiplication by $2i$)



$|2i| = 2, \quad \arg(2i) = \pi/2$



Multiplication by $2i$ rotates by $\pi/2$ and scales by 2

Example. (Raising to a power) Let's compute $(1+i)^6$ and $\left(\frac{1+i\sqrt{3}}{2}\right)^3$.

Solution: $1+i$ has magnitude $= \sqrt{2}$ and arg $= \pi/4$, so $1+i = \sqrt{2}\mathrm{e}^{i\pi/4}$. Raising to a power is now easy:

$$(1+i)^6 = \left(\sqrt{2}\mathrm{e}^{i\pi/4}\right)^6 = 8\mathrm{e}^{6i\pi/4} = 8\mathrm{e}^{3i\pi/2} = -8i$$

Similarly, $\dfrac{1+i\sqrt{3}}{2} = \mathrm{e}^{i\pi/3}$, so $\left(\dfrac{1+i\sqrt{3}}{2}\right)^3 = (1\cdot \mathrm{e}^{i\pi/3})^3 = \mathrm{e}^{i\pi} = -1$

2.6.3 Complexification or complex replacement

Example. Use complex replacement to compute

$$I = \int dx\ \mathrm{e}^{x} \cos(2x)$$

Solution: We have Euler's formula

$$\mathrm{e}^{2ix} = \cos(2x) + i \sin(2x)$$

$$I_c = \int dx\ (\mathrm{e}^{x} \cos 2x + i\mathrm{e}^{x} \sin 2x)\ , \qquad I = \mathrm{Re}(I_c).$$

Computing I_c is straightforward:

$$I_c = \int dx\ \mathrm{e}^{x}\mathrm{e}^{i2x} = \int dx\ \mathrm{e}^{x(1+2i)} = \frac{\mathrm{e}^{x(1+2i)}}{1+2i}$$

To find real part in rectangular coordinates

$$
\begin{aligned}
I_c &= \frac{e^{x(1+2i)}}{1+2i} \cdot \frac{1-2i}{1-2i} \\
&= \frac{e^x(\cos(2x) + i\,\sin(2x))(1-2i)}{5} \\
&= \frac{1}{5} e^x(\cos(2x) + 2\sin(2x) + i(-2\cos(2x) + \sin(2x)))
\end{aligned}
$$

So,

$$I = \mathrm{Re}(I_c) = \frac{1}{5} e^x(\cos(2x) + 2\sin(2x)).$$

We can use polar coordinates to simplify the expression for I_c:

$1 + 2i = re^{i\phi}$, where $r = \sqrt{5}$ and $\phi = \arg(1+2i) = \tan^{-1}(2)$ in the first quadrant.

$$I_c = \frac{e^{x(1+2i)}}{\sqrt{5}e^{i\phi}} = \frac{e^x}{\sqrt{5}} e^{i(2x-\phi)} = \frac{e^x}{\sqrt{5}}(\cos(2x-\phi) + i\,\sin(2x-\phi))$$

Thus,

$$I = \mathrm{Re}(I_c) = \frac{e^x}{\sqrt{5}} \cos(2x-\phi)$$

2.6.4 Nth roots

$$\boxed{z^N = c} \qquad c = Re^{i\phi} \text{ and } z = re^{i\theta}$$

$$r^N e^{iN\theta} = Re^{i\phi}$$

$$\Rightarrow \qquad r = R^{1/N} \ , \qquad N\theta = \phi + 2\pi n \ , \qquad n = 0, \pm 1, \pm 2, \ldots$$

Solving for θ, we have

$$\theta = \frac{\phi}{N} + 2\pi \frac{n}{N}.$$

Example. Find all 5 fifth roots of 2.

Solution: For $c = 2$, we have $R = 2$ and $\phi = 0$, so the fifth roots of 2 are

$$z_n = 2^{1/5} \mathrm{e}^{2n\pi i/5}, \text{ where } n = 0, \pm 1, \pm 2, \dots$$

This means, we have 5 different roots corresponding to $n = 0, 1, 2, 3, 4$.

$$z_n = 2^{1/5},\ 2^{1/5}\mathrm{e}^{2\pi i/5},\ 2^{1/5}\mathrm{e}^{4\pi i/5},\ 2^{1/5}\mathrm{e}^{6\pi i/5},\ 2^{1/5}\mathrm{e}^{8\pi i/5}$$

Example. Find the 4 fourth roots of 1.

Solution: Need to solve $z^4 = 1$, so $\phi = 0$. So the 4 distinct fourth roots are in polar form

$$z_n = 1,\ \mathrm{e}^{i\pi/2},\ \mathrm{e}^{i\pi},\ \mathrm{e}^{i3\pi/2}$$

and in Cartesian representation

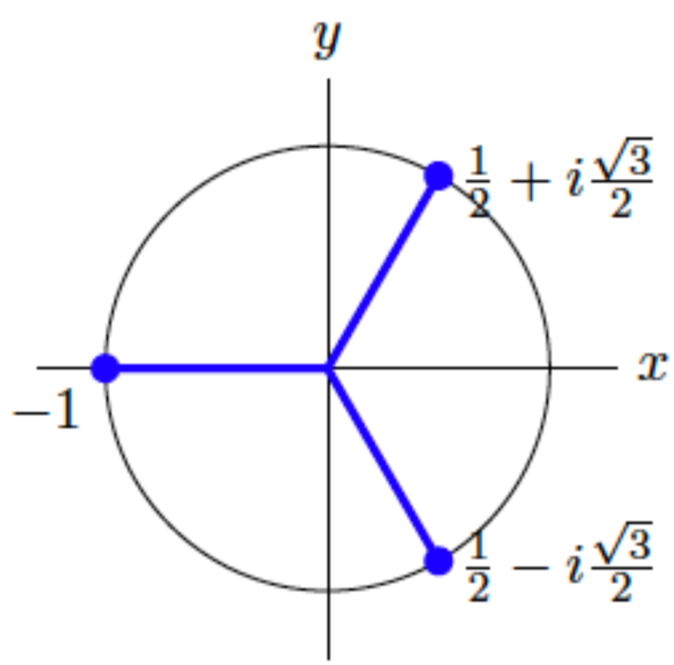
$$z_n = 1,\ i,\ -1,\ -i$$

Example.

