5.2 Stokes equations

5.2.1 Motivation

Consider an object of characteristic length L, moving at absolute velocity $U = |\boldsymbol{U}|$ through (relative to) an incompressible, homogeneous Newtonian fluid of constant viscosity μ and constant density ϱ. The object can be imagined as a moving boundary (condition), which induces a flow field $\boldsymbol{u}(t, \boldsymbol{x})$ in the fluid. The ratio of the inertial (dynamic) pressure ϱU^2 and viscous shearing stress $\mu U/L$ can be characterized by the Reynolds number[4]

$$\mathcal{R} = \frac{|\varrho(\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u})|}{|\mu \nabla^2 \boldsymbol{u}|} \simeq \frac{\varrho U^2/L}{\mu U/L^2} = \frac{UL\varrho}{\mu} = \frac{UL}{\nu}. \tag{5.11}$$

For example, when considering swimming in water ($\nu = 10^{-6}\,\mathrm{m}^2/\mathrm{s}$), one finds for fish or humans:

$$L \simeq 1\,\mathrm{m},\ U \simeq 1\ \mathrm{m/s} \qquad \Rightarrow \qquad \mathcal{R} \simeq 10^6,$$

whereas for bacteria:

$$L \simeq 1\,\mu\mathrm{m},\ U \simeq 10\ \mu\mathrm{m/s} \qquad \Rightarrow \qquad \mathcal{R} \simeq 10^{-5}.$$

If the Reynolds number is very small, $\mathcal{R} \ll 1$, the nonlinear NSEs (5.8) can be approximated by the linear *Stokes equations*[5]

$$0 = \nabla \cdot \boldsymbol{u}, \tag{5.12a}$$

$$0 = \mu \nabla^2 \boldsymbol{u} - \nabla p + \boldsymbol{f}. \tag{5.12b}$$