Quasi-smooth Derived Manifolds

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Abstract

The category **Man** of smooth manifolds is not closed under arbitrary fiber products; for example the zeroset of a smooth function on a manifold is not necessarily a manifold, and the non-transverse intersection of submanifolds is not a manifold. We describe a category **dMan**, called the category of *derived manifolds* with the following properties: 1. **dMan** contains **Man** as a full subcategory; 2. **dMan** is closed under taking zerosets of arbitrary smooth functions (and consequently fiber products over a smooth base); and 3. every compact derived manifold has a fundamental homology class which has the desired properties.

In order to accomplish this, we must incorporate homotopy theoretic methods (e.g. model categories or ∞-categories) with basic manifold theory. Jacob Lurie took on a similar project in his thesis, namely incorporating homotopy theory and algebraic geometry. We derive much of our inspiration from that and subsequent works by Lurie. For example, we define a derived manifold to be a *structured space* that can be covered by principal derived manifolds, much as a derived scheme is a structured space that can be covered by affine derived schemes.

We proceed to define a cotangent complex on derived manifolds and use it to show that any derived manifold is isomorphic to the zeroset of a section of a vector bundle $E \to \mathbb{R}^N$. After defining *derived manifolds with boundary* we begin to explore the notion of *derived cobordism* over a topological space K. We prove two crucial facts: that any two manifolds which are derived cobordant are in fact (smoothly) cobordant, and that any derived manifold is derived cobordant over K to a smooth manifold. In other words, the cobordism ring agrees with the derived cobordism ring. Finally, we define the fundamental class on a derived manifold $\mathcal{X}$ by equating it with the fundamental class of a smooth manifold which is derived cobordant to $\mathcal{X}$.

The main attraction of derived manifolds is that they provide a robust way of intersecting submanifolds. That is, the intersection product (as well as other constructions such as the Euler class of a vector bundle) is functorially defined within the category **dMan** itself, rather than just homologically.

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Introduction

0.1 Background

This dissertation falls at the intersection of homotopy theory, derived algebraic geometry, intersection theory, and synthetic differential geometry. Its purpose in some sense is to functorialize Thom's Transversality theorem, so that submanifolds do not have to be perturbed in order for one to be able to work with their intersection.

Often one wants to take the limit of a diagram of manifolds. The simplest type of limit that does not always exist in the category of manifolds is the fiber product

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[d, "c"] \\ N \arrow[r, "d"'] & P. \end{tikzcd}$$

For example, one may want

1. to intersect two submanifolds $M, N \subset P$,
2. to look at the fiber of a smooth morphism $d : N \to P$ of manifolds over a point $\bullet$ (here, $M = \bullet$), or
3. to take the zeroset of a section $d : N \to P$ of a vector bundle $p : P \to N$ (here $M = N$ is the base, and c is the zero section).

None of these fiber products generally exist in the category of manifolds (though they do exist in the category of topological spaces).

This would present a problem, except for Thom's transversality theorem. It says that though these fiber products do not exist in general, the subset of diagrams for which a fiber product does exist is *dense* in the set of all such diagrams. One can tell whether a fiber product exists (and is in some sense "correct") in the category of manifolds by looking at the induced morphisms on tangent spaces $c_* : T_M \to T_P$ and $d_* : T_N \to T_P$. If at every point of $c(M) \cap d(N)$, the subspace of T_P spanned by the images of c_* and d_* is all of T_P, then we say that c and d are transverse, and a (correct) fiber product exists. Thom's transversality theorem states that the space of morphisms $M \to P$ that are transverse to d is dense in $\mathrm{Hom}_{\mathbf{Man}}(M, P)$. This is one of the fundamental theorems in topology.

