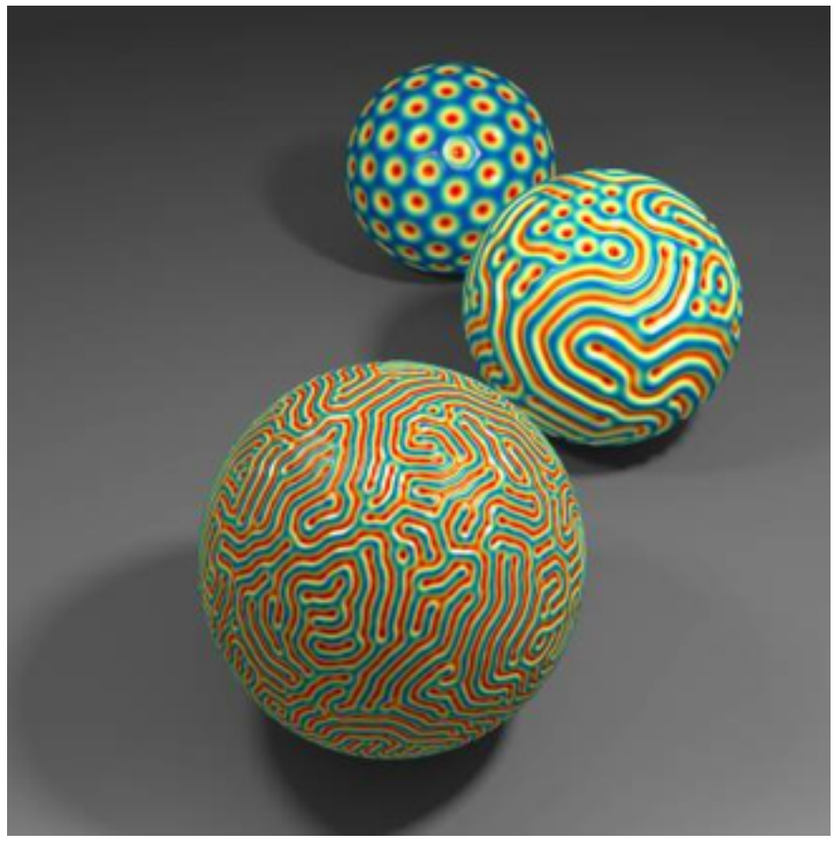
Wrinkling transitions in curved soft bilayer materials

Jörn Dunkel





Joint work with

MIT MATHEMATICS

elasticity, geometry & statistics LABORATORY
understanding the mechanics of thin objects


Norbert Stoop

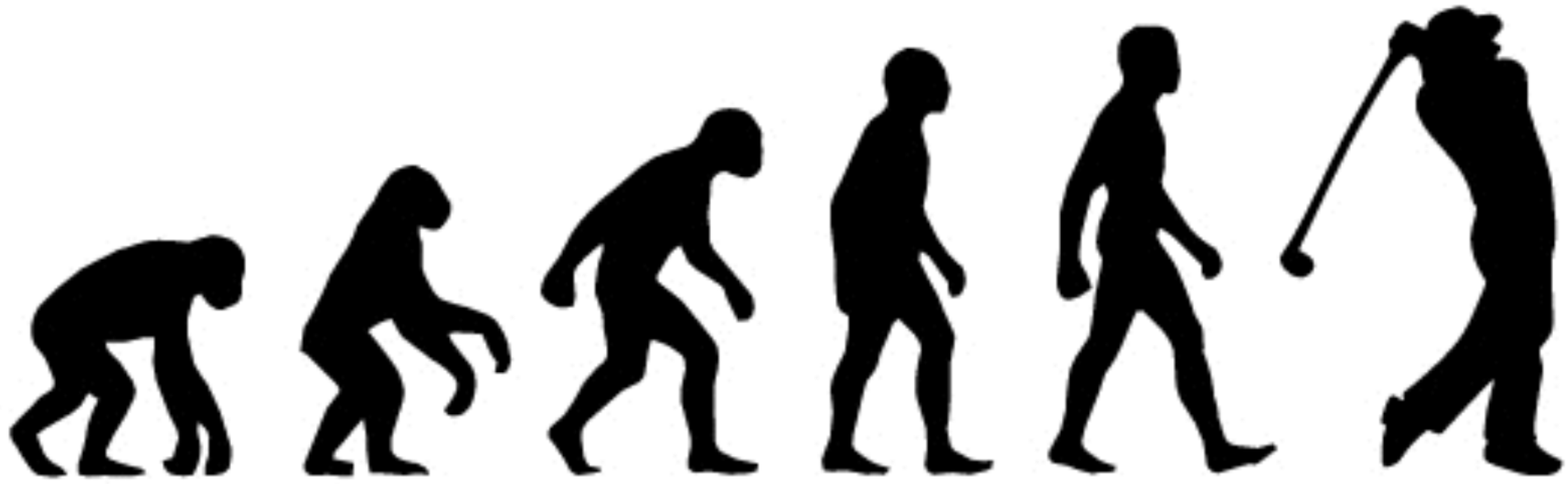

Pedro Reis



Denis Terwagne

\+ Romain Lagrange (CEA) + Francisco Lopez-Jimenez (MIT Mech Eng)