Find the 2 square roots of 4.

Solution: $z^2 = 4\mathrm{e}^{i\,2\pi n}$. So, $z_n = 2\mathrm{e}^{i\,\pi n}$ with $n = 0, 1$. So the two roots are $2\mathrm{e}^0 = +2$ and $2\mathrm{e}^{i\pi} = -2$ as expected.

2.6.5 The geometry of Nth roots



Cube roots of -1

$$z^3 = -1$$

$$z \cdot z \cdot z \cdot 1 = -1$$

2.7 Inverse Euler formula

$$\mathrm{e}^{it} = \cos(t) + i\sin(t) \quad \text{and} \quad \mathrm{e}^{-it} = \cos(t) - i\sin(t).$$

By adding and subtracting we get:

$$\cos(t) = \frac{\mathrm{e}^{it} + \mathrm{e}^{-it}}{2} \quad \text{and} \quad \sin(t) = \frac{\mathrm{e}^{it} - \mathrm{e}^{-it}}{2i}.$$

Please take note of these formulas we will use them frequently!

2.8 de Moivre's formula

For positive integers n we have de Moivre's formula:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

Proof. This is a simple consequence of Euler's formula:

$$(\cos(\theta) + i\sin(\theta))^n = (\mathrm{e}^{i\theta})^n = \mathrm{e}^{in\theta} = \cos(n\theta) + i\sin(n\theta)$$

2.9 Representing complex multiplication as matrix multiplication

Consider two complex numbers $z_1 = a + bi$ and $z_2 = a + bi$ and their product

$$z_1 z_2 = (a + bi)(x + iy) = (ax - by) + i(bx + ay) =: w$$

Now let's define two matrices

$$Z_1 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \qquad Z_2 = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

$$Z_1 Z_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} ax - by & -(bx + ay) \\ bx + ay & ax - by \end{bmatrix} = W$$

$$Z = \begin{bmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ \operatorname{Im} z & \operatorname{Re} z \end{bmatrix} = \operatorname{Re} z \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \operatorname{Im} z \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

i.e., the imaginary unit i corresponds to the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $i^2 = -1$ becomes

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Polar form (decomposition). Writing $z = re^{i\theta} = r(\cos\theta + i \sin\theta)$, we find

$$Z = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$$

multiplication by i corresponds to the rotation with angle $\theta = \pi/2$ and $r = 1$

3 Complex functions

3.1 The exponential function

We have Euler's formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$. We can extend this to the complex exponential function e^z.

Definition. For $z = x + iy$ the complex exponential function is defined as

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)).$$

all the usual rules of exponents hold:

1. $e^0 = 1$
2. $e^{z_1+z_2} = e^{z_1}e^{z_2}$
3. $(e^z)^n = e^{nz}$ for positive integers n.
4. $(e^z)^{-1} = e^{-z}$
5. $e^z \neq 0$

It will turn out that the property $\dfrac{de^z}{dz} = e^z$ also holds, but we can't prove this yet because we haven't defined what we mean by the complex derivative $\dfrac{d}{dz}$.

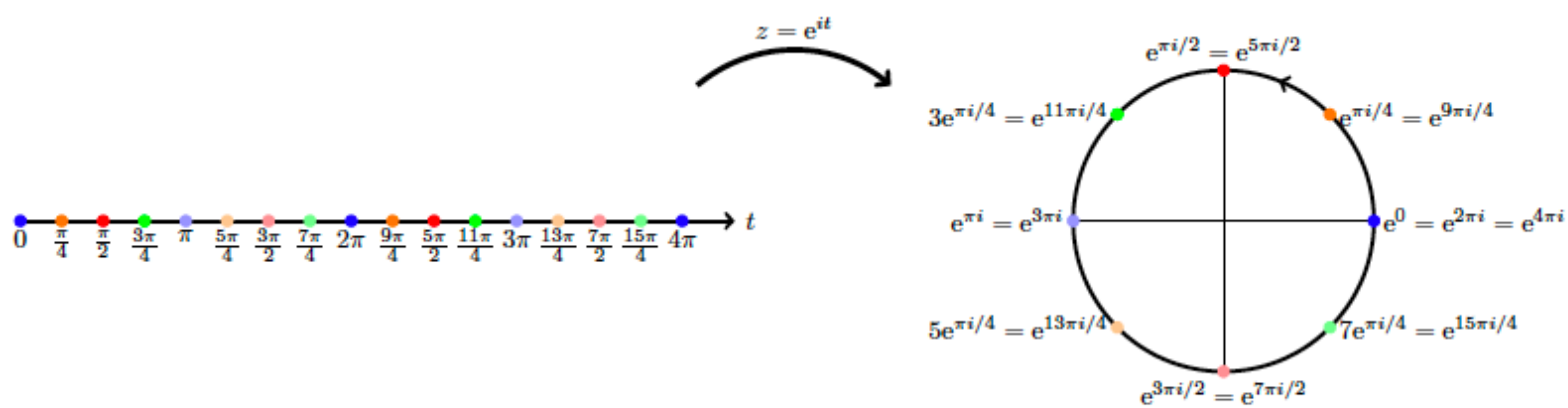
6. $|e^{i\theta}| = 1$

   Proof.
   $$|e^{i\theta}| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1.$$

7. $|e^{x+iy}| = e^x$ (as usual $z = x + iy$ and x, y are real).