As useful as this theorem is, it has significant drawbacks. First, it is not functorial: one must choose a transverse replacement, and this can cause problems. Second, there are instances (e.g. in Floer homology) in which the submanifolds one starts with are presented in a particularly convenient or symmetric way, and the process of making them transverse destroys this symmetry. Third, and most importantly, transversality completely fails in the equivariant setting. For example, consider the circle action on the 2-sphere S^2 given by rotation about an axis. The only equivariant maps from a point to S^2 are given by the north and south pole. Thus, the S^1-equivariant intersection of a point with itself on S^2 is a point, which is of codimension 2 instead of the desired codimension 4. This is one illustration of how intersection theory fails in the equivariant setting.

Another approach to taking limits in the category of manifolds might be to attempt to follow in the footsteps of scheme theory. Arbitrary fiber products do exist in the category of schemes without need of a transversality requirement. This is part of what led Lawvere ([17]) to suggest adapting the ideas of scheme theory to the setting of differential geometry. By this time, he had already invented the notion of theories and models [2, 3.3]. A theory $\mathcal{T}$ is a category with a set of objects $\{T^0, T^1, T^2, \ldots\}$, indexed by the natural numbers, in which each object T^n is the n-fold Cartesian product of T^1. A model of $\mathcal{T}$ is a product-preserving functor from $\mathcal{T}$ to **Sets**, and a morphism of models is a natural transformation of functors.

Let $\mathcal{R}$ denote the theory whose objects are the affine spaces $\{\mathbb{Z}^0, \mathbb{Z}^1, \mathbb{Z}^2, \ldots\}$ and whose morphisms are polynomial maps (with integer coefficients). Then the category of $\mathcal{R}$ models is isomorphic to the category of commutative rings. One might agree that manifolds are related to smooth functions in the same way that schemes are related to polynomials. This led Kock, Dubuc, and others to develop a theory of "Synthetic Differential Geometry" by use of so-called C^∞-rings. Let $\mathcal{T}$ denote the theory whose objects are Euclidean spaces $\{\mathbb{R}^0, \mathbb{R}^1, \mathbb{R}^2, \ldots\}$ and whose morphisms are smooth functions $f : \mathbb{R}^i \to \mathbb{R}^j$. The models of this theory are called C^∞-rings. Synthetic differential geometry is the study of C^∞-rings and C^∞-schemes, which contains the category of manifolds as a full subcategory, but also includes the ring of germs or the ring of jets at a point in a manifold, as well as far more general objects (see [22].)

Fiber products indeed exist in the category of C^∞-schemes, which would appear to solve our problem. Unfortunately, one of our main purposes for needing these fiber products was for doing intersection theory (because fiber products generalize intersections), and intersection theory is conspicuously missing from the literature on C^∞-schemes.

Moreover, even if one could do intersection theory with C^∞-schemes, it would not be quite right. Most conspicuously, the codimension of a non-transverse intersection of submanifolds could be different from the sum of their codimensions. In these situations, data is somehow lost when we naively perform the intersection. The easiest example, and one that runs throughout this thesis, is that of intersecting a point with itself in the affine line. Let us quickly recall the situation in algebraic geometry. Let $k[x] \to k$ denote the map sending x to 0. One has $k \otimes_{k[x]} k = k$, which means geometrically that the intersection of the origin with itself is equal to the origin. This is perhaps fine geometrically, but it has problems intersection-theoretically. Most glaringly, its dimension is too big (it has dimension 0 instead of dimension -1). This dimension problem is corrected if we work in the derived category, because Serre's intersection formula gives the expected result. One might say that the issues are resolved because tensor product is a flat functor in the derived category.

The above issues are carefully fleshed out in the introduction to Jacob Lurie's thesis, [18]. There, he describes a solution to the intersection theoretic problems outlined above. According to Lurie, the problem stems from the fact that when one quotients a ring (resp. C^∞-ring) by an ideal, one is *setting elements equal to each other*. He points out that this is not in line with the basic philosophy of category theory: instead of setting two objects equal to each other, one should introduce an isomorphism connecting them. Another way to think of it is that the process of taking quotients often results in "information loss"; one can achieve the same result without losing information by simply adding an isomorphism, or in our case, a path between two vertices in a certain simplicial set. Two paths between the same two vertices is not redundant information (as it would be if we were to quotient twice by the same element of a ring) – each new path increases the number of generators of π_1 of the simplicial set.