… humans evolved to play golf



… golf balls evolved to fly farther

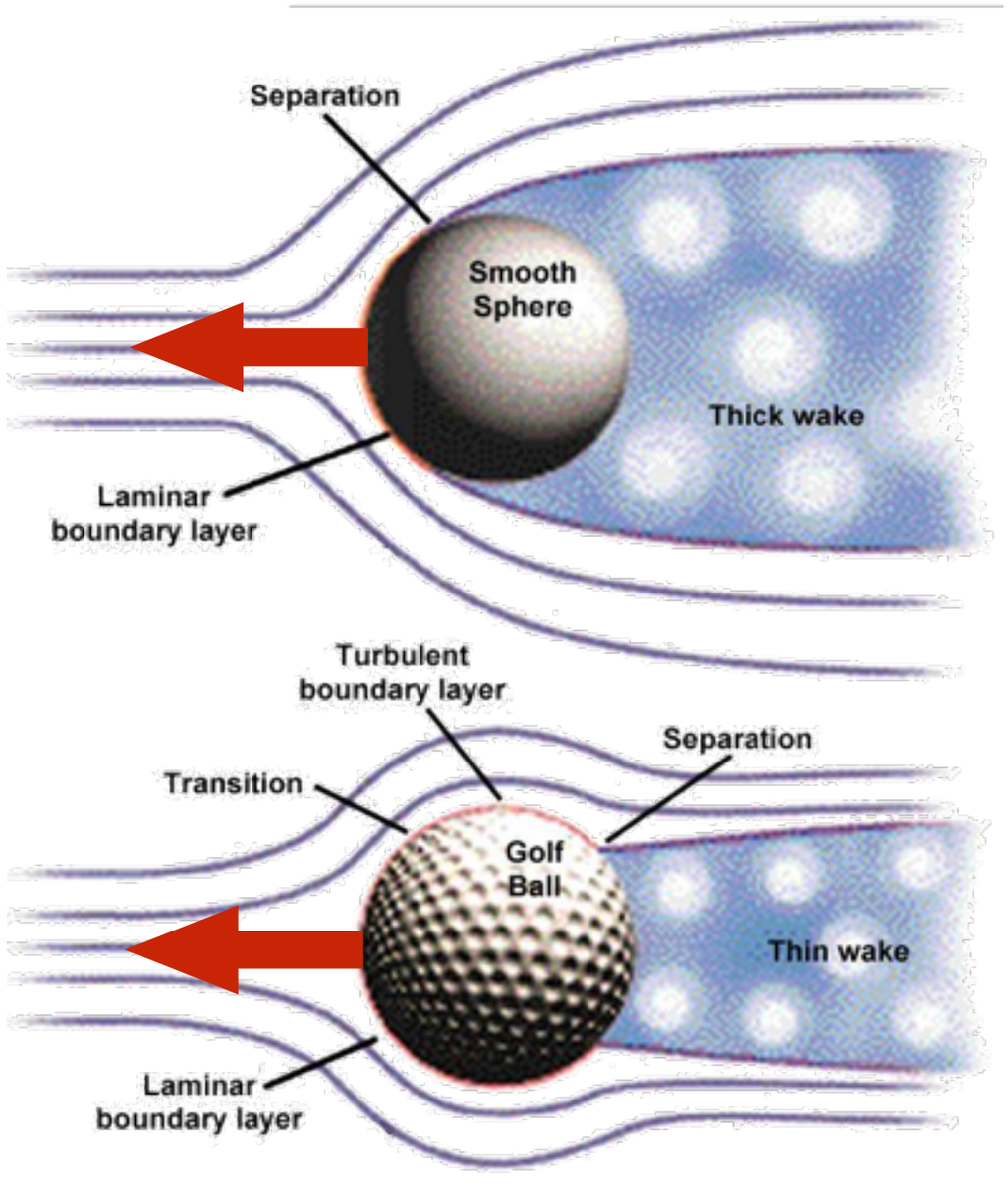




site.rockbottomgolf.com

Drag reduction

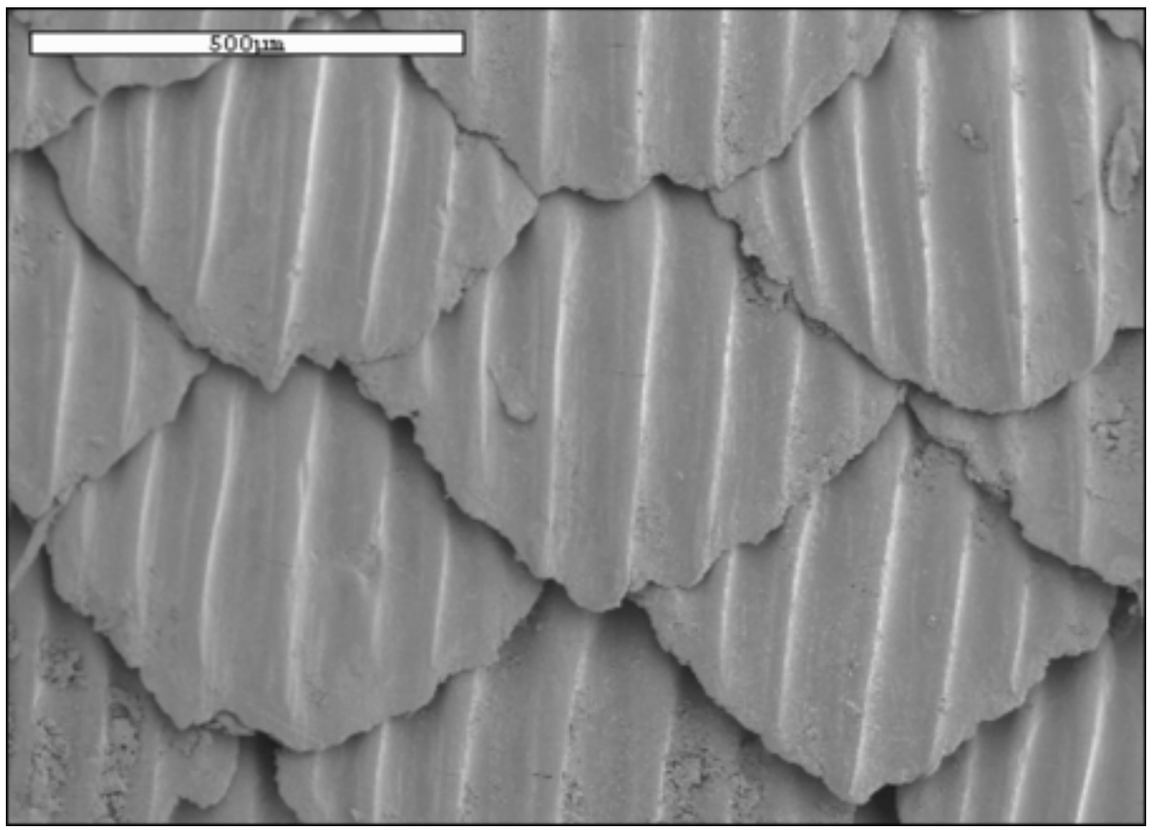




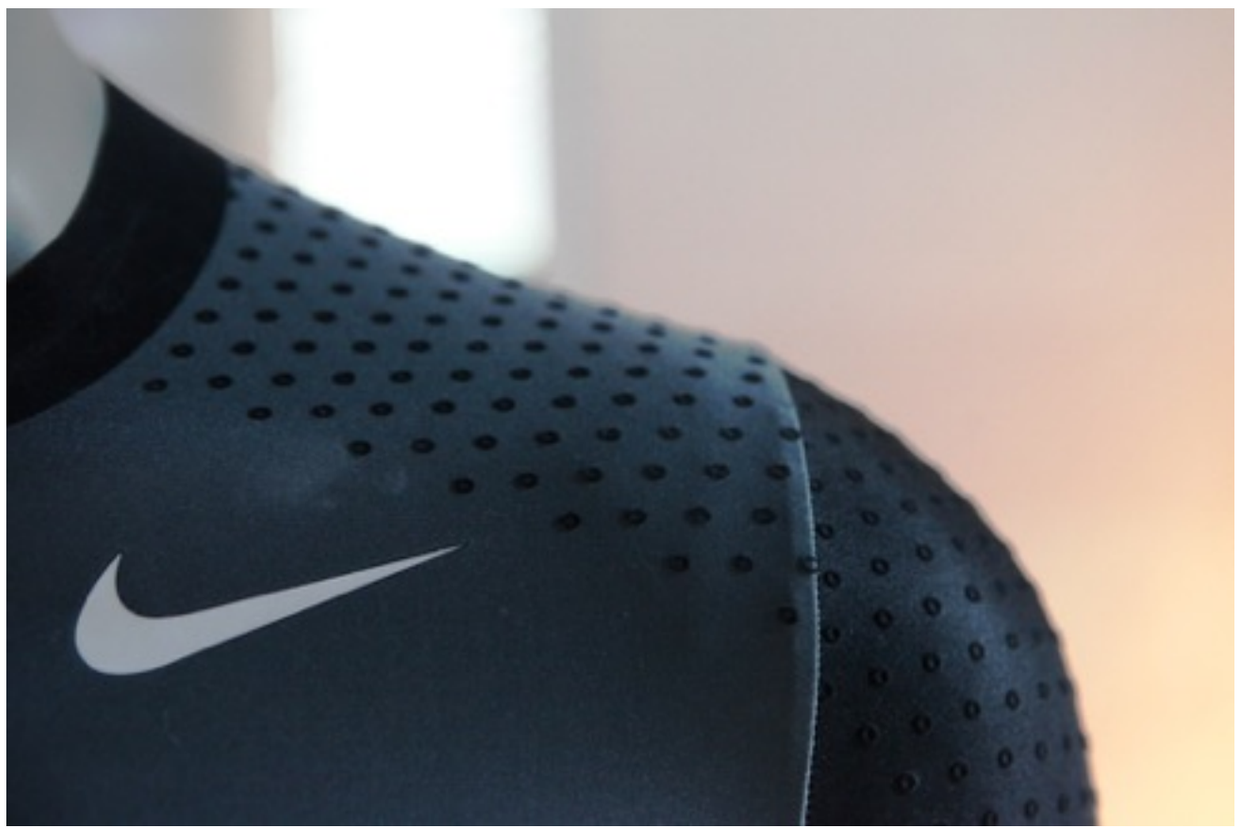
sharkopedia.discovery.com







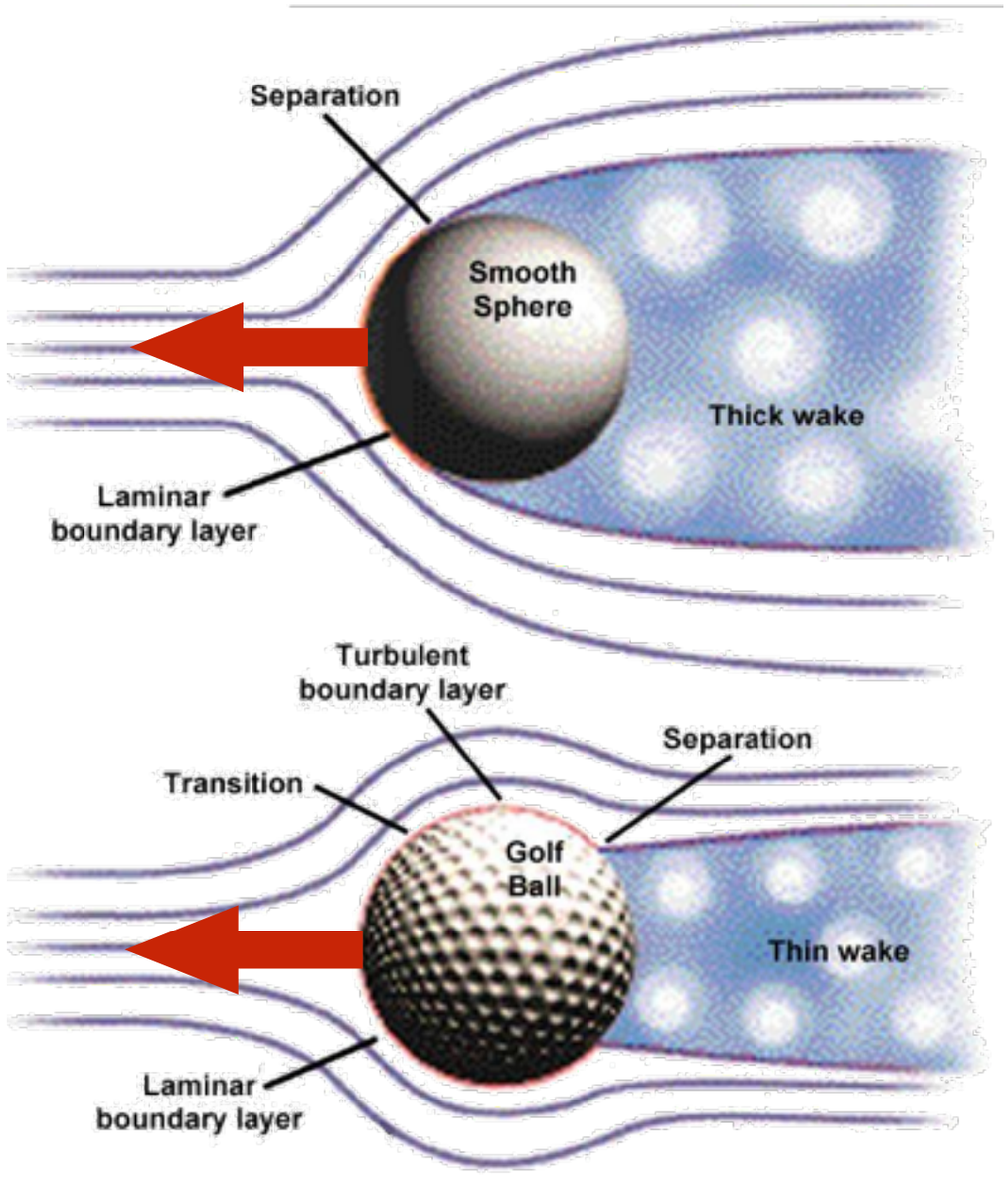
www.maritimejournal.com



NIKE Pro Turbospeed Suit

Drag reduction





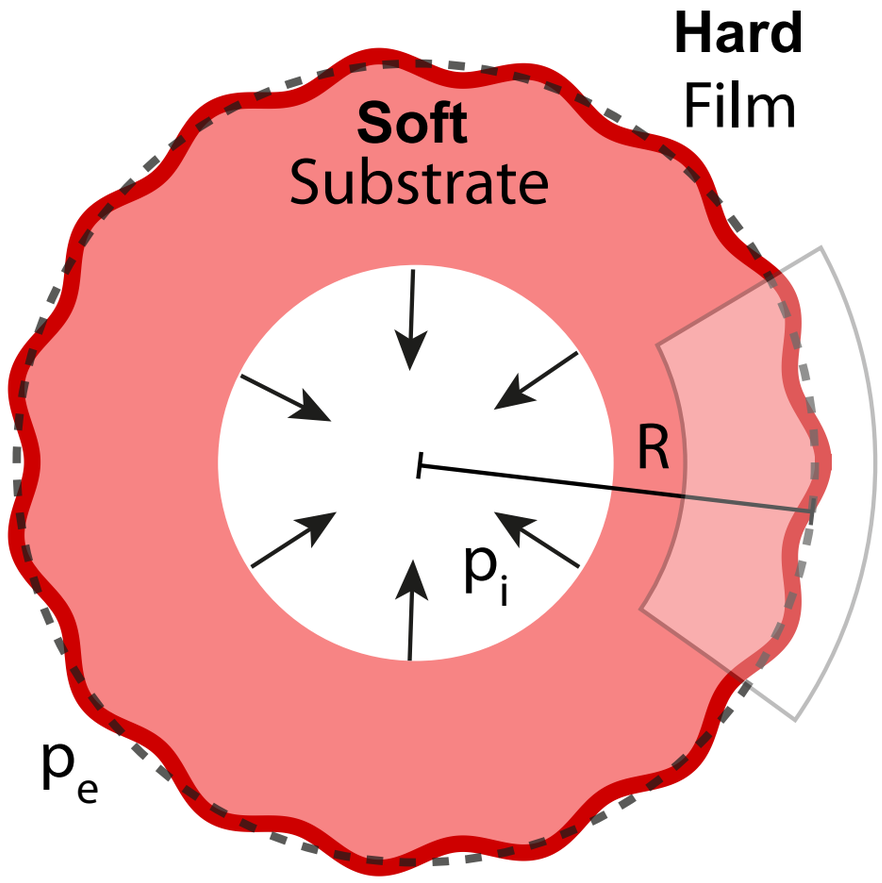
Pedro: **Soft materials** with **tunable surface patterns** ?

Experiment





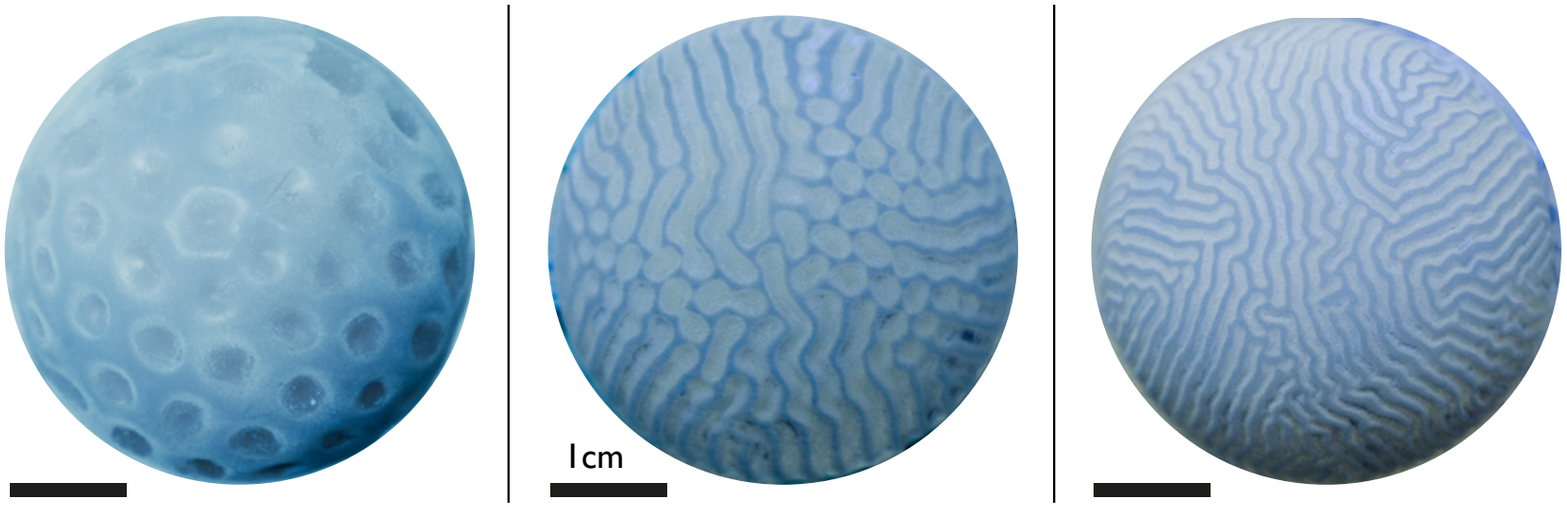
Denis Terwagne & Pedro Reis

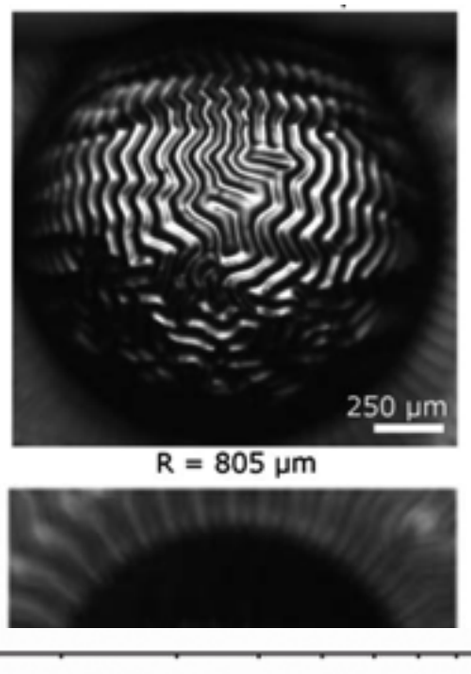




film thickness $h \ll R$

Curvature / stress-induced wrinkling transitions





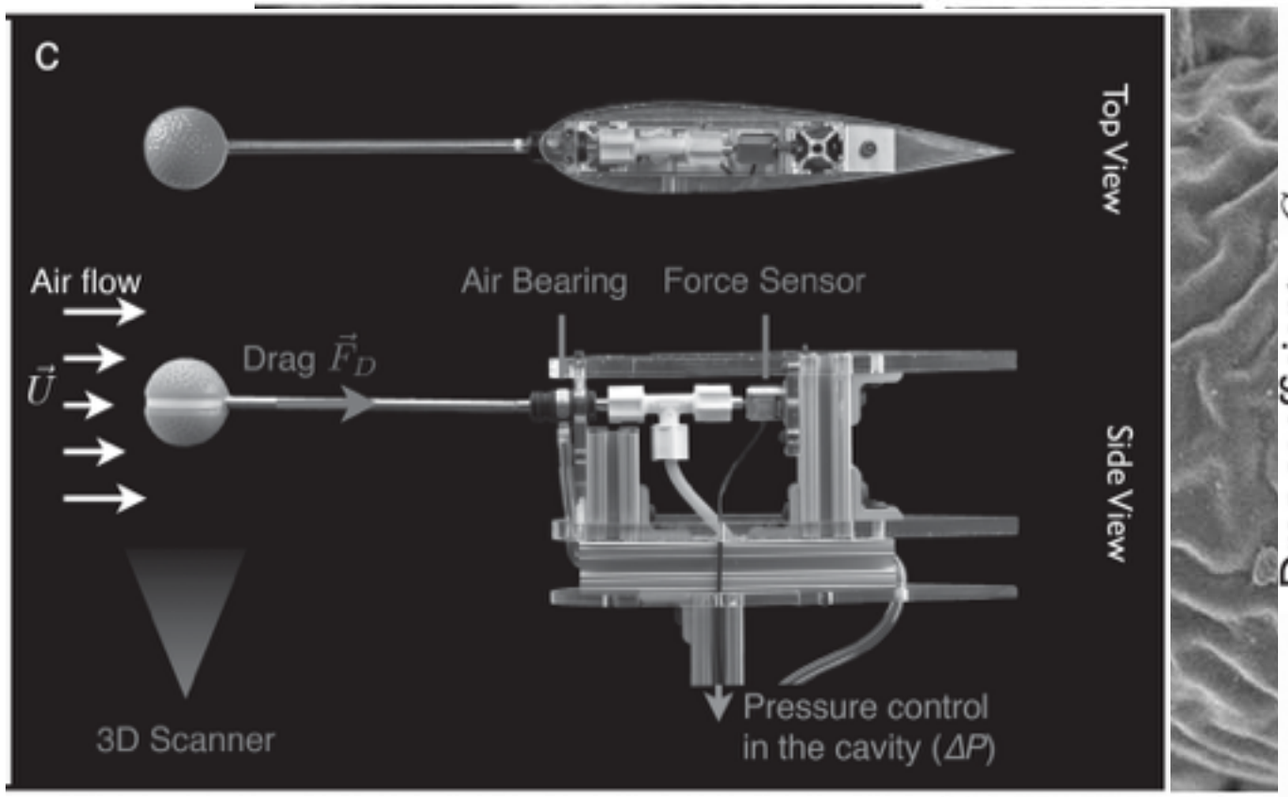
Kolinski & Rubinstein, Proc. Natl. Acad. Sci. (2008)

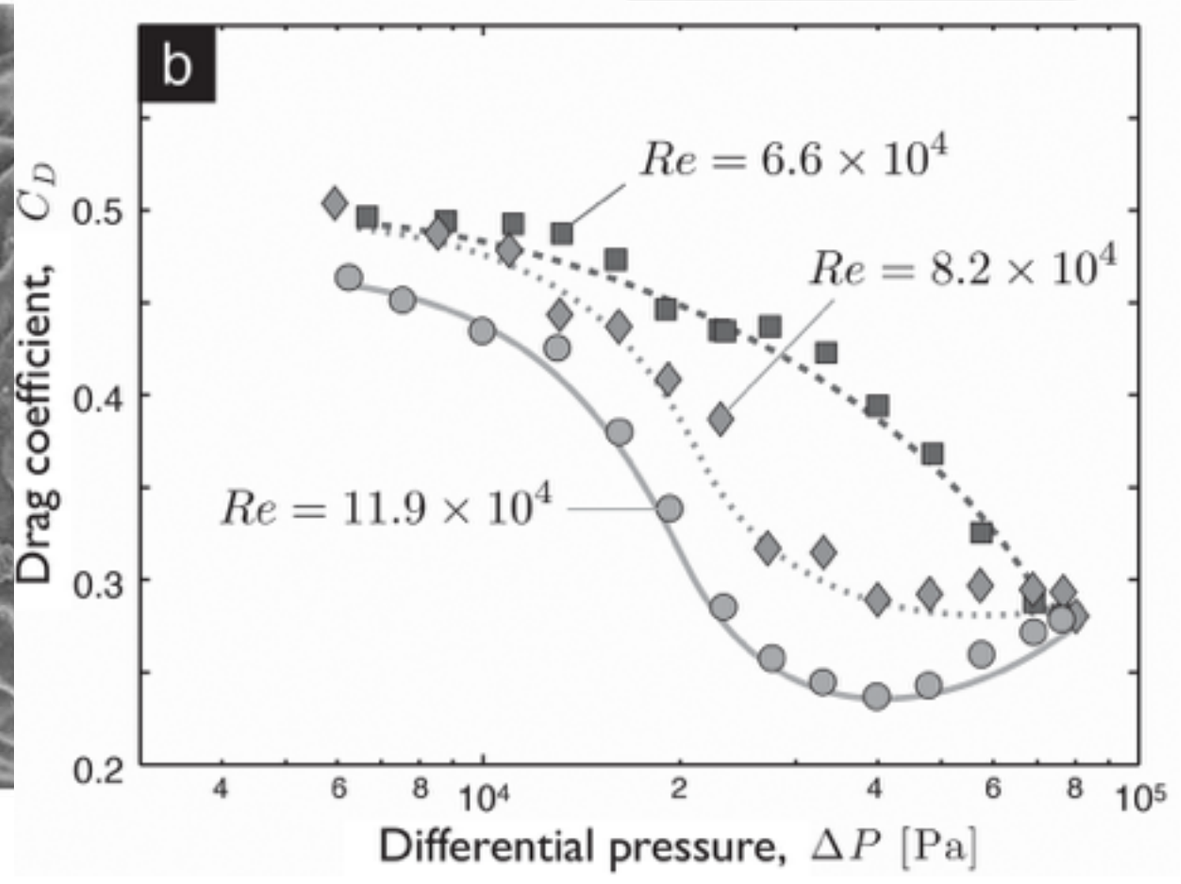
Kolinski, Mahadevan & Rubinstein, Europhys. Lett. (2008)

Kolinski, Mahadevan & Rubinstein, Phys. Rev. E (2009)

Kolinski, Mahadevan & Rubinstein, Phys. Rev. Lett. (2009)

Kolinski, Mahadevan & Rubinstein, Soft Matter (2013)





D Terwagne, M Brojan and P M Reis "Smart Morphable Surfaces for Aerodynamic Drag Control" *Adv. Mater.* 26:38, 6608 (2014)