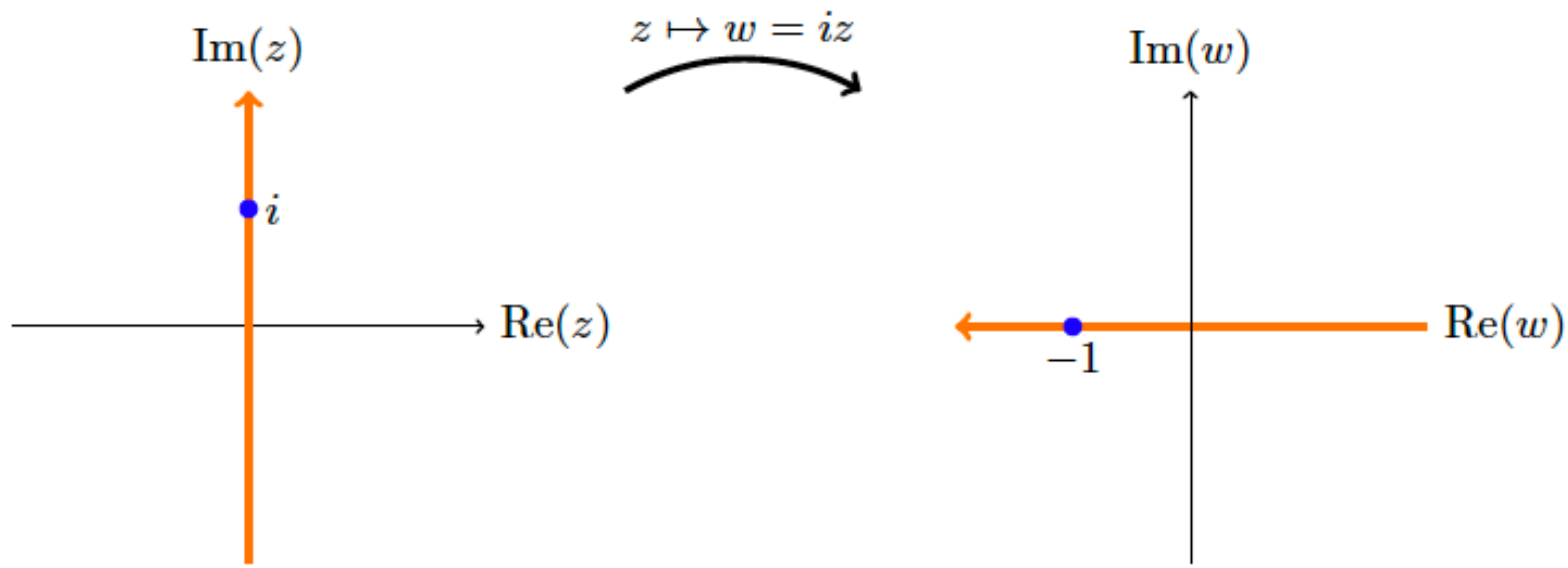
   Proof. You should be able to supply this. If not: ask a teacher or TA.

8. The path e^{it} for $0 < t < \infty$ wraps counterclockwise around the unit circle. It does so infinitely many times. This is illustrated in the following picture.



The map $t \to e^{it}$ wraps the real axis around the unit circle.

3.2 Complex functions as mappings



The image of the imaginary axis under $z \mapsto iz$.