This brings us to homotopical algebra. Homotopical algebra is the study of categories equipped with a class of quasi-isomorphisms or weak equivalences. This is a basic setting in which one can "do homotopy theory." Introduced by Quillen in 1967 ([24]), homotopical algebra includes homological algebra (with weak equivalences given by quasi-isomorphisms), the homotopy theory of topological spaces (with weak equivalences given by maps which induce isomorphism on homotopy groups), spectra, simplicial commutative rings, etc.

The localization of model categories was first introduced by Bousfield in [4] as a way of adding weak equivalences to a model category. He used localization to add homology isomorphisms (i.e. morphisms of topological spaces which induce isomorphisms on homology groups) to the set of weak equivalences in the category of topological spaces. For any homology theory h_*, one may wish to consider a map of topological spaces $f : X \to Y$ to be a weak equivalence if and only if $h_*(f)$ is an isomorphism, and to accomplish this one should use localization of model categories. Since Bousfield, localization has become a central tool in homotopy theory. Many mathematical constructions can be achieved homotopy-theoretically by replacing equality with weak equivalence, limits and colimits by homotopy limits and homotopy colimits, etc.

For example, in [6], Dugger et al. showed that Jardine's model category of simplicial presheaves on a Grothendieck site $\mathcal{C}$ ([16]) was in fact a localization of the functor category $\mathbf{sSets}^{\mathcal{C}}$. The localization was simply that which forces local presheaves to satisfy "hypercover descent". In other words, Jardine's simplicial presheaves could be called "homotopy sheaves" or "derived sheaves". (All this is spelled out in Section 2.3.) In the same vein, Hollander has shown [14] that all definitions of a stack (e.g. as a category fibered in groupoids, as a presheaf of groupoids, as a sheaf of groupoids, and as a lax presheaf of groupoids) are Quillen equivalent to a very basic model category: the localization of $\mathbf{Grpd}^{\mathcal{C}}$ at the set of Čech hypercovers.

We use the same derived approach to achieve our aim of making smooth intersection theory more robust. We follow Lurie's thesis for as long as we can. In particular, we use the concepts of admissible morphisms, and structured spaces (which we call local smooth-ringed spaces) from [18, 4.1]. However, instead of using simplicial commutative rings, we use a variant of C^∞-rings, which we call smooth rings and define in Chapter 2.

A fibrant smooth ring is a functor from the category of manifolds to the category of simplicial sets which preserves those limits which do happen to exist and are "correct" (i.e. satisfy a transversality condition) in manifolds. For example, if $s : M \to N$ and $a : P \to N$ are smooth maps of manifolds and s is a submersion, then the fiber product X in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[d, "s"] \\ P \arrow[r, "a"'] & N \end{tikzcd}$$

exists and is correct in the category of manifolds. A fibrant smooth ring F : $\mathbf{Man} \to \mathbf{sSets}$ is a functor which preserves this fiber product (up to homotopy), i.e. a functor for which the diagram

$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & F(M) \arrow[d] \\ F(P) \arrow[r] & F(N) \end{tikzcd}$$

is a homotopy limit. For example, if M is any manifold then the representable functor $H_M := \mathrm{Hom}(M, -)$ is a smooth ring because representable functors preserve limits.