5 orders of magnitude in length … similar patterns & transitions

Soft surface patterns

Fingerprints

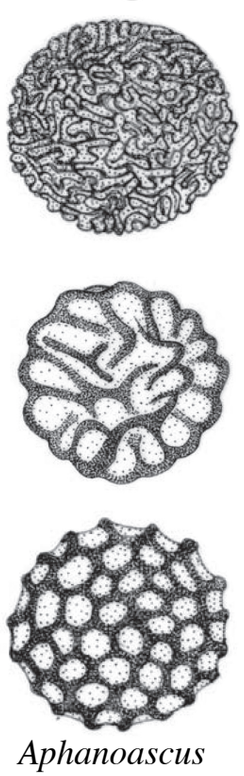


Brain corals

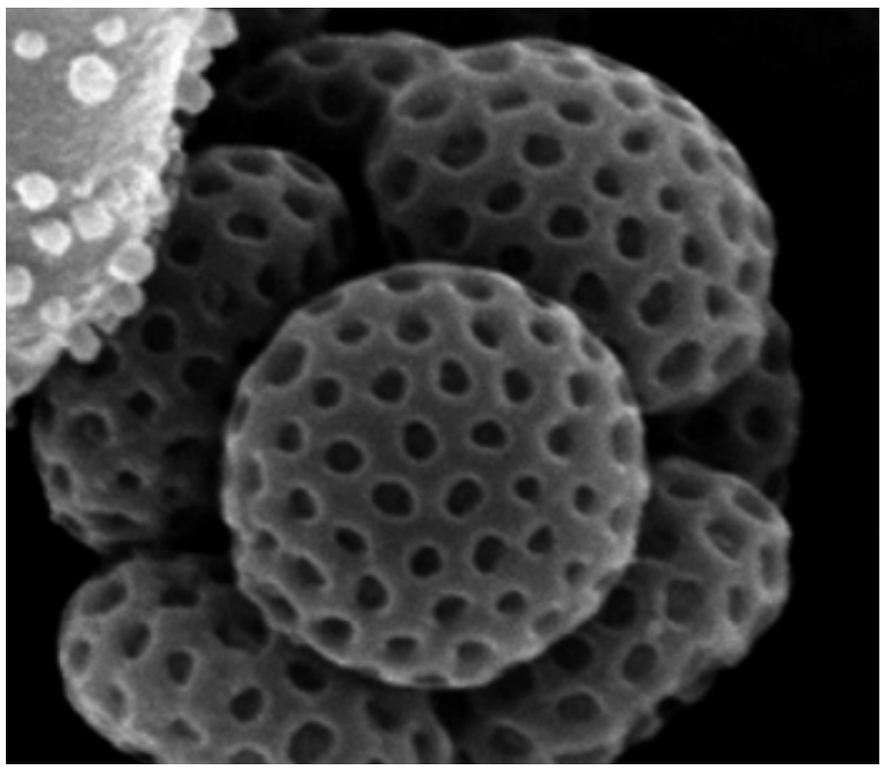




Fungal sporulation

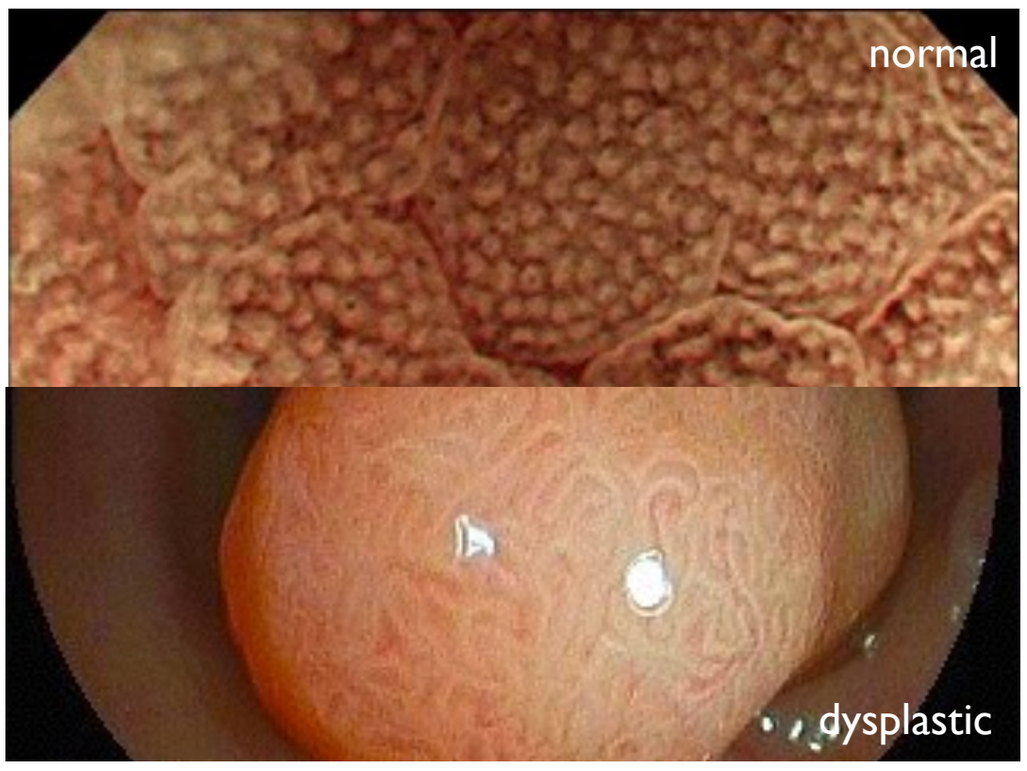


Aphanoascus



Courtesy of Nico Dam
Netherlands' Mycological Society

Colon cancer



Courtesy of Bas Oldenburg
University Medical Center Utrecht

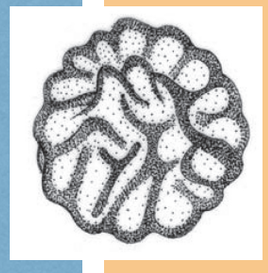
Soft surface patterns

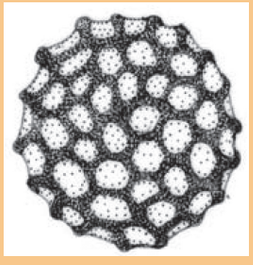
Labyrinthine







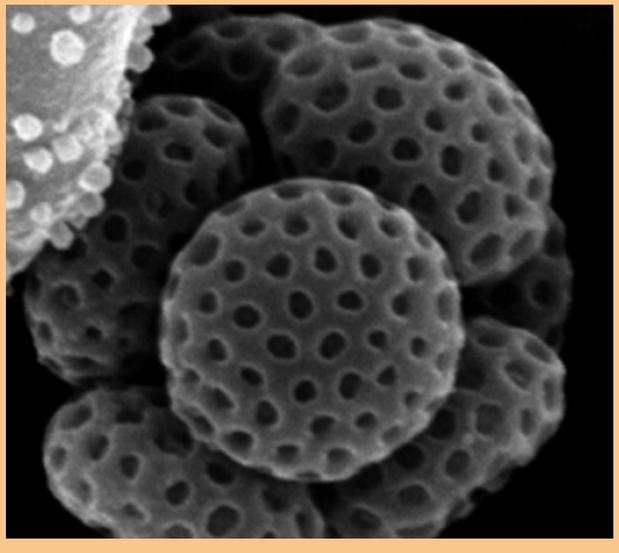






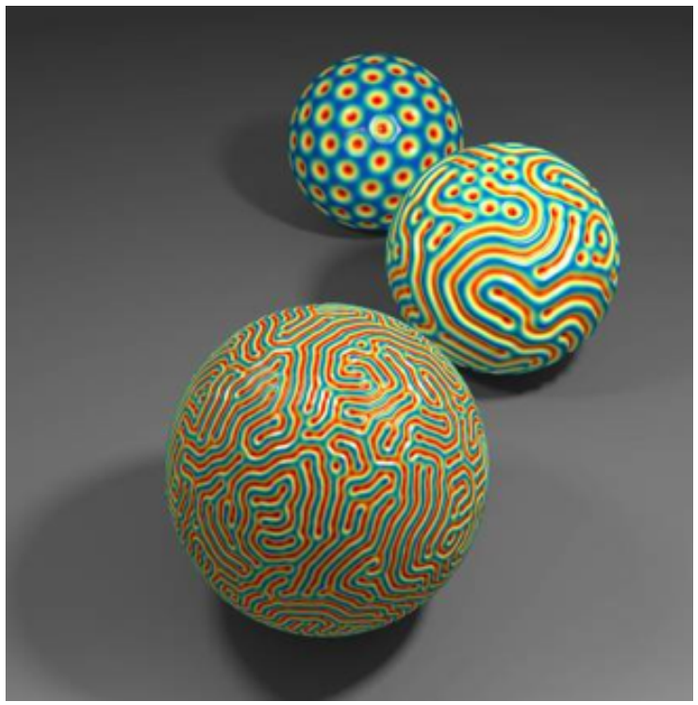
Crystalline

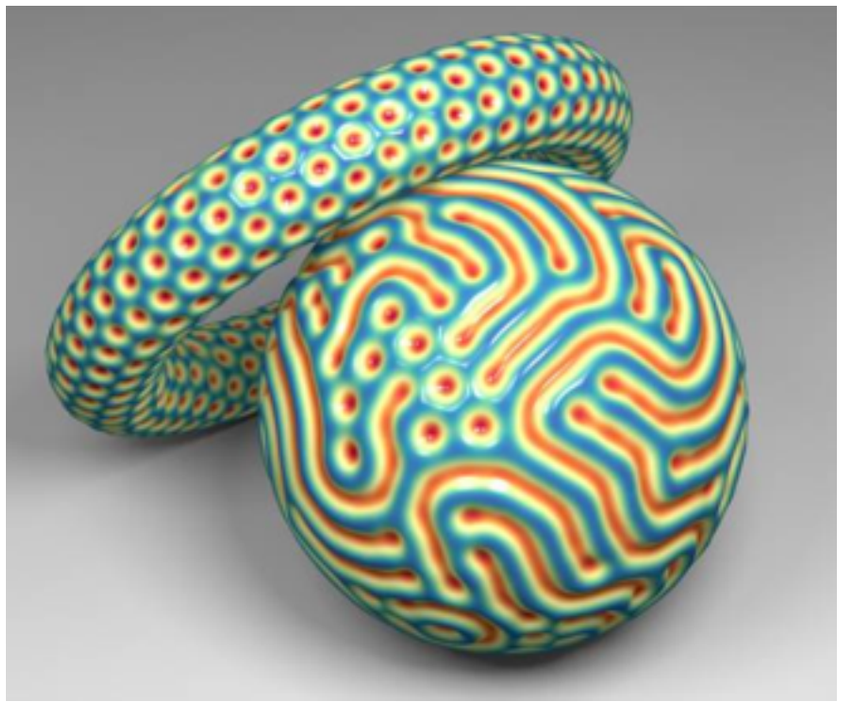






How can we describe such
generic patterns and transitions
between them mathematically ?

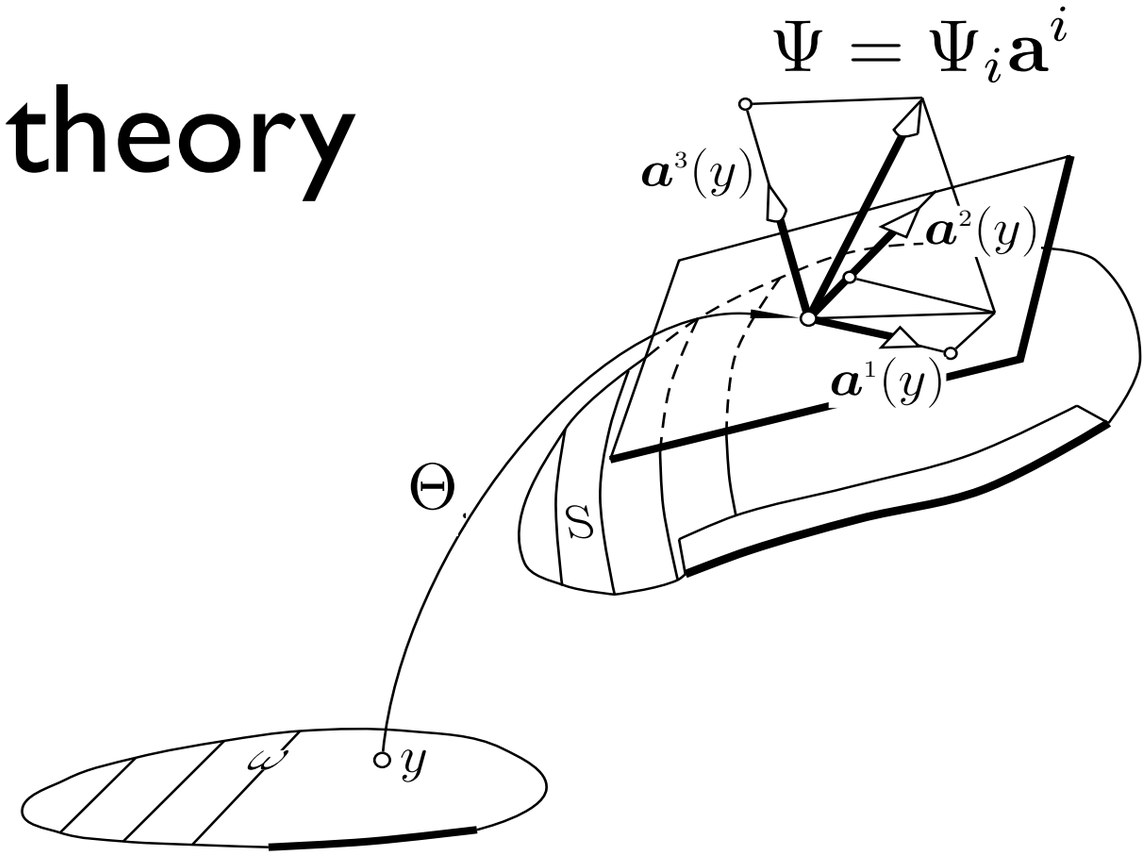




Nature Materials 2015

Covariant Koiter shell theory

E_f	Film Young modulus
h	Film thickness
R	Substrate radius
ν	Poisson ratio
$\Psi = \Psi_i \mathbf{a}^i$	Displacement field



$$\mathcal{E}_{\mathrm{KS}}(\Psi) = \mathcal{E}_b(\Psi) + \mathcal{E}_s(\Psi) + \mathcal{E}_f(\Psi)$$

bending energy | stretching energy | ext. forces (pressure)

effective model: substrate coupling via external forces

$$\mathcal{E}_s = \frac{E_f}{2(1-\nu^2)} \int_\omega d\omega \; \frac{h}{2} H^{\alpha\beta\gamma\delta} G_{\gamma\delta}(\Psi) G_{\alpha\beta}(\Psi)$$

$$\mathcal{E}_b = \frac{E_f}{2(1-\nu^2)} \int_\omega d\omega \; \frac{h^3}{24} H^{\alpha\beta\gamma\delta} R_{\gamma\delta}(\Psi) R_{\alpha\beta}(\Psi)$$

metric tensor



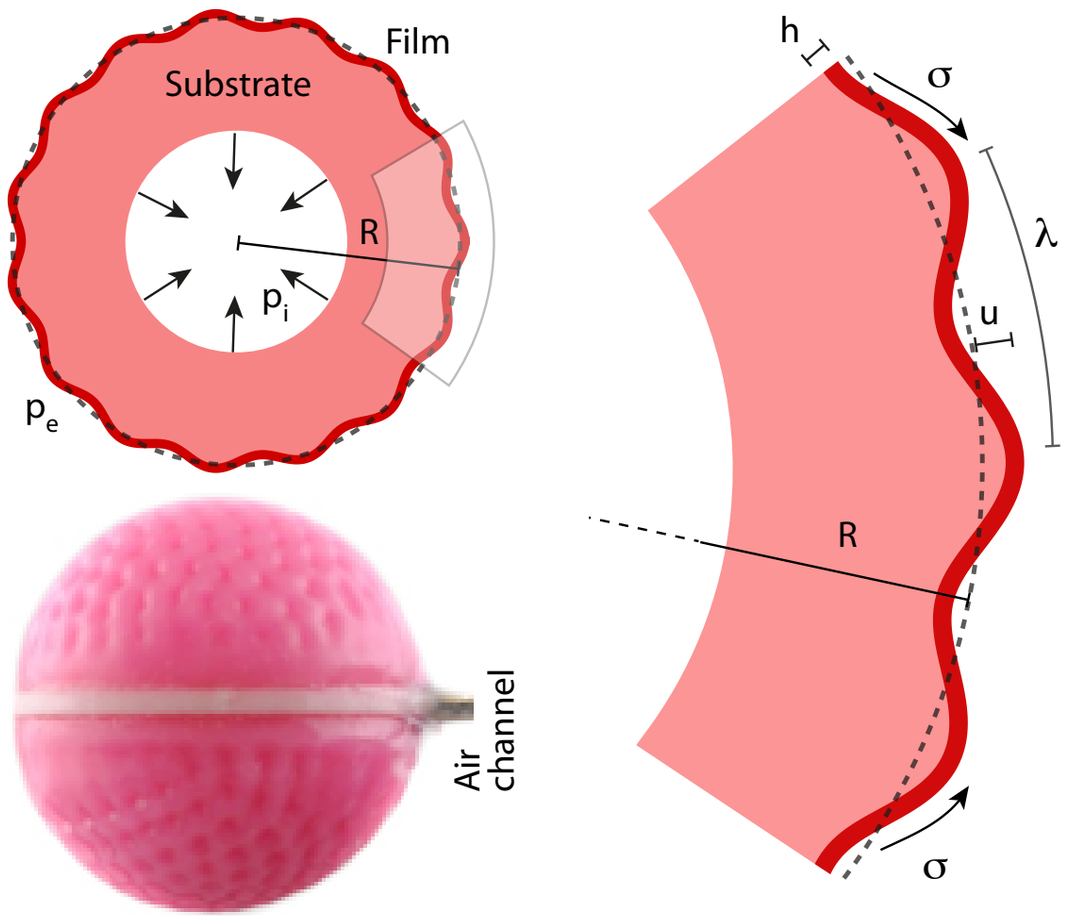
Generalized Swift-Hohenberg theory



Norbert Stoop

$$\partial_t u = \gamma_0 \Delta u - \gamma_2 \Delta^2 u - au - bu^2 - cu^3$$

$$+ \Gamma_1 \left[(\nabla u)^2 + 2u\Delta u \right] + \Gamma_2 \left[u(\nabla u)^2 + u^2 \Delta u \right]$$