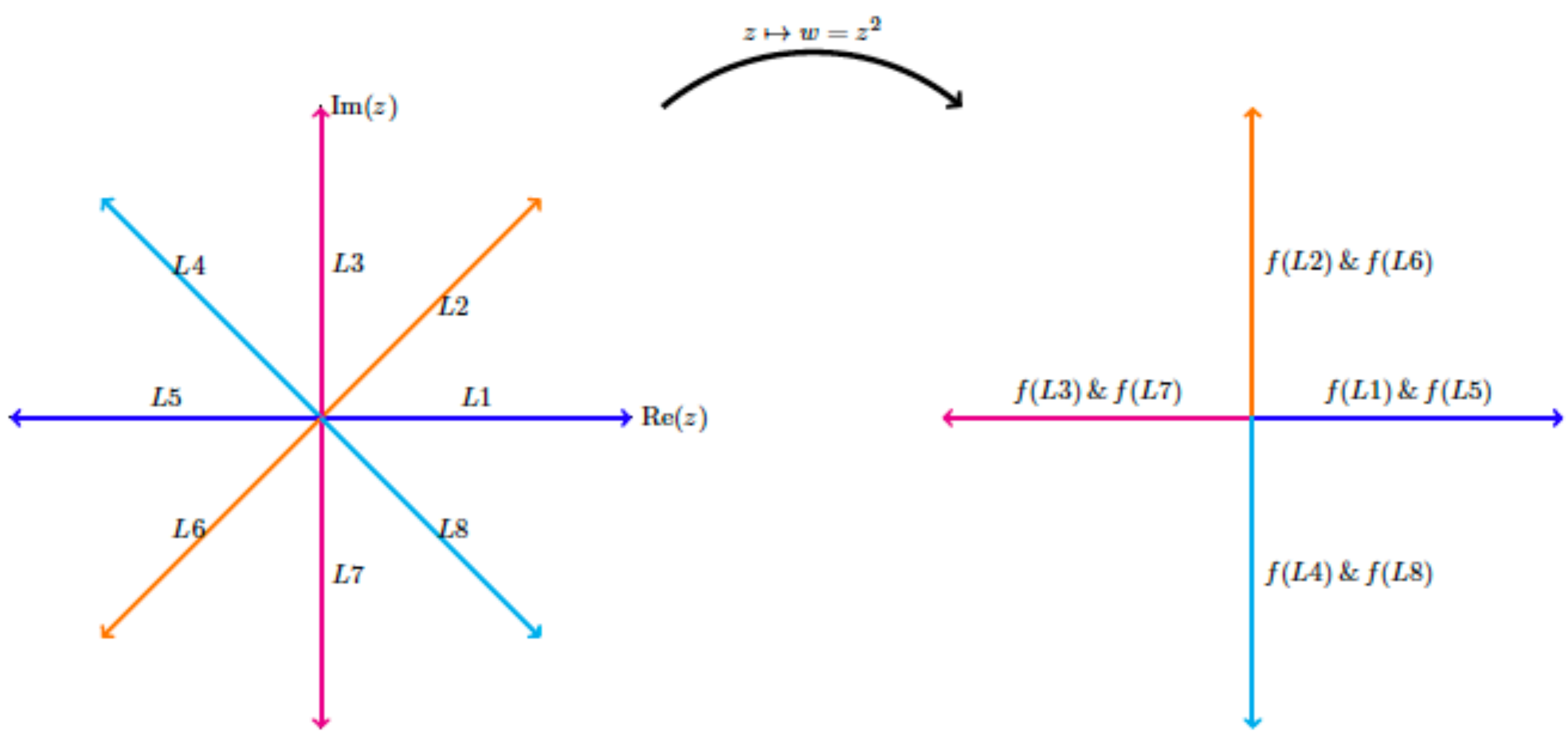
- A function $w = f(z)$ will also be called a mapping of z to w.
- Alternatively we will write $z \mapsto w$ or $z \mapsto f(z)$. This is read as "z maps to w".
- We will say that "w is the image of z under the mapping" or more simply "w is the image of z".
- If we have a set of points in the z-plane we will talk of the image of that set under the mapping. For example, under the mapping $z \mapsto iz$ the image of the imaginary z-axis is the real w-axis.

Example. The mapping $w = z^2$.

1. The ray L_2 at $\pi/4$ radians is mapped to the ray $f(L_2)$ at $\pi/2$ radians.

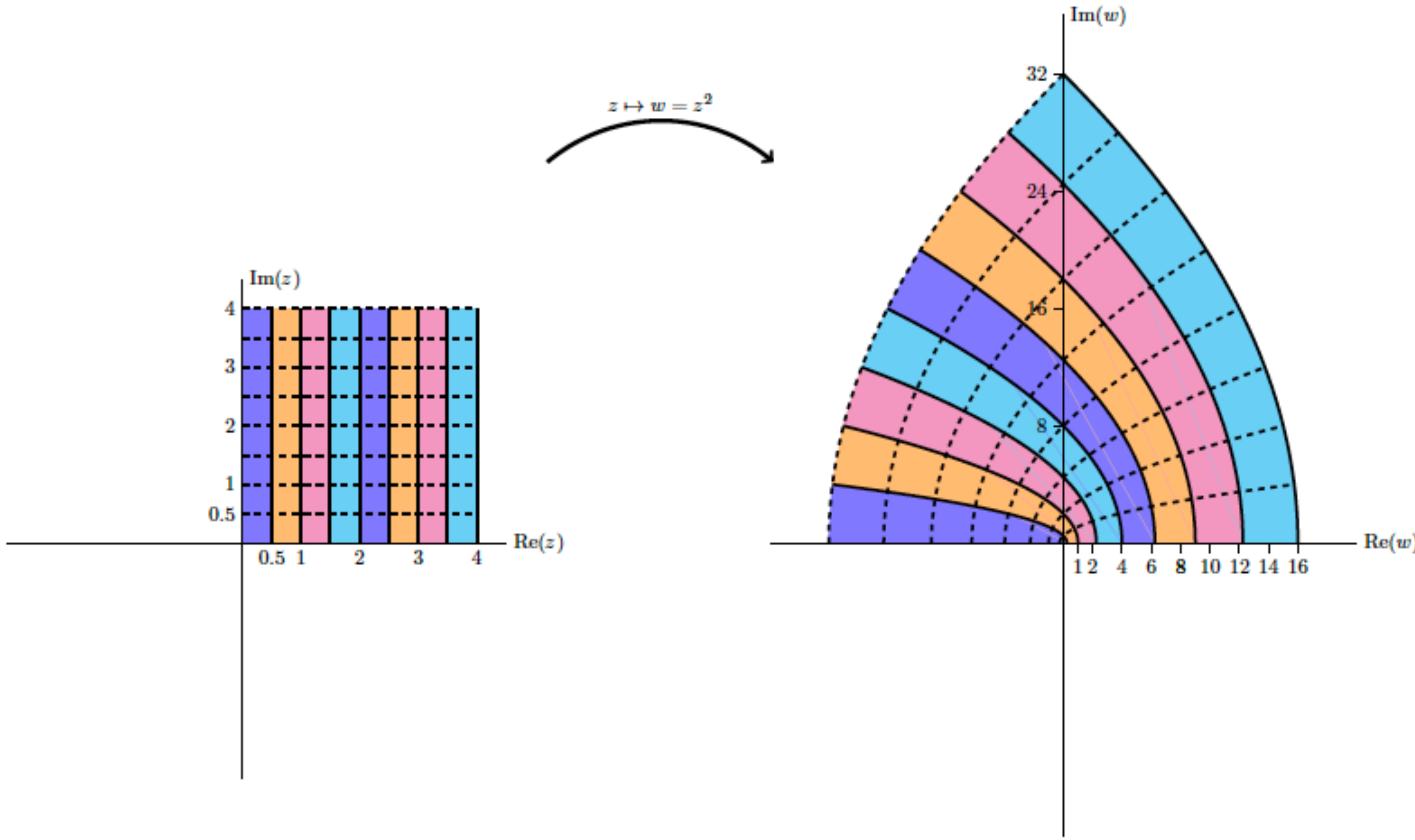
2. The rays L_2 and L_6 are both mapped to the same ray. This is true for each pair of diametrically opposed rays.