Small deformations of a **sphere**

$$(a_{\alpha\beta}) = \begin{pmatrix} (R\sin\theta_2)^2 & 0 \\ 0 & R^2 \end{pmatrix}$$

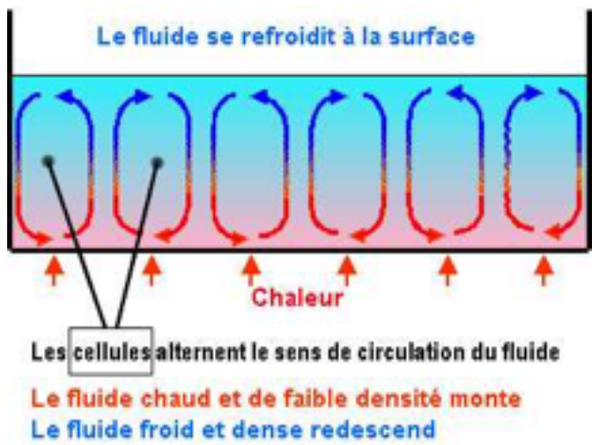
$$\triangle\psi = \nabla_\alpha \nabla^\alpha \psi = a^{\gamma\delta}\psi_{,\gamma\delta} - a^{\gamma\delta}\Gamma^\lambda_{\gamma\delta}\psi_{,\lambda}$$

Nature Materials 2015

Generalized Swift-Hohenberg theory

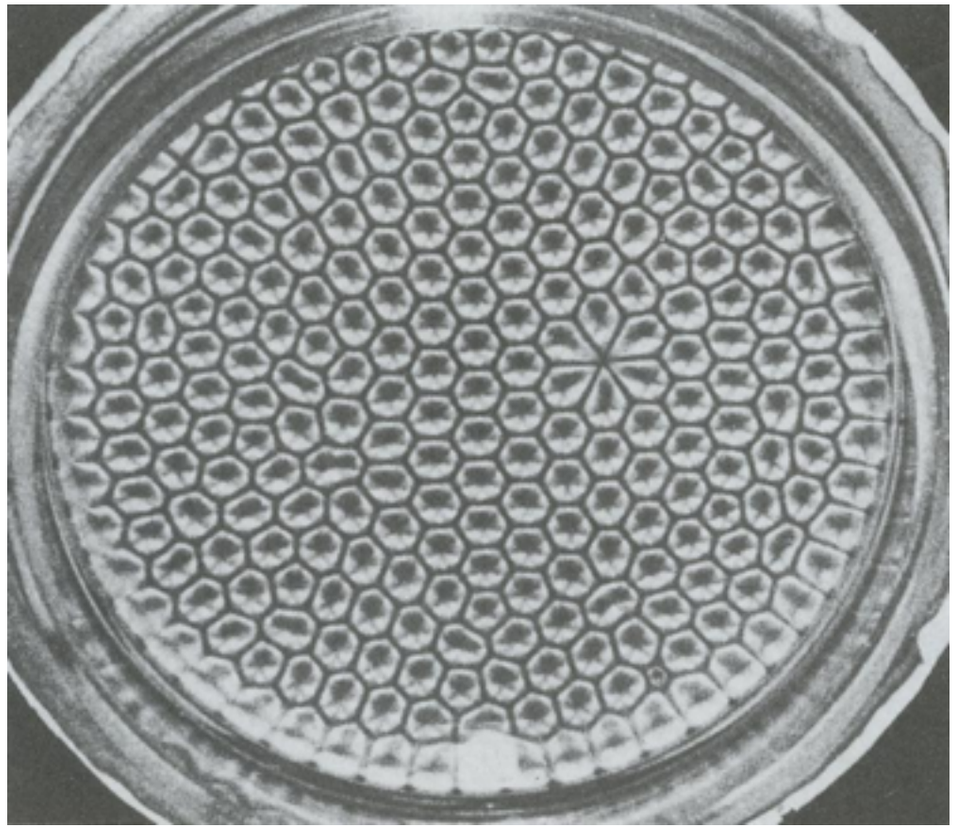
$$\partial_t u = \gamma_0 \Delta u - \gamma_2 \Delta^2 u - au - bu^2 - cu^3$$

$$+ \Gamma_1 \left[(\nabla u)^2 + 2u\Delta u \right] + \Gamma_2 \left[u(\nabla u)^2 + u^2 \Delta u \right]$$



fr.wikipedia.org

Rayleigh-Benard convection



Van Dyke (1982) *An Album of Fluid Motion*

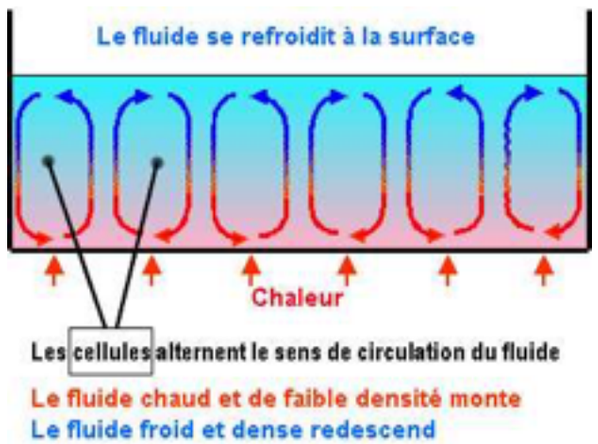
Universality!

J Swift and P C Hohenberg
Phys Rev A **15**, 319 (1977)

Generalized Swift-Hohenberg theory

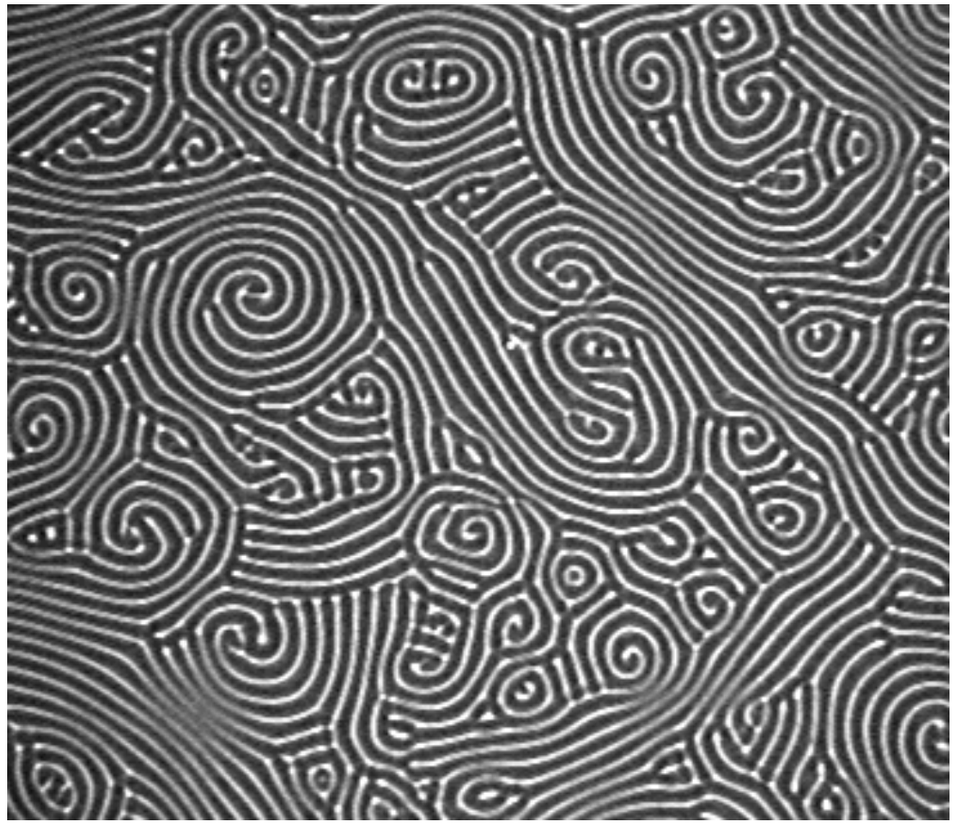
$$\partial_t u = \gamma_0 \Delta u - \gamma_2 \Delta^2 u - au - bu^2 - cu^3$$

$$+ \Gamma_1 \left[(\nabla u)^2 + 2u\Delta u \right] + \Gamma_2 \left[u(\nabla u)^2 + u^2 \Delta u \right]$$



fr.wikipedia.org

Rayleigh-Benard convection



Morris, Bodenschatz, Cannel & Ahlers, 1993

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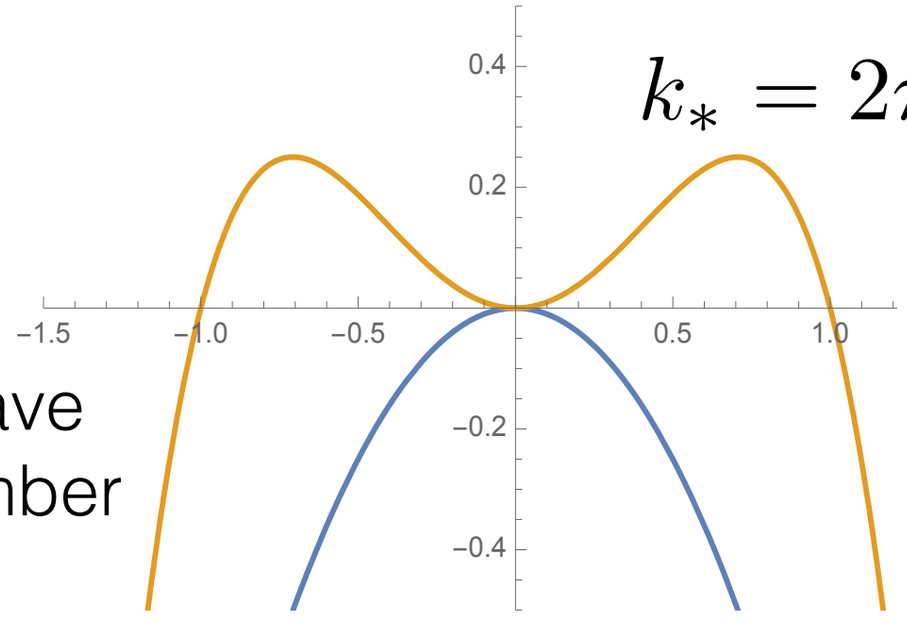
Universality idea



$$\partial_t u = \gamma_0 \Delta u - \gamma_2 \Delta^2 u - au - bu^2 - cu^3 + \Gamma_1 \left[(\nabla u)^2 + 2u\Delta u\right] + \Gamma_2 \left[u(\nabla u)^2 + u^2 \Delta u\right]$$

Fourier space

dispersion



$k_* = 2\pi/\Lambda_*$

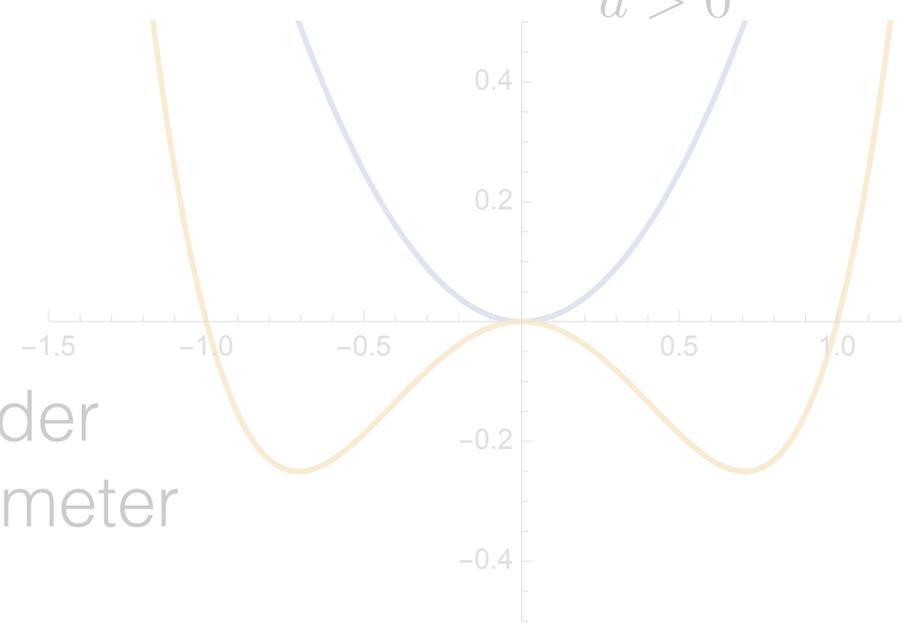
wave number

$$-\gamma_0 k^2 - \gamma_2 k^4$$

$$\gamma_0 \nabla^2 - \gamma_2 (\nabla^2)^2$$

Order-parameter space

potential



order parameter

$$au^2 + cu^4$$

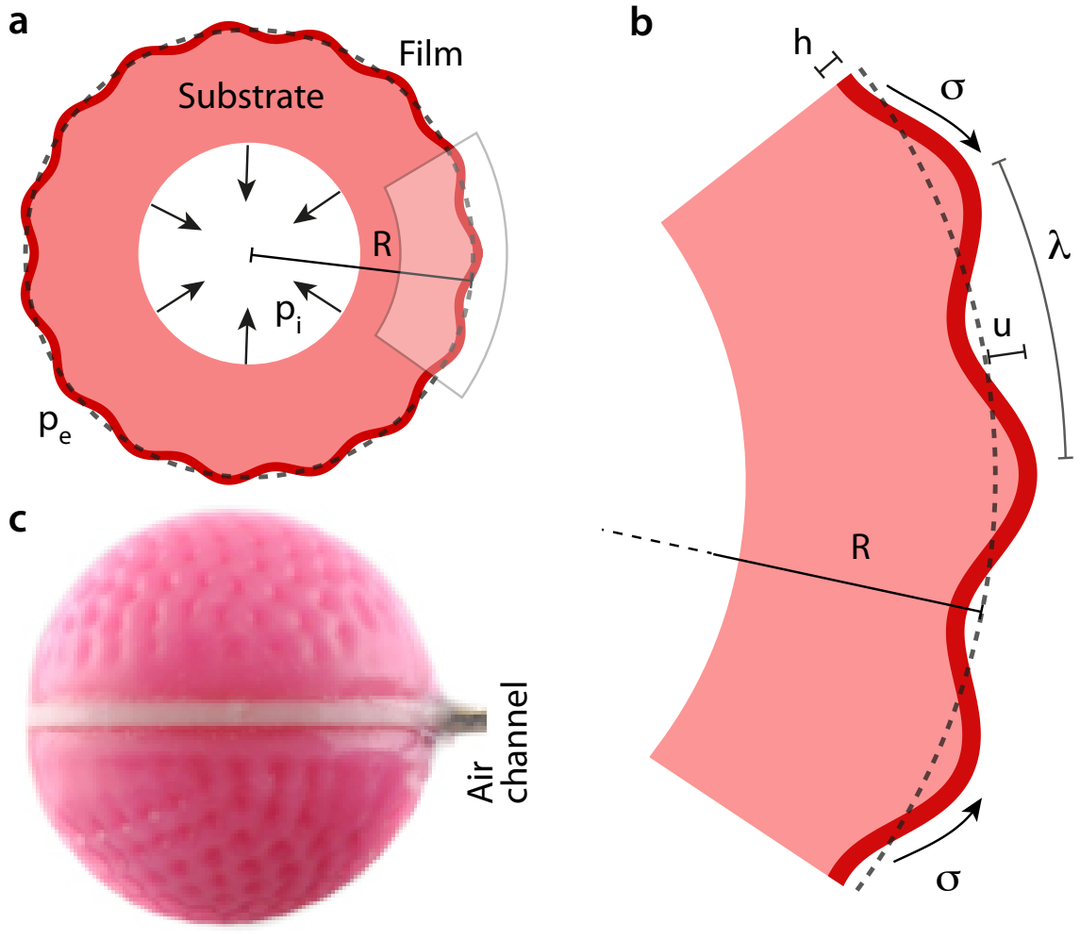
Where is the pattern transition ?



Norbert Stoop

$$\partial_t u = \gamma_0 \Delta u - \gamma_2 \Delta^2 u - au - bu^2 - cu^3$$

$$+ \Gamma_1 \left[(\nabla u)^2 + 2u\Delta u \right] + \Gamma_2 \left[u(\nabla u)^2 + u^2 \Delta u \right]$$



$$\gamma_0 = \frac{\kappa^2}{3} - \frac{1}{6}\sqrt{\eta^{4/3} + 24(1+\nu)\kappa^2 + 16\kappa^4}$$

$$a = \frac{\eta^{4/3}}{12} + \frac{6(1+\nu) - \eta^{2/3}}{3}\kappa^2 + \frac{\kappa^4}{3} + \tilde{a}_2 \Sigma_e$$

$$b = 3(1+\nu)\kappa^3$$

$$c = \frac{2(1+\nu)\eta^{2/3}}{3} c_1 + (1+\nu)\kappa^4$$

$$\Gamma_1 = \frac{1+\nu}{2}\kappa$$

$$\Gamma_2 = \frac{1+\nu}{2}\kappa^2$$

$$\tilde{a}_2 = -\frac{\eta^{4/3}(c + 3|\gamma_0|\Gamma_2)}{48\gamma_0^2}$$

TABLE I: List of parameters for Eq. (1) in units $h = 1$, with $\eta = 3E_s/E_f$, $\gamma_2 = 1/12$, $\Sigma_e = (\sigma/\sigma_c) - 1$ and $\kappa = h/R$. The only remaining fitting parameter of the model is c_1.

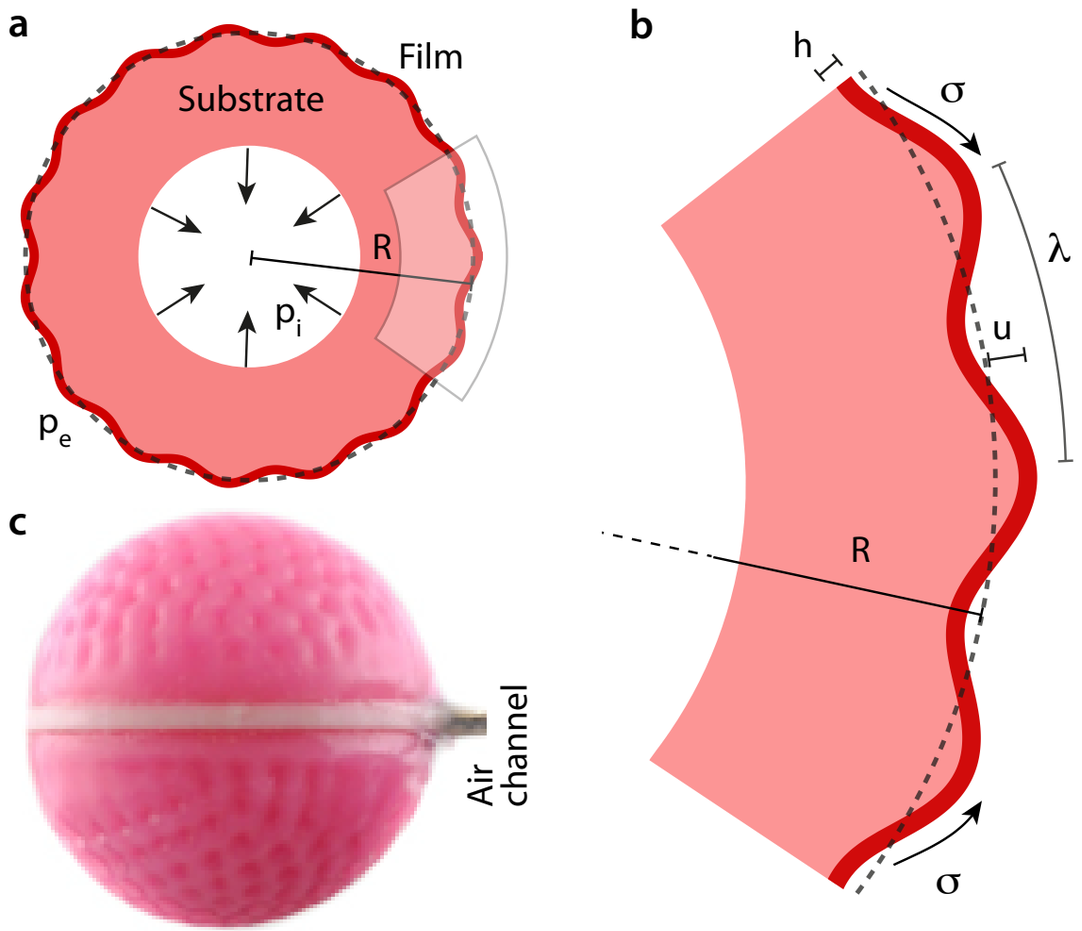
Nature Materials 2015

Where is the pattern transition ?



Norbert Stoop

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$$+ \Gamma_1 \left[(\nabla u)^2 + 2u\Delta u\right] + \Gamma_2 \left[u(\nabla u)^2 + u^2 \Delta u\right]$$



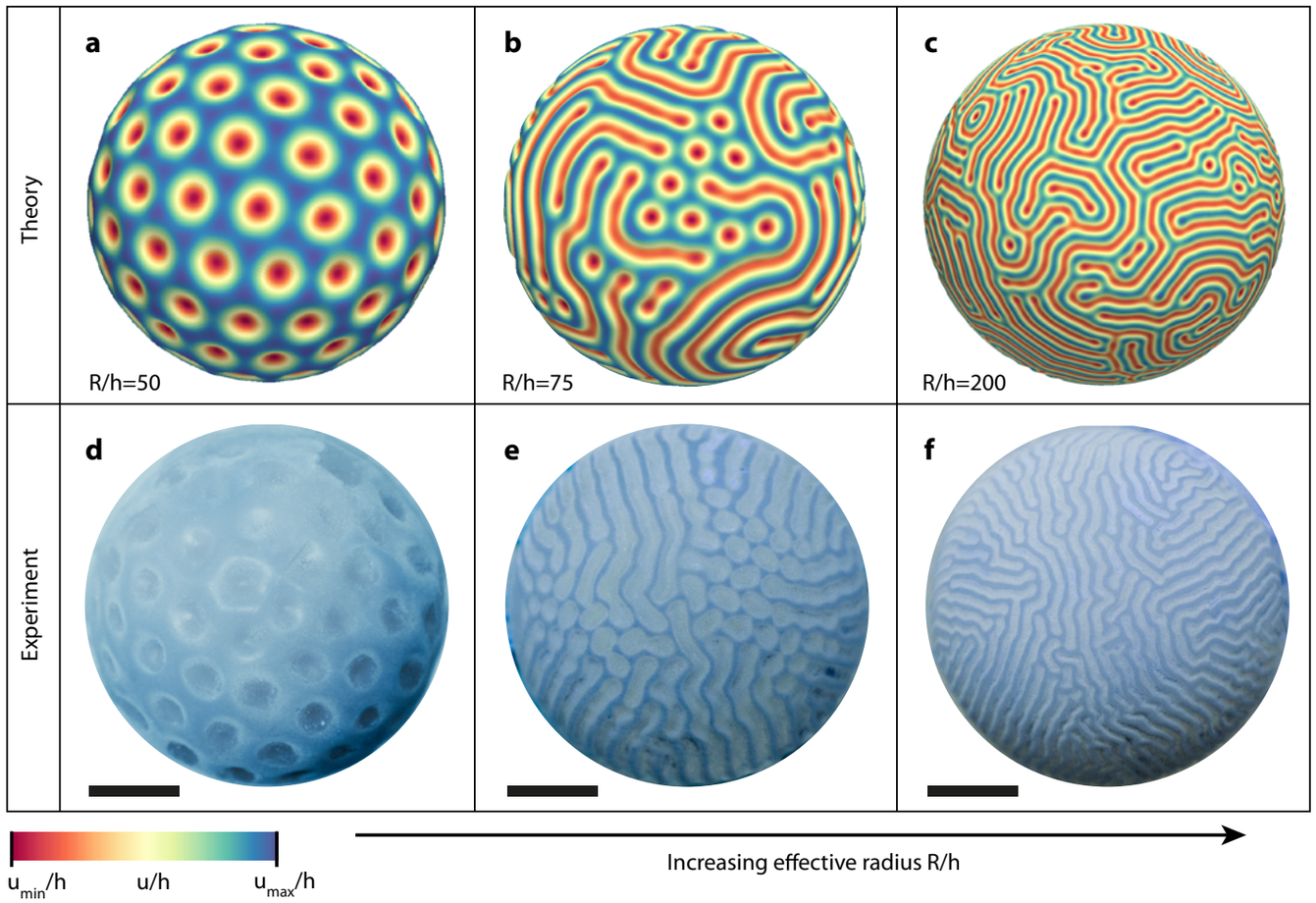
symmetry-breaking depends on curvature:

$$b \sim \frac{h^3}{R^3} \qquad \Gamma_1 \sim \frac{h}{R}$$

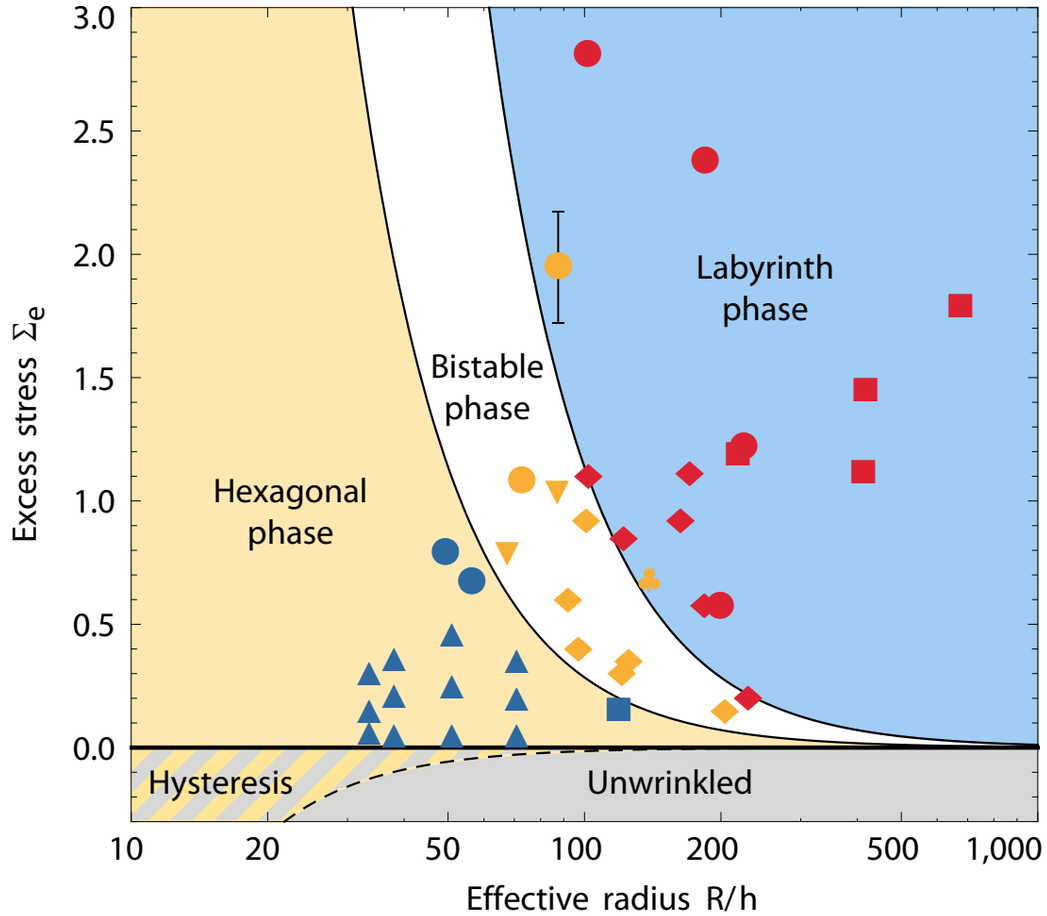
Nature Materials 2015

Theory correctly predicts

morphology



phase transitions



Nature Materials 2015

Arbitrarily curved surfaces

$$\begin{aligned}\mu\partial_t u = \; & \gamma_0\triangle u - \gamma_2\triangle^2 u - au - bu^2 - cu^3 + \\ & \frac{h}{2}\left\{(\nu-1)\left[b^{\alpha\beta}\nabla_\alpha u\nabla_\beta u + 2u\nabla_\beta\left(b^{\alpha\beta}\nabla_\alpha u\right)\right] + \right. \\ & \left. 2\nu\left[\mathcal{H}(\nabla u)^2 - 2\nabla\cdot(\mathcal{H}u\nabla u)\right]\right\} + \\ & \frac{h}{2}\left[(1-\nu)u\nabla_\beta\left(uc^{\alpha\beta}\nabla_\alpha u\right) - \nu\mathcal{R}u(\nabla u)^2 + \right. \\ & \left. \nu\nabla\cdot(\mathcal{R}u^2\nabla u)\right]\end{aligned}$$

Mean curvature

$$\mathcal{H} = \frac{1}{2}b^\gamma_\gamma$$

$$\mathcal{K} = \det(\{a_{\alpha\beta}\})/\det(\{b_{\alpha\beta}\})$$

Gauss curvature

$$\mathcal{R} \equiv b^{\alpha\beta}b_{\alpha\beta} = c^\gamma_\gamma = 4\mathcal{H}^2 - 2\mathcal{K}$$

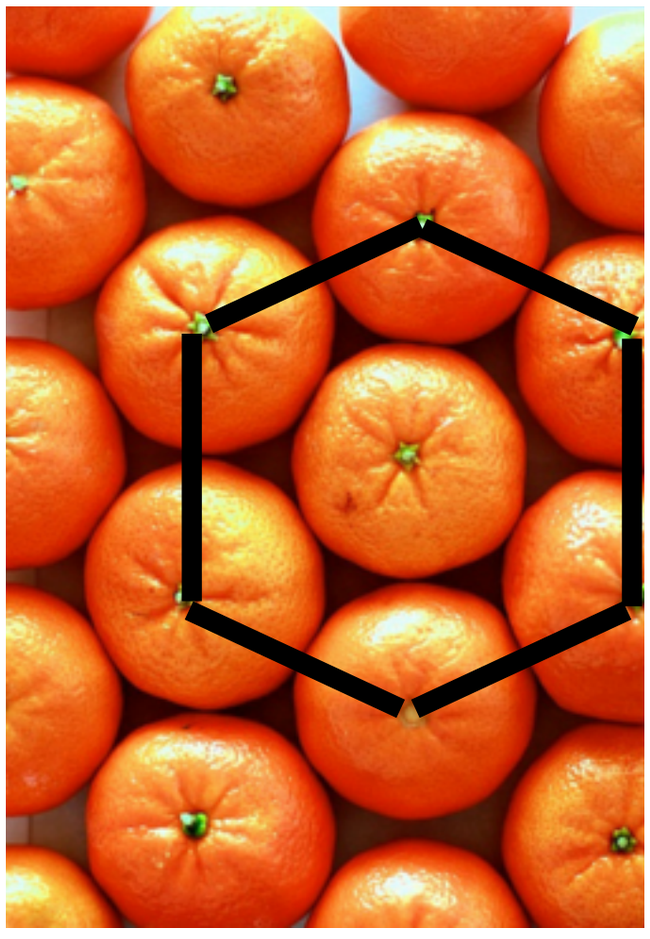
Arbitrarily curved surfaces

$$
\begin{aligned}
\mu\partial_t u = \; & \gamma_0\triangle u - \gamma_2\triangle^2 u - au - bu^2 - cu^3 + \\
& \frac{h}{2}\left\{(\nu-1)\left[b^{\alpha\beta}\nabla_\alpha u\nabla_\beta u + 2u\nabla_\beta\left(b^{\alpha\beta}\nabla_\alpha u\right)\right] + \right. \\
& \left. 2\nu\left[\mathcal{H}(\nabla u)^2 - 2\nabla\cdot(\mathcal{H}u\nabla u)\right]\right\} + \\
& \frac{h}{2}\left[(1-\nu)u\nabla_\beta\left(uc^{\alpha\beta}\nabla_\alpha u\right) - \nu\mathcal{R}u(\nabla u)^2 + \right. \\
& \left. \nu\nabla\cdot(\mathcal{R}u^2\nabla u)\right]
\end{aligned}
$$