3. A ray at angle θ is mapped to the ray at angle 2θ.



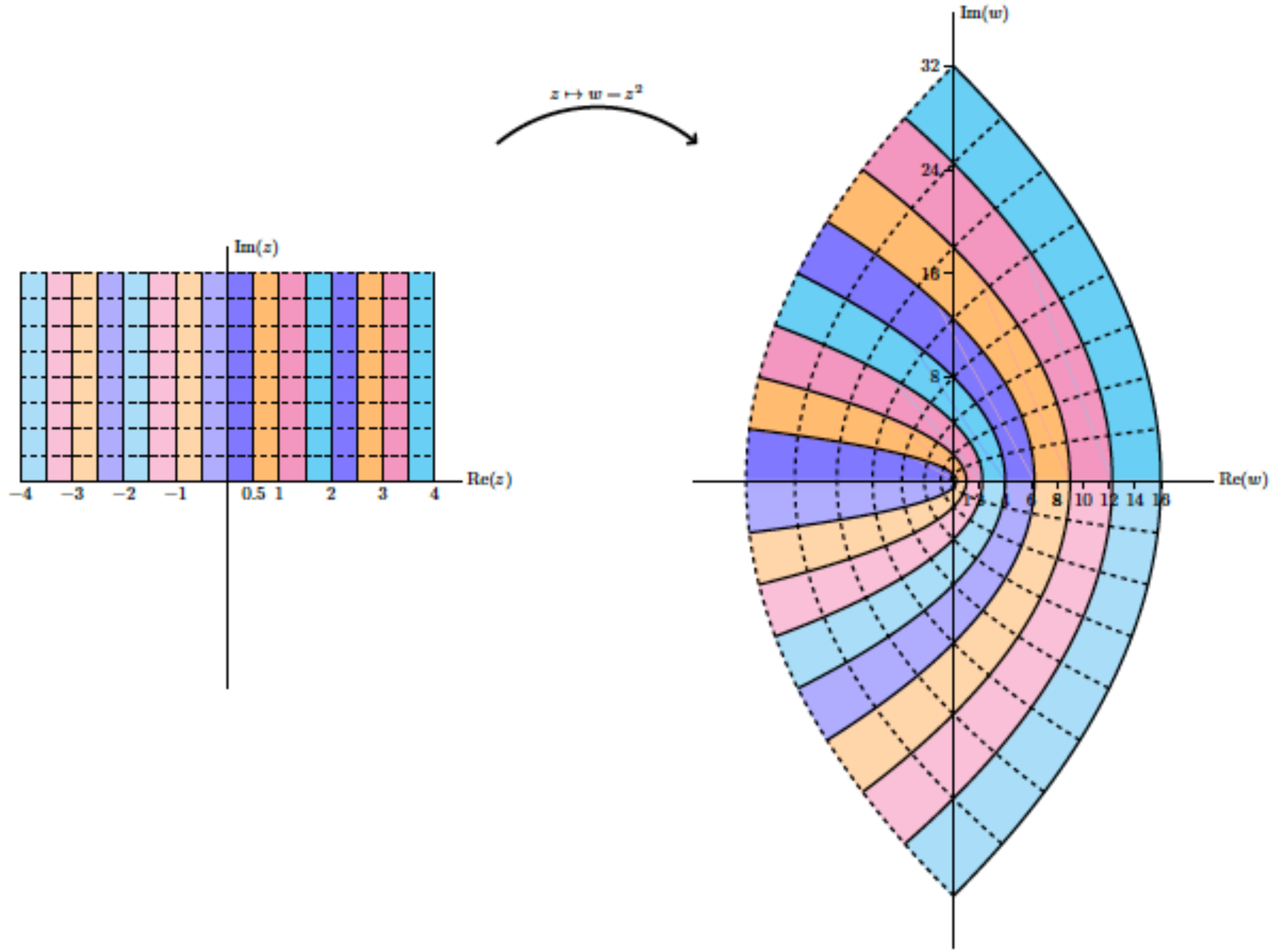
$f(z) = z^2$ maps rays from the origin to rays from the origin.

The next figure gives another view of the mapping. Here we see vertical stripes in the first quadrant are mapped to parabolic stripes that live in the first and second quadrants.



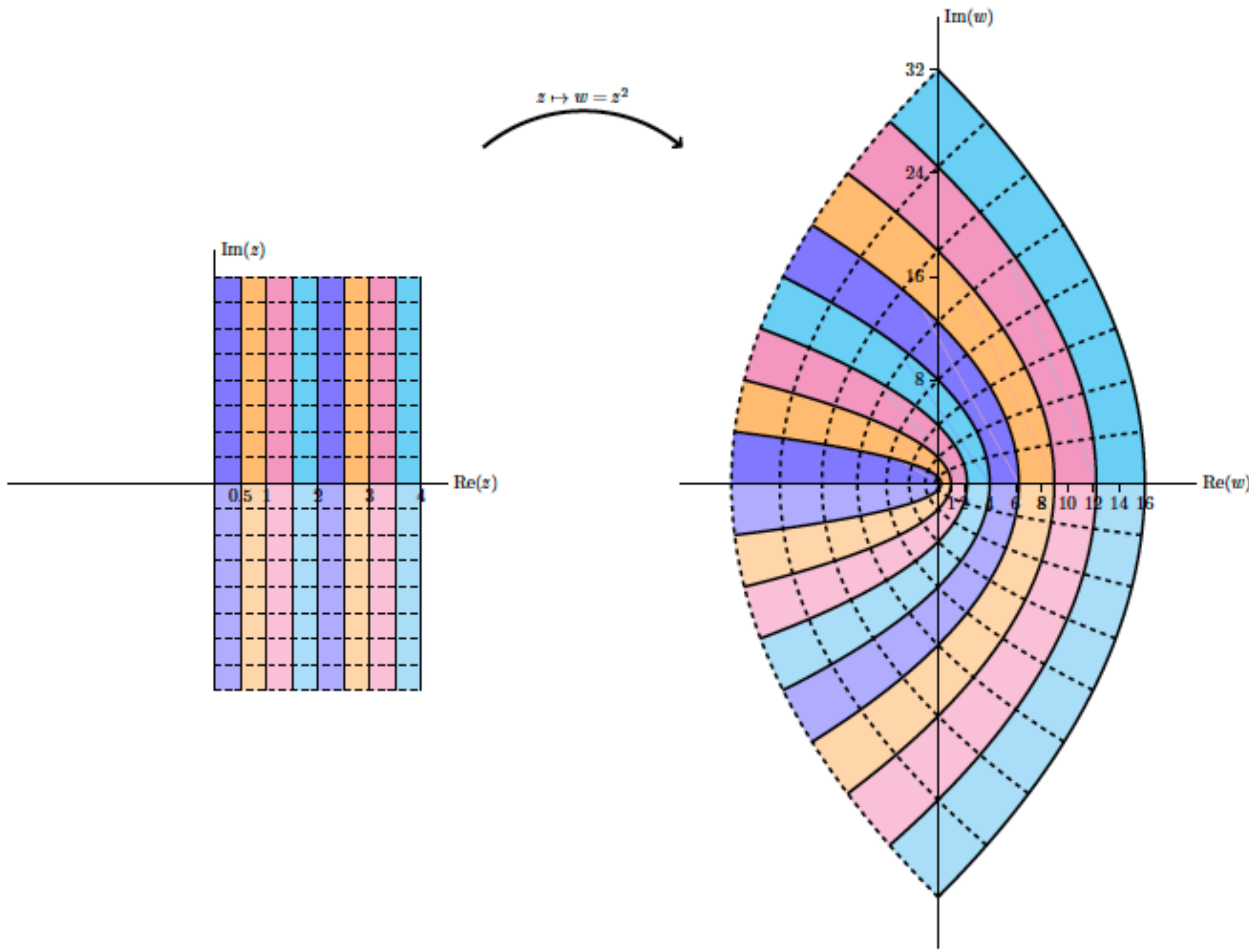
$z^2 = (x^2 - y^2) + i2xy$ maps vertical lines to left facing parabolas.

The next figure is similar to the previous one, except in this figure we look at vertical stripes in both the first and second quadrants. We see that they map to parabolic stripes that live in all four quadrants.