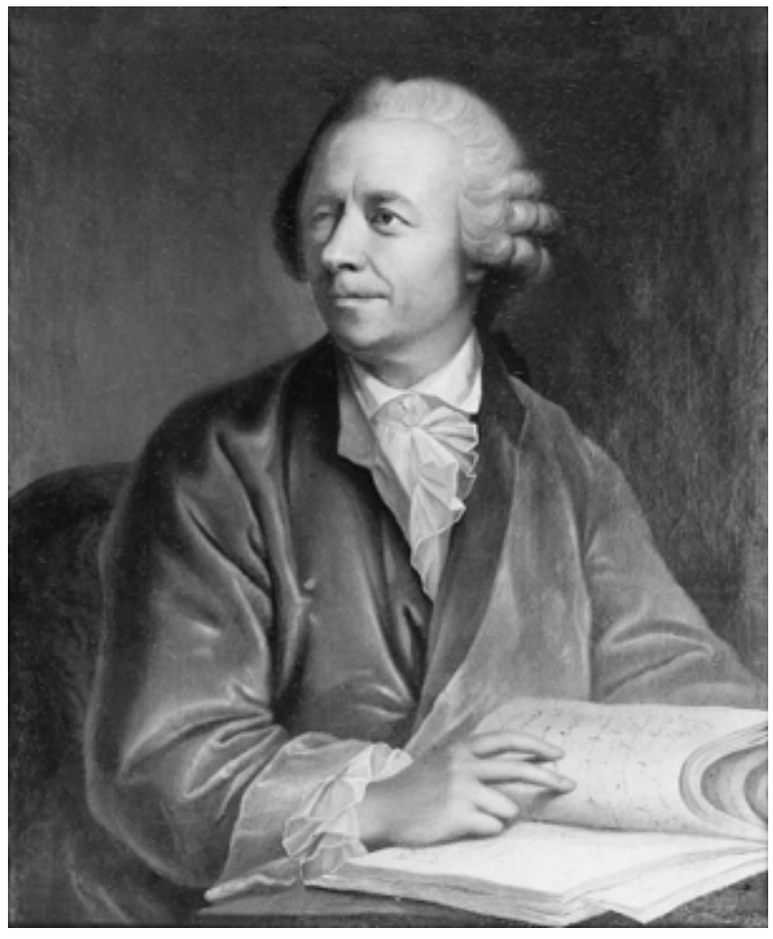


Topological defects





20 hexagons
12 pentagons

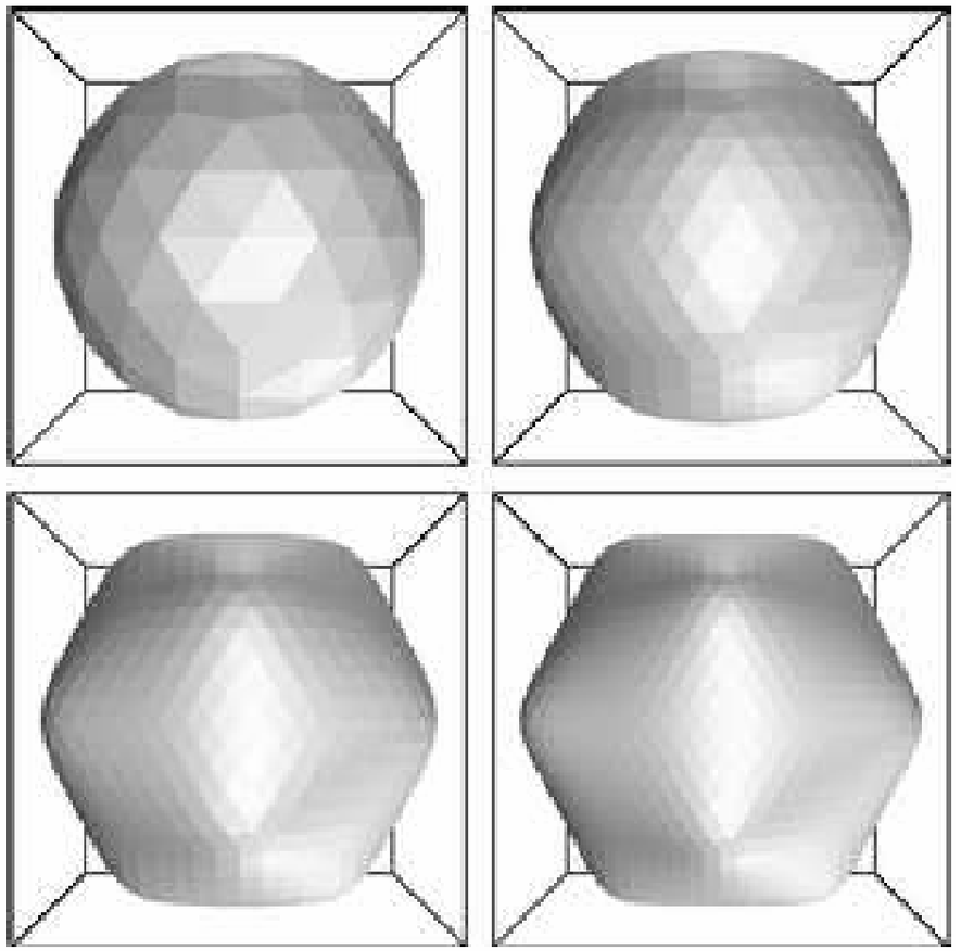


Leonard Euler

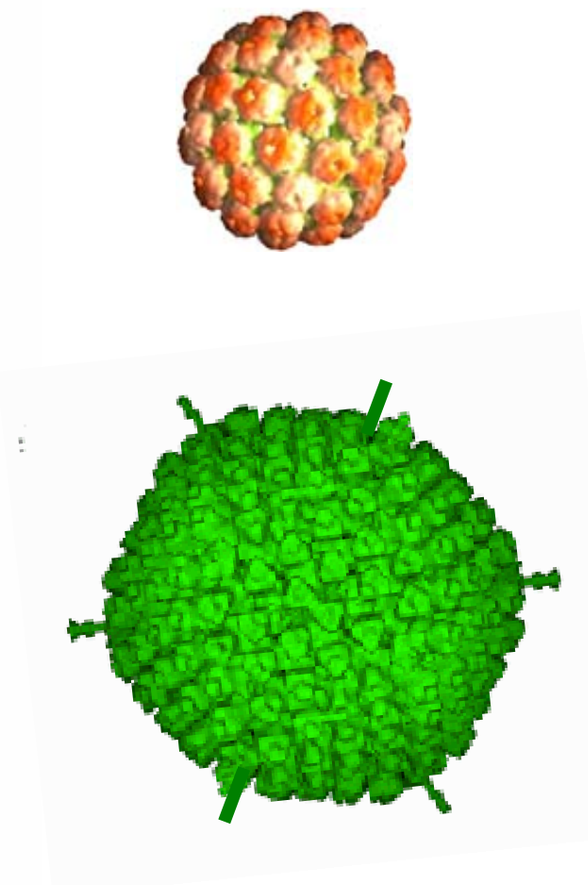
$$V - E + F = 2$$

Why important ?

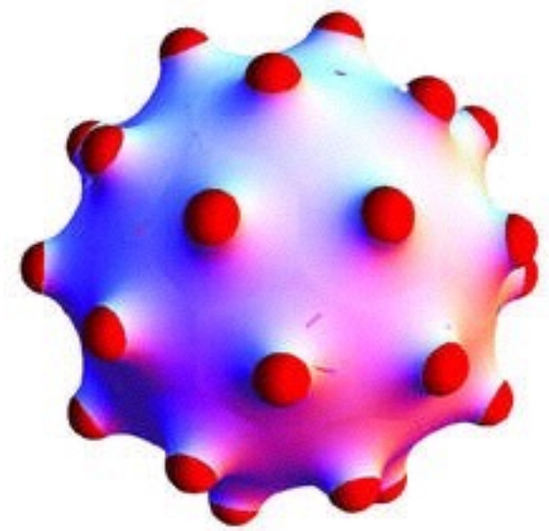
defects nucleate size-induced shape transition in viral capsids



J Lidmar, L Mirny and DR Nelson
Phys. Rev. E 68, 051910 (2003)

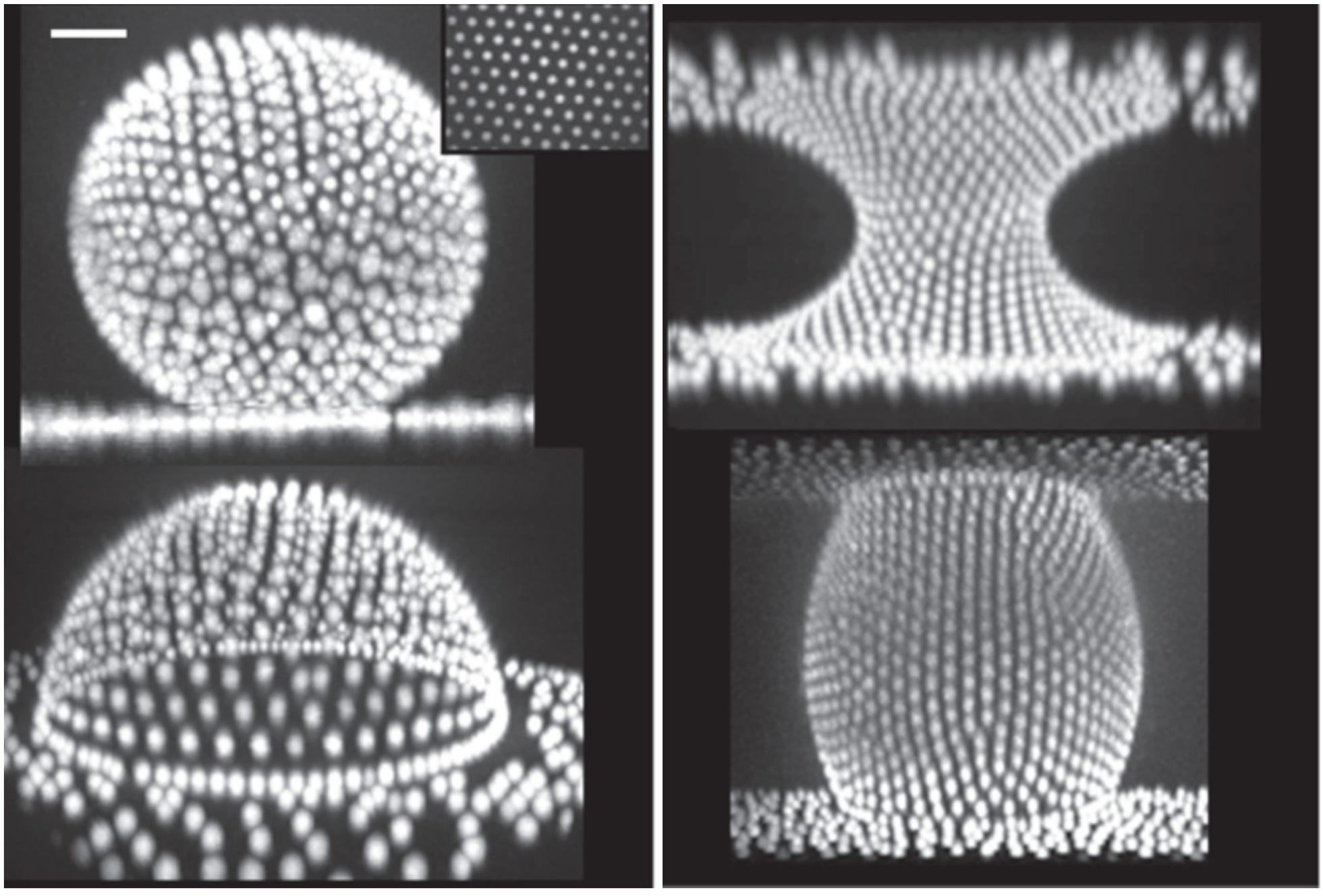


defects define tubular growth sites



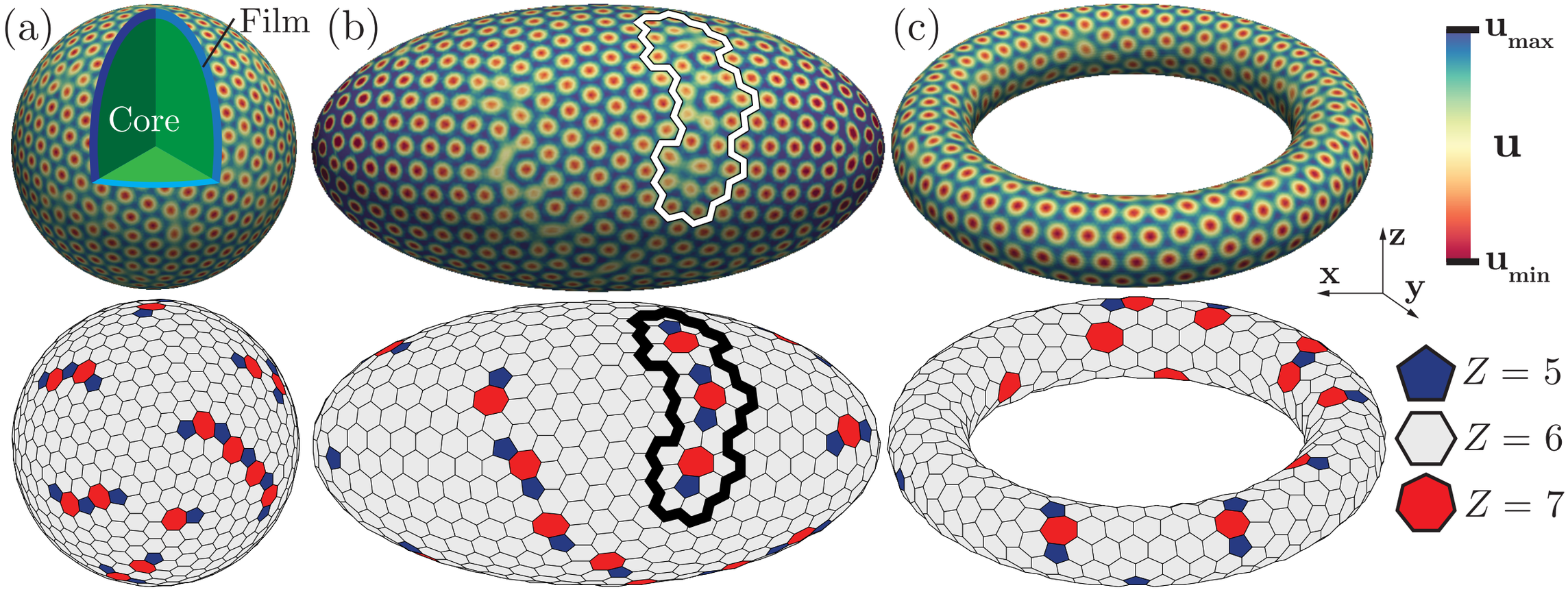
T Auth and G Gompper
Phys. Rev. E 80, 031901 (2009)

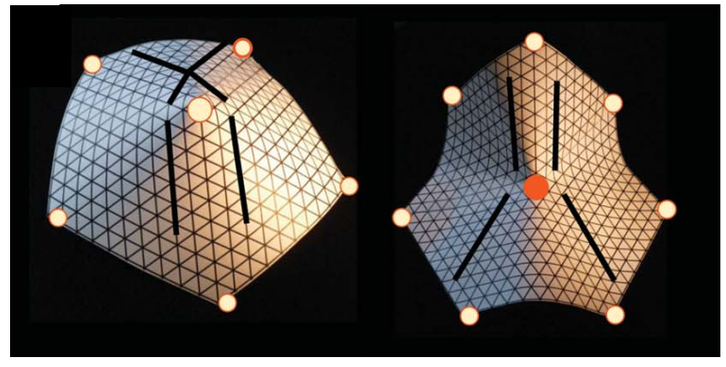
Interface crystallography



Irvine et al (2010) Nature

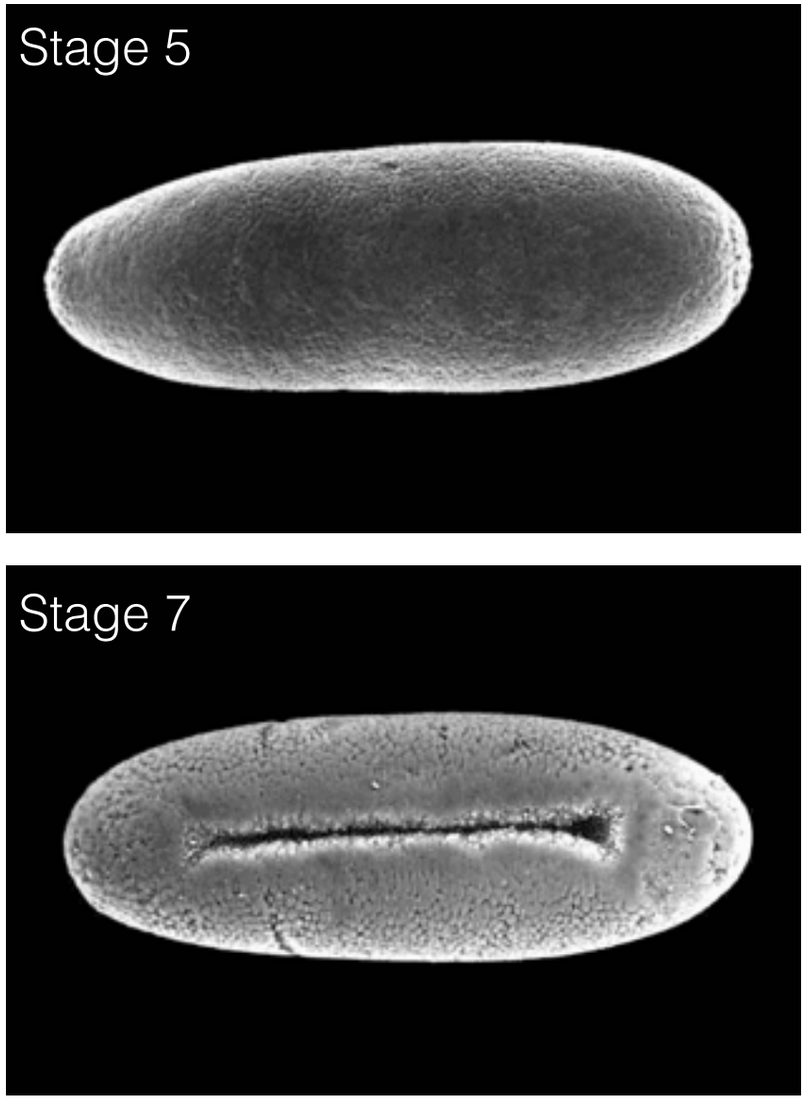
Elastic surface crystals





Physical Review Letters 2016

Future goal:
Apply similar ideas to understand pattern formation during embryogenesis

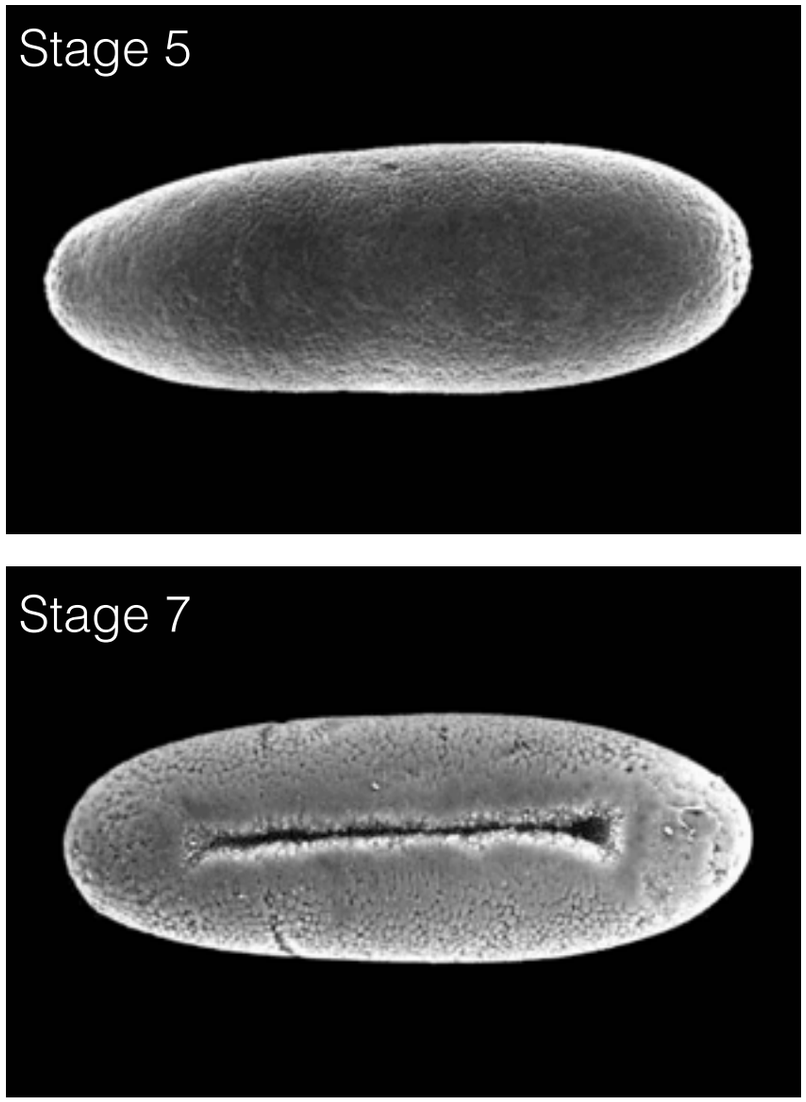




http://flybase.org

Collaboration: Martin Lab (MIT Biology)

Future goal:
Apply similar ideas to understand pattern formation during embryogenesis



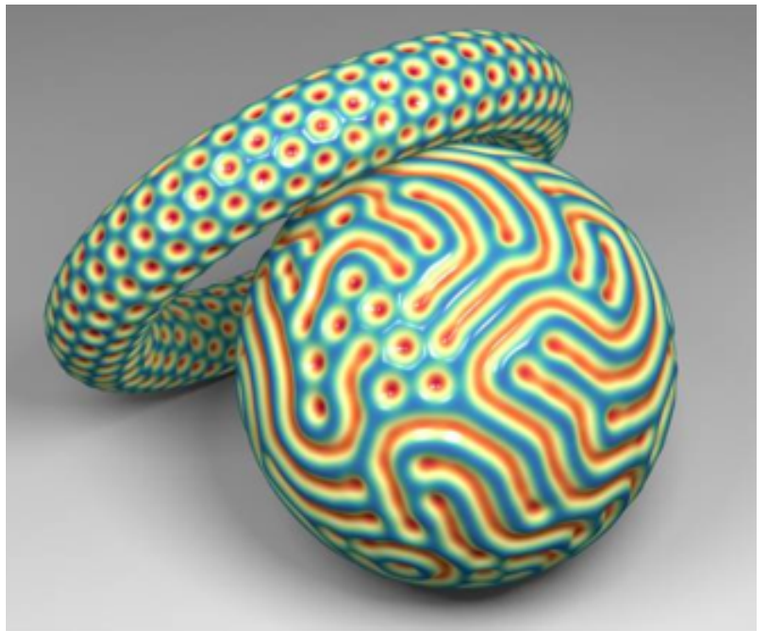
http://flybase.org



Image: Martin Lab (MIT Biology)

Thank you!





N Stoop, R Lagrange, D Terwagne, P M Reis and J D, Nature Materials 14: 337-342, 2015

F Lopez Jimenez, N Stoop, R Lagrange, J D and P M Reis, Physical Review Letters 116: 104301, 2016

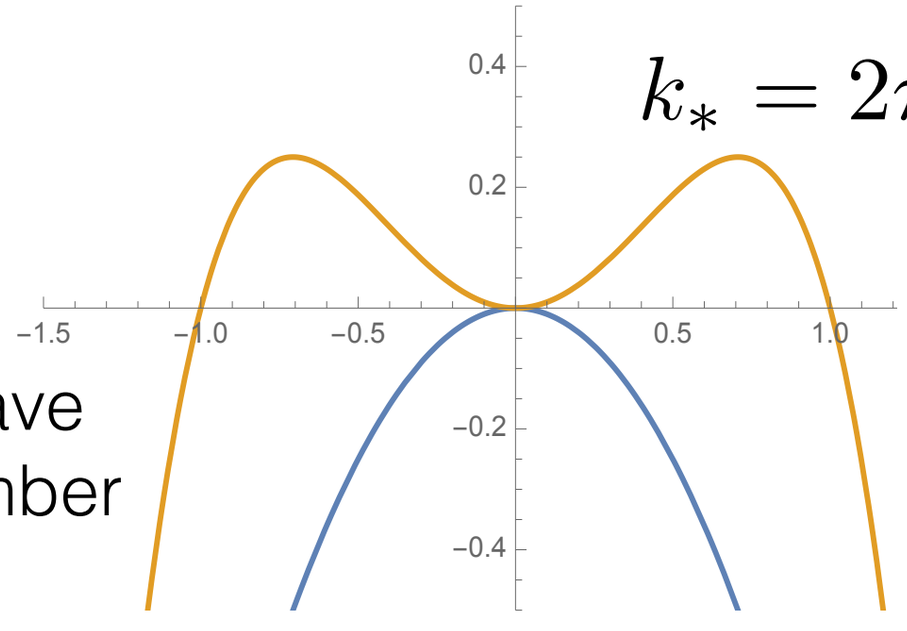
Universality idea



$$\partial_t u = \gamma_0 \Delta u - \gamma_2 \Delta^2 u - au - bu^2 - cu^3 + \Gamma_1 \left[(\nabla u)^2 + 2u\Delta u\right] + \Gamma_2 \left[u(\nabla u)^2 + u^2 \Delta u\right]$$

Fourier space

dispersion



$k_* = 2\pi/\Lambda_*$

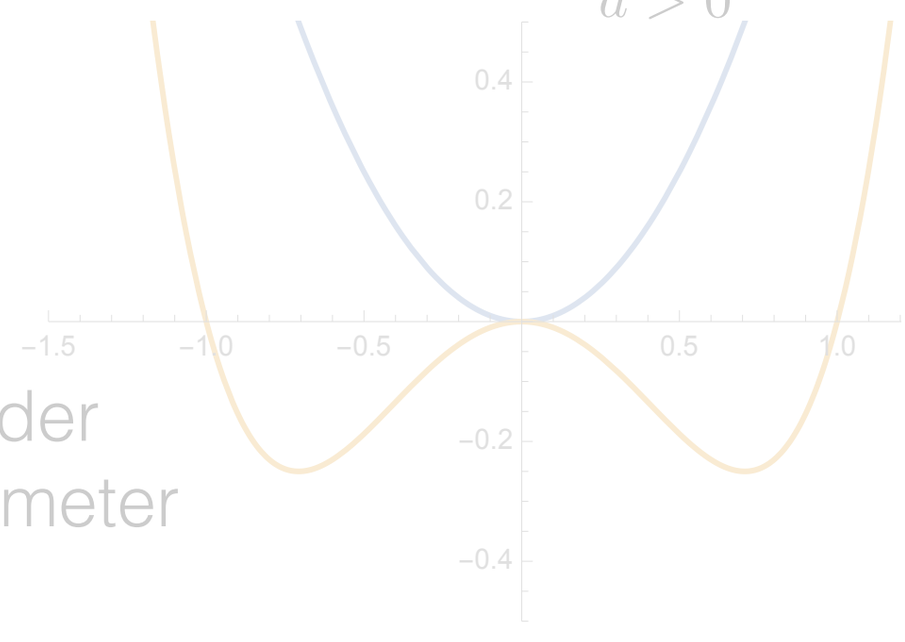
wave number

$$-\gamma_0 k^2 - \gamma_2 k^4$$

$$\gamma_0 \nabla^2 - \gamma_2 (\nabla^2)^2$$

Order-parameter space

potential



order parameter

$$au^2 + cu^4$$

$$\mathcal{E}_{\mathrm{KS}}(\mathbf{\Psi}) = \mathcal{E}_b(\mathbf{\Psi}) + \mathcal{E}_s(\mathbf{\Psi}) + \mathcal{E}_f(\mathbf{\Psi})$$

with bending energy

$$\mathcal{E}_b = \frac{E_f}{2(1-\nu^2)} \int_\omega d\omega \; \frac{h^3}{24} C^{\alpha\beta\gamma\delta} R_{\gamma\delta}(\mathbf{\Psi}) R_{\alpha\beta}(\mathbf{\Psi})$$

stretching energy

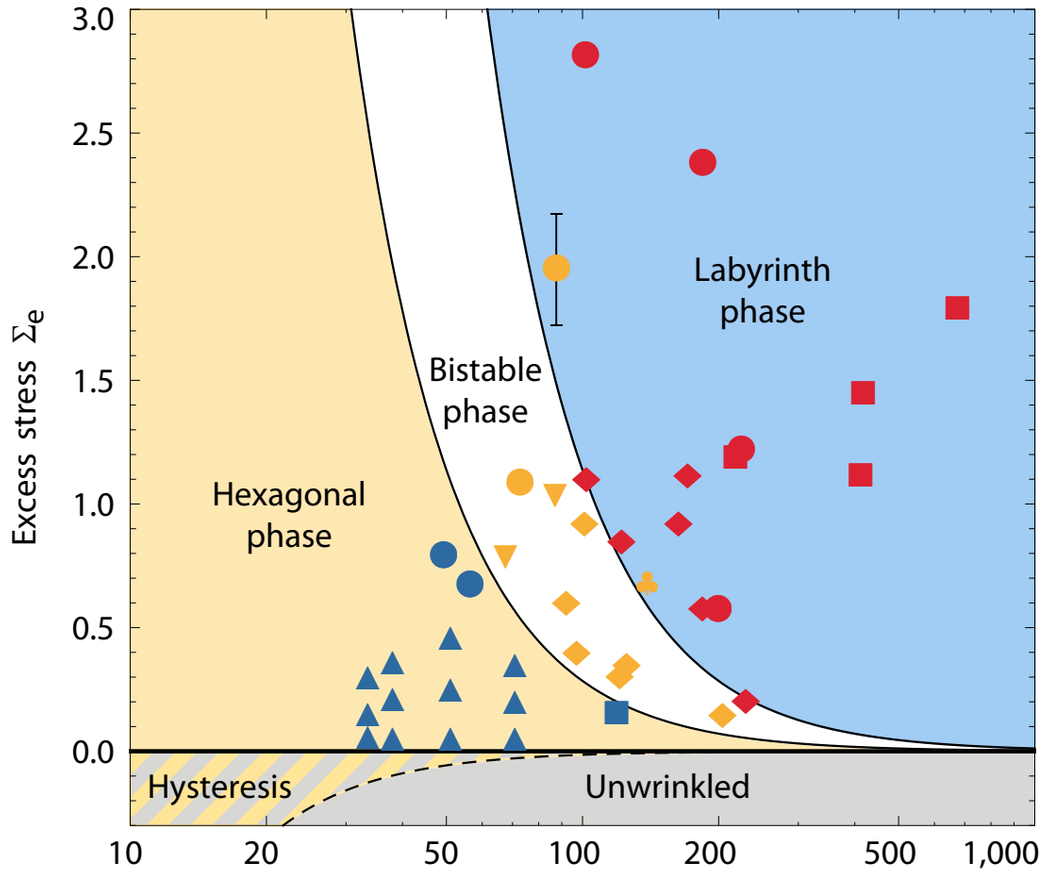
$$\mathcal{E}_s = \frac{E_f}{2(1-\nu^2)} \int_\omega d\omega \; \frac{h}{2} C^{\alpha\beta\gamma\delta} G_{\gamma\delta}(\mathbf{\Psi}) G_{\alpha\beta}(\mathbf{\Psi})$$