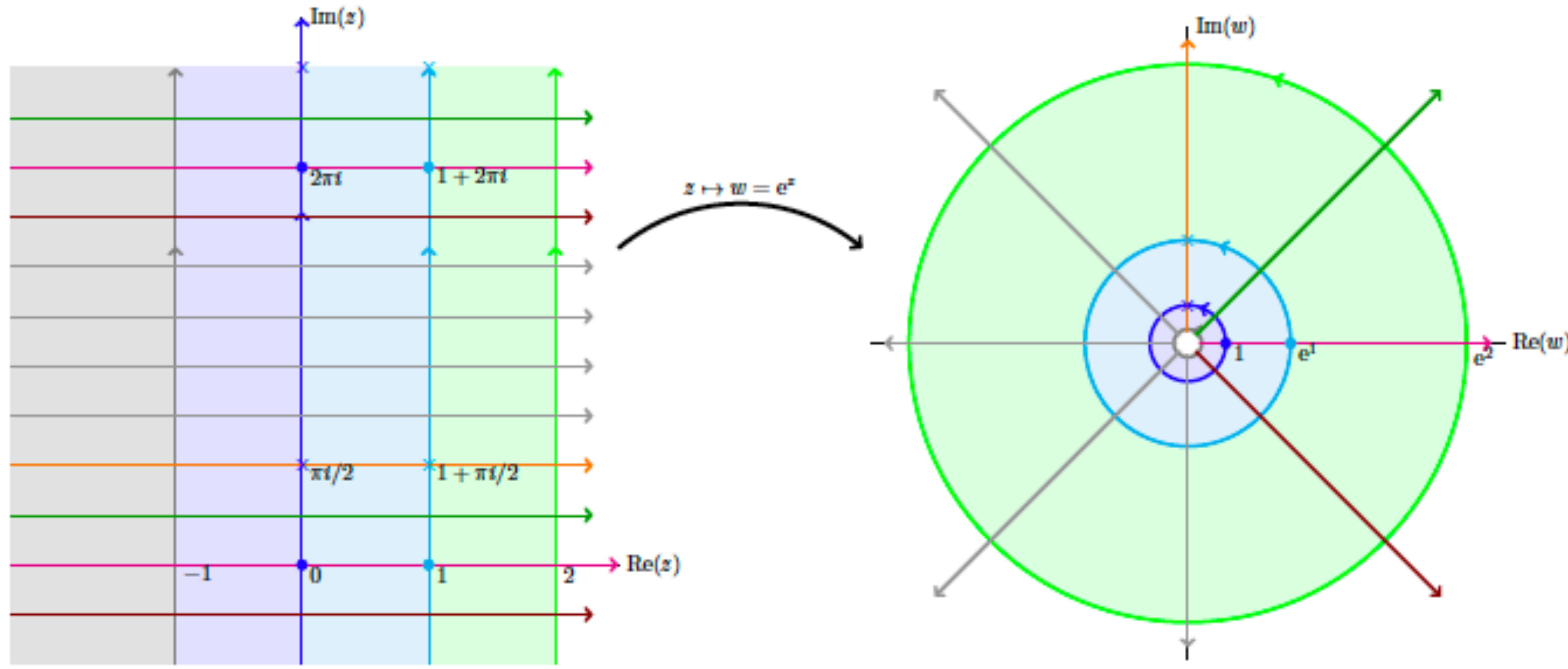
$f(z) = z^2$ maps the first two quadrants to the entire plane.

The next figure shows the mapping of stripes in the first and fourth quadrants. The image map is identical to the previous figure. This is because the fourth quadrant is minus the second quadrant, but $z^2 = (-z)^2$.



Vertical stripes in quadrant 4 are mapped identically to vertical stripes in quadrant 2.

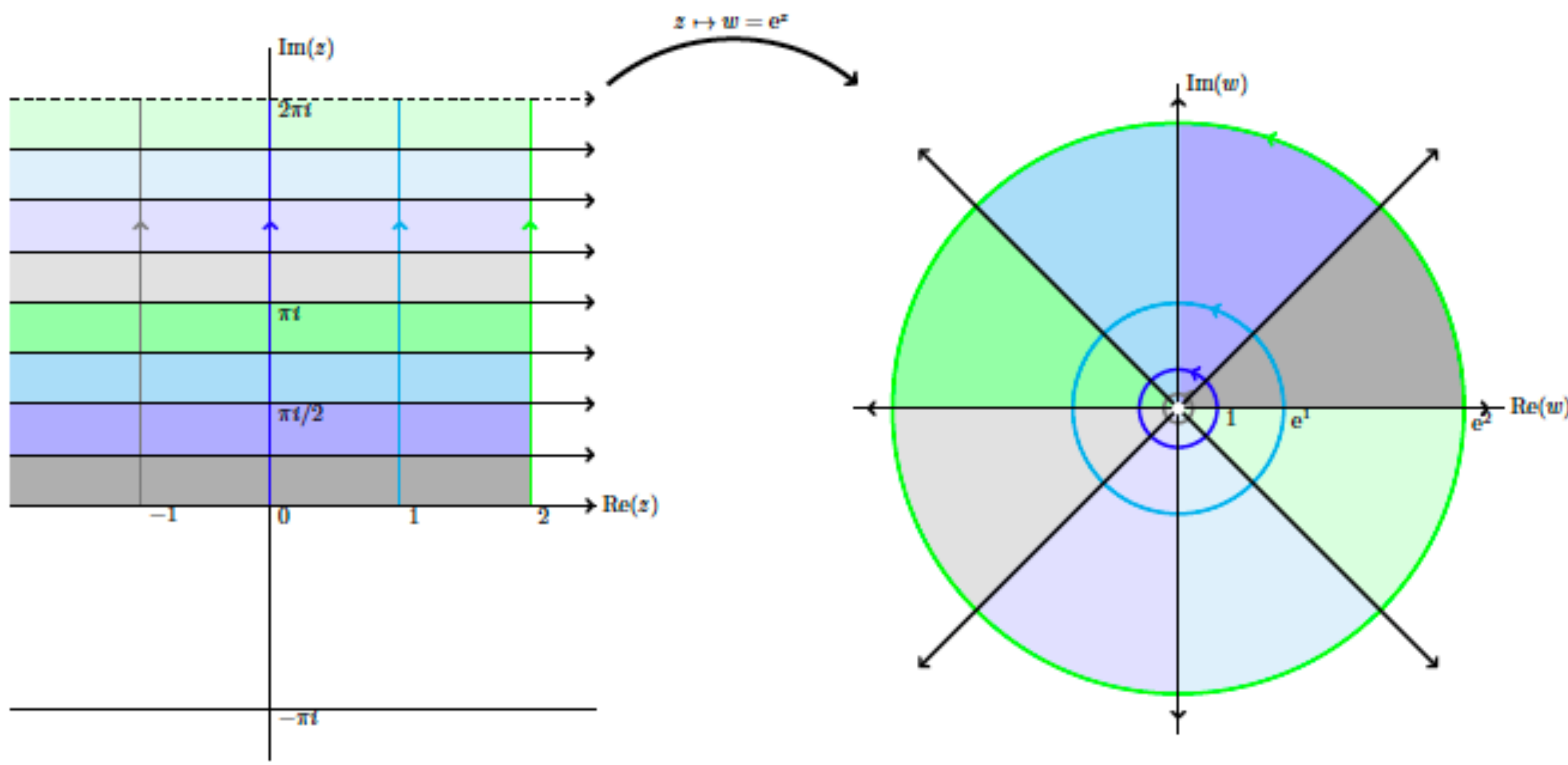
Example. The mapping $w = \mathrm{e}^z$.