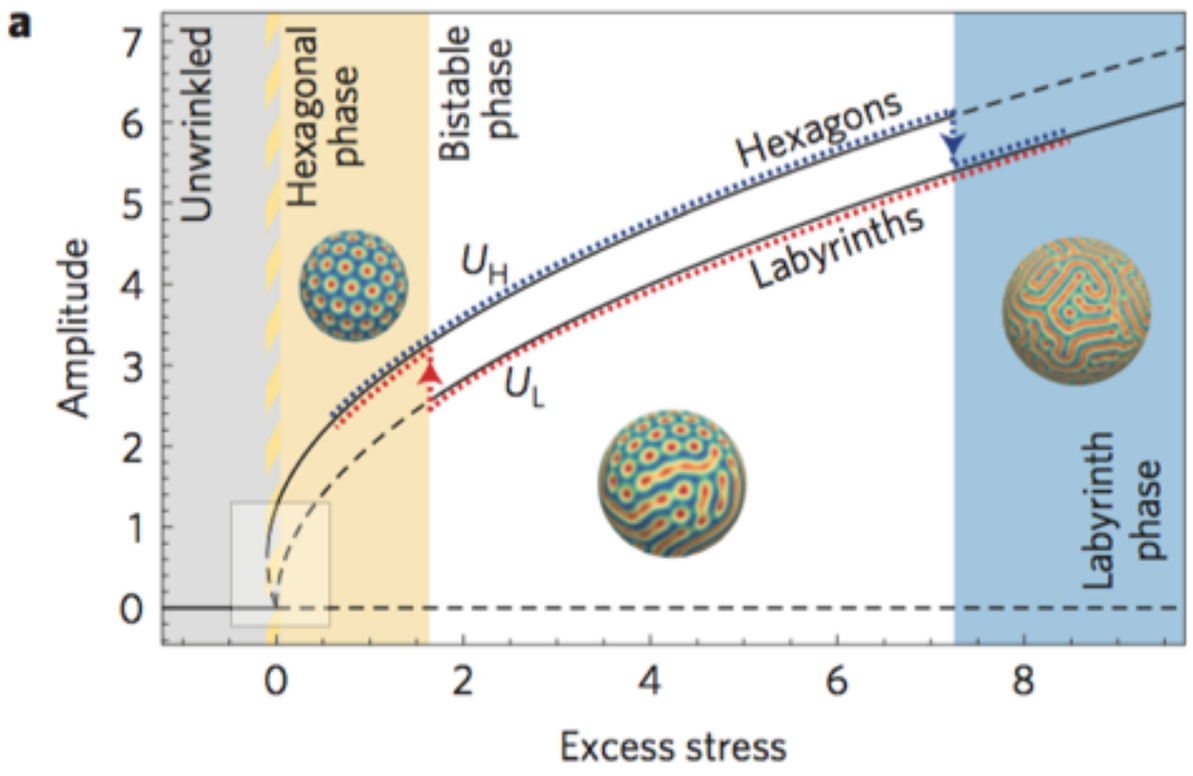
and energy contributions

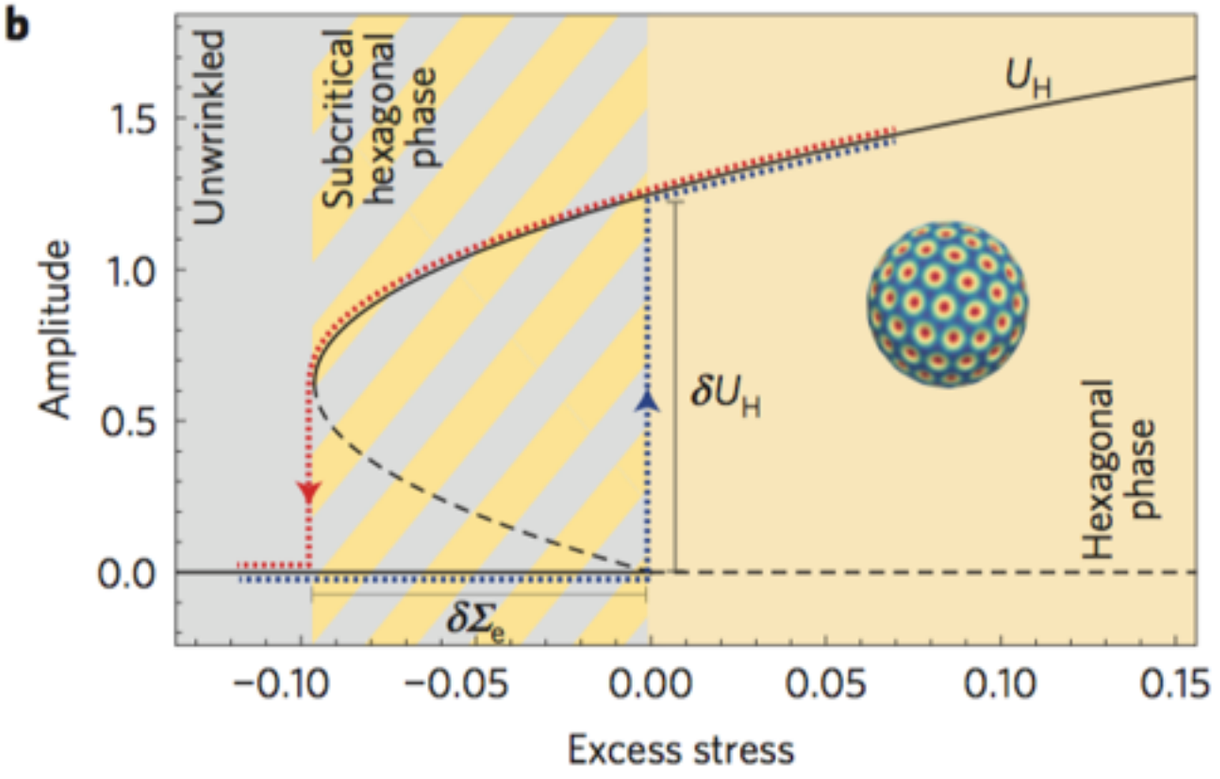
$$\mathcal{E}_f = -\int_\omega d\omega \; p^i \Psi_i$$

Kirchhoff-St Venant material (generalized Hooke's law)

$$C^{\alpha\beta\gamma\delta} = (1-\nu)(a^{\alpha\delta}a^{\beta\gamma} + a^{\alpha\gamma}a^{\beta\delta}) + 2\nu a^{\alpha\beta}a^{\gamma\delta}$$


Excess stress Σ_e
3.0
2.5
2.0
1.5
1.0
0.5
0.0
Labyrinth phase
Bistable phase
Hexagonal phase
Hysteresis
Unwrinkled
10
20
50
100
200
500
1,000
Effective radius R/h

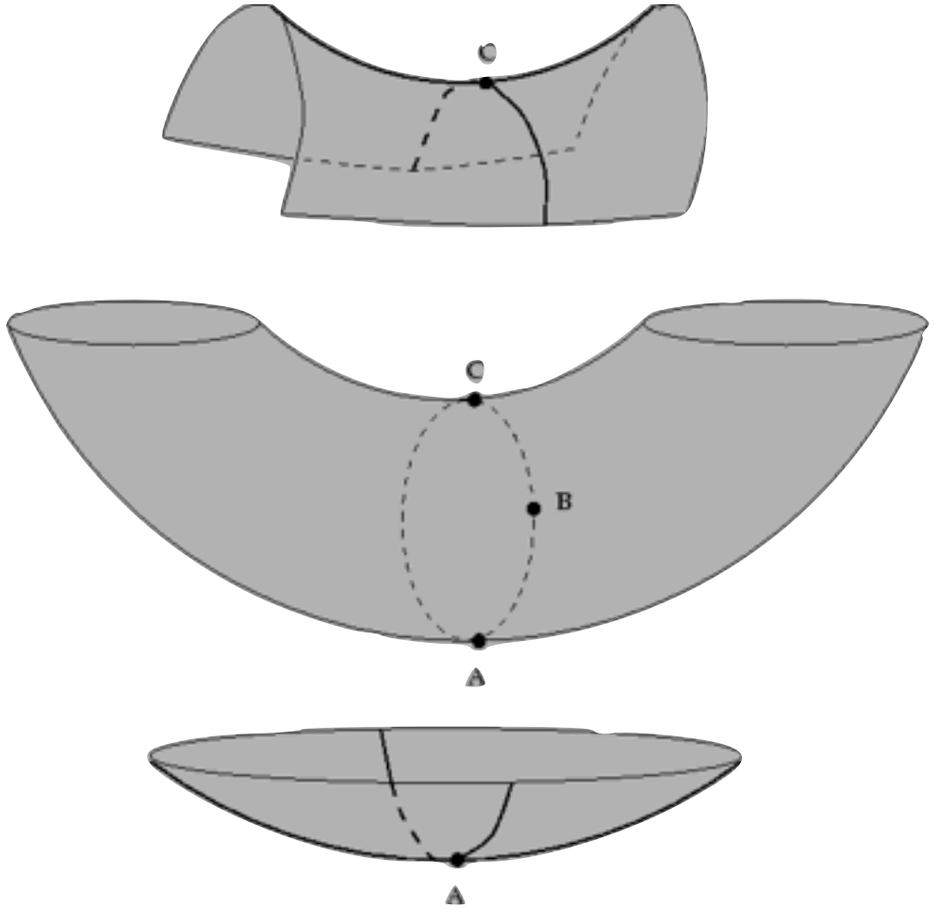

a
Amplitude
Unwrinkled
Hexagonal phase
Bistable phase
Hexagons
Labyrinths
U_H
U_L
Labyrinth phase
Excess stress

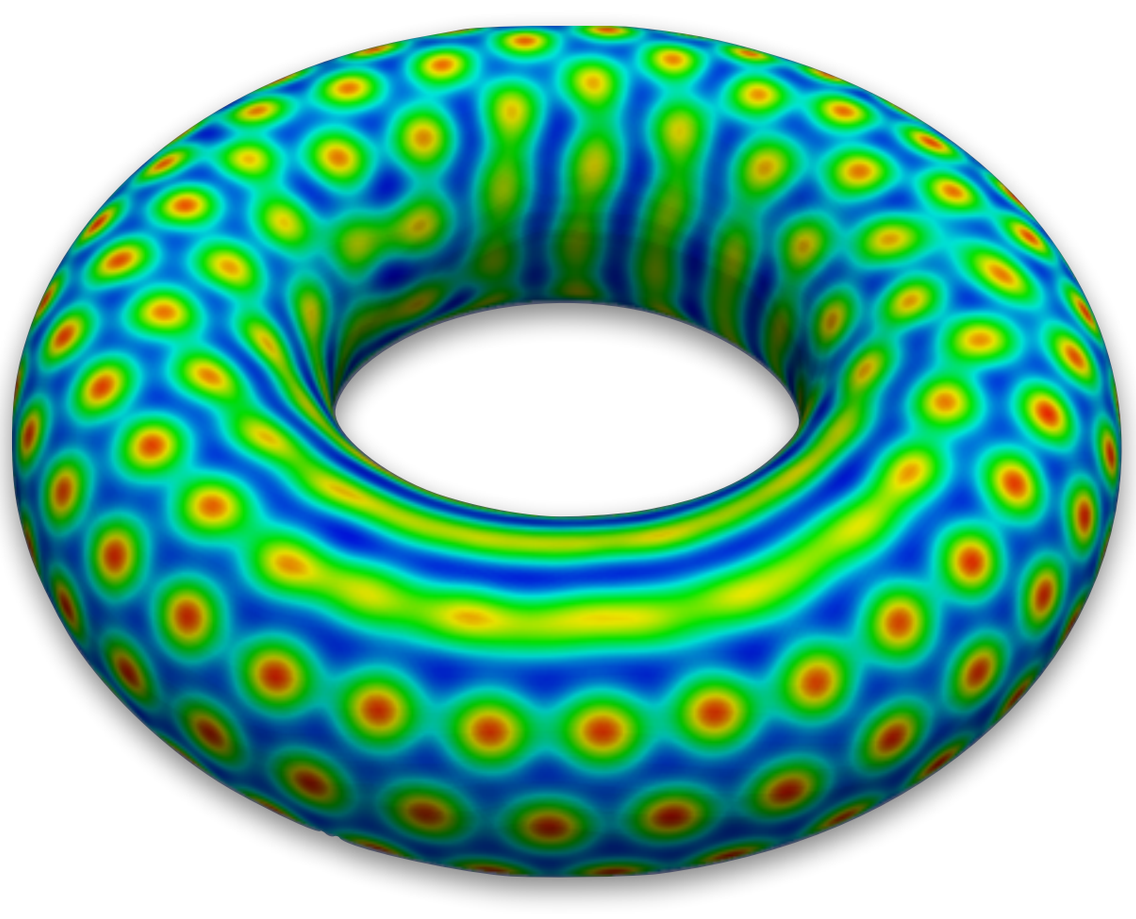

b
Amplitude
Unwrinkled
Subcritical hexagonal phase
U_H
δU_H
$\delta\Sigma_e$
Hexagonal phase
Excess stress

Torus

$$-\frac{\delta\bar{\mathcal{E}}}{\delta u} = \gamma_0\triangle u - \gamma_2\triangle^2 u - au - bu^2 - cu^3 + \tag{34}$$

$$\frac{h}{2}\left\{(\nu-1)\left[b^{\alpha\beta}\nabla_\alpha u\nabla_\beta u + 2u\nabla_\beta\left(b^{\alpha\beta}\nabla_\alpha u\right)\right] + 2\nu\left[\mathcal{H}(\nabla u)^2 - 2\nabla\cdot(\mathcal{H}u\nabla u)\right]\right\}$$

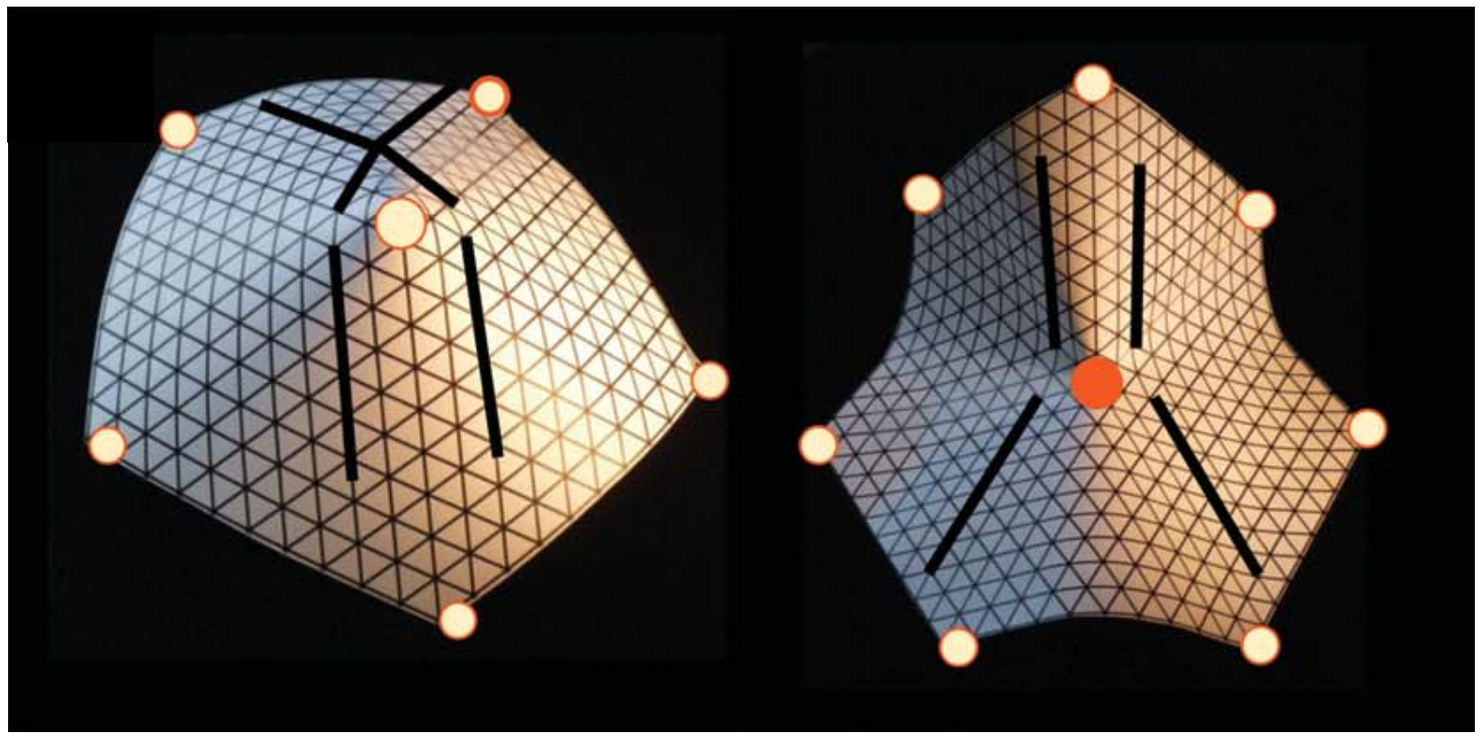


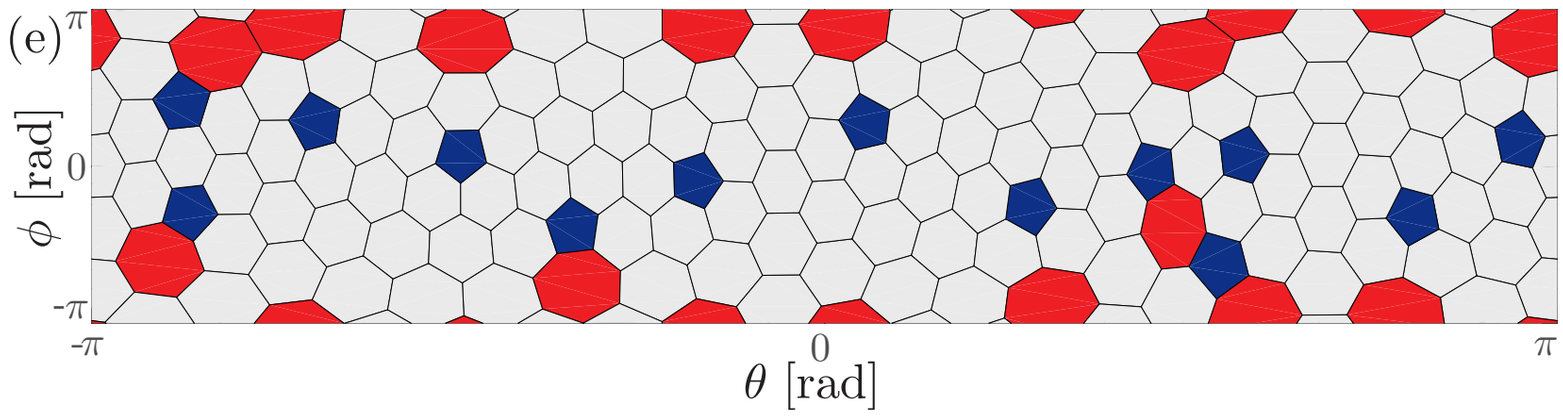


Defects on the torus



- Charge separation due to Gaussian curvature





- Electrostatic analogy (M. Bowick et al, Phys. Rev. E 69, 2004)