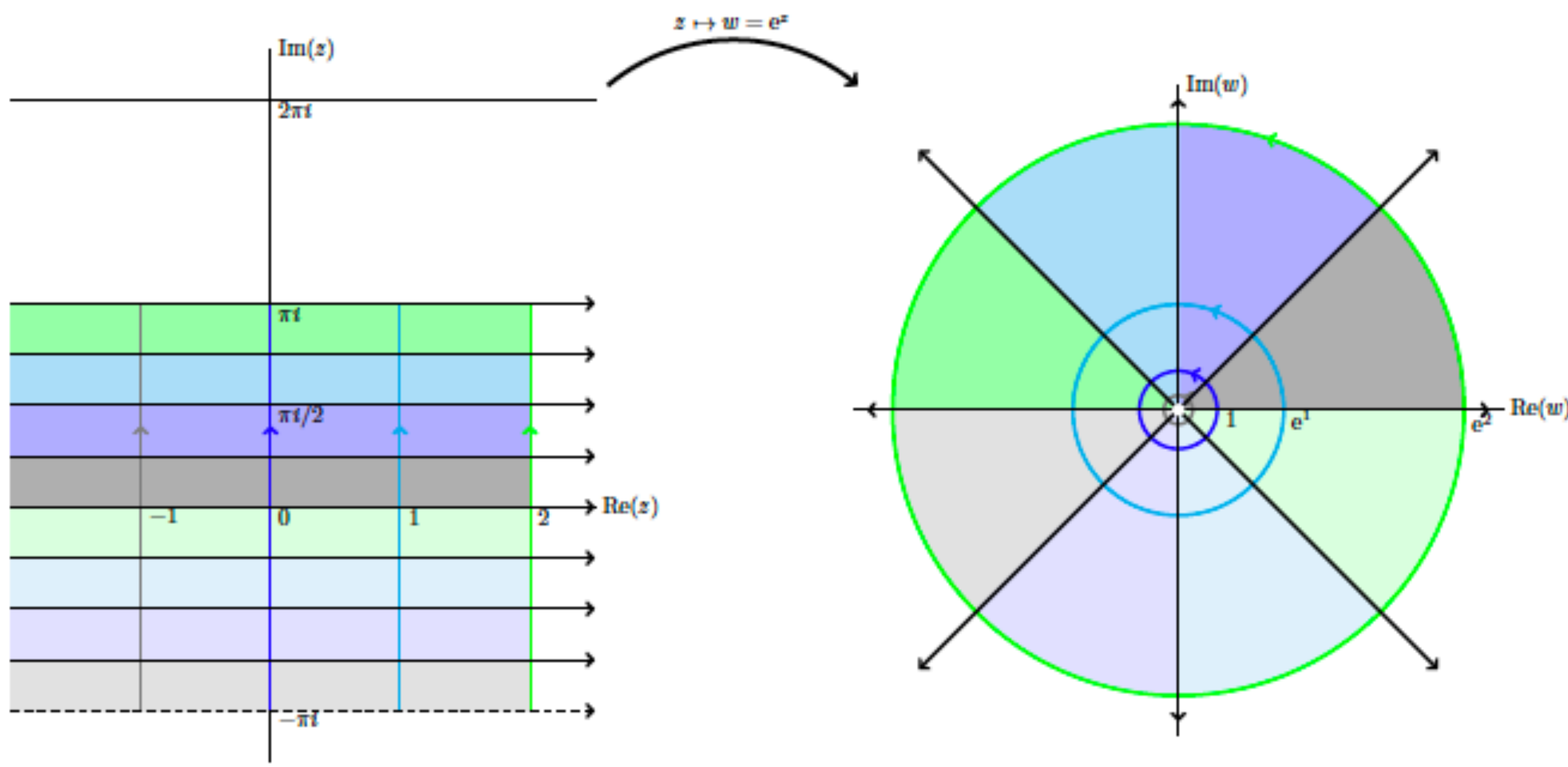
Notice that vertical lines are mapped to circles and horizontal lines to rays from the origin.

Because the plane minus the origin comes up frequently we give it a name:

Definition. The punctured plane is the complex plane minus the origin. In symbols we can write it as $\mathbf{C} - \{0\}$ or $\mathbf{C}/\{0\}$.



The horizontal strip $0 \le y < 2\pi$ is mapped to the punctured plane



The horizontal strip $-\pi < y \leq \pi$ is mapped to the punctured plane