The model category $\mathbf{SR}$ of smooth rings is the localization of $\mathbf{sSets}^{\mathbf{Man}}$ which declares that fibrant objects preserve such submersion pullbacks. That is, there is a set Ψ of morphisms in $\mathbf{sSets}^{\mathbf{Man}}$ such that localizing at Ψ forces fibrant objects to preserve submersion pullbacks. Note that since, for any manifold M, the terminal map $t : M \to \bullet$ is a submersion, any smooth ring is in particular product-preserving (up to weak equivalence). We have seen product-preserving functors before, when discussing theories and C^∞-rings. Indeed, if we start with a smooth ring $F : \mathbf{Man} \to \mathbf{sSets}$, restrict our domain category from $\mathbf{Man}$ to the subcategory of Euclidean spaces $\mathbb{R}^n$ and then take π_0, we recover the notion of C^∞-rings. This will in fact be useful for several calculations. After working out the basic properties of smooth rings, we move on to define smooth-ringed spaces, which are the analogues of ringed spaces from algebraic geometry. A smooth-ringed space is a pair $(X, \mathcal{O}_X)$ in which X is a topological space and $\mathcal{O}_X$ is a homotopy sheaf of smooth rings on X.

In Chapter 3, we define local smooth-ringed spaces to be smooth-ringed spaces that satisfy a locality condition, making them analogous to local ringed spaces. (A local smooth ring is a smooth ring whose underlying C^∞-ring is local.) Finally, we define our version of homotopical smooth schemes, called derived manifolds, in Chapter 4. They are local smooth-ringed spaces which are locally equivalent to a fiber product $\mathcal{X}$ of a diagram of manifolds $M \xrightarrow{f} P \xleftarrow{g} N$, taken in the category of smooth-ringed spaces. There is no (e.g. transversality) requirement on f and g. A "derived manifold" does not have boundary unless so stated.

We proceed in Chapter 5 to develop a theory of cotangent complexes on derived manifolds, following [25] and [19]. The cotangent complex is crucial in proving one of our main results (Theorem 6.1.12). Namely, that any compact derived manifold $\mathcal{X}$ is the (derived) zeroset of a section of a smooth vector bundle $p : E \to M$. (Conversely, the zeroset of any section of a smooth vector bundle is a derived manifold).

After defining derived manifolds with boundary, we begin discussing derived cobordism and its relationship with cobordism. By use of the above theorem, we prove that any derived manifold is derived cobordant to a smooth manifold, and that any two smooth manifolds that are derived cobordant are in fact smoothly cobordant. This is tantamount to proving that the derived cobordism ring of derived manifolds is isomorphic to the cobordism ring of manifolds. This is our main theorem and can be found in Chapter 6.

Finally, we define homology and cohomology for derived manifolds. If $\mathcal{X}$ is a compact derived manifold of dimension n, the n-th homology group $H_n(\mathcal{X})$ is not typically 1-dimensional. However, it does have a distinguished element $[\mathcal{X}]$. We call this element the *fundamental class* of $\mathcal{X}$ because it behaves like the fundamental class of a manifold:

- if $\mathcal{X}$ is a manifold then $[\mathcal{X}]$ is indeed the fundamental class of M;
- the intersection product of two (not necessarily transverse) submanifolds is equal to the fundamental class of their (derived) intersection (which is always a derived manifold); and
- the (not necessarily transverse) zero section of a vector bundle $E \to M$ is a derived submanifold of M whose fundamental class gives the Poincaré dual of the Euler class $e(E)$.

0.2 Notation

The symbol $[i] (i \in \mathbb{N})$ will have different meanings depending on context. As a category, $[i]$ is the linear order category with $i+1$ objects, and this is our main usage. For example $[0]$ is the terminal category with one object and one (identity) morphism, and $[1]$ is the category with two objects and a unique non-identity arrow connecting them. We also sometimes use the notation $[i]$ to denote a simplicial set with a similar flavor. Namely, we may use $[i] \in \mathbf{sSets}$ to denote the "subdivided interval" with $i+1$ vertices, i (directed) edges, and no non-degenerate higher simplices. It will be clear from context which usage we intend.

For a category $\mathcal{C}$, the category whose objects are the morphisms of $\mathcal{C}$ and whose morphisms are commutative squares in $\mathcal{C}$ can be expressed by $\mathcal{C}^{[1]}$. When that is inconvenient, we may write $\mathrm{Mor}(\mathcal{C})$ for this category.

We denote the terminal object in the category of sets and the terminal object in the category of topological spaces by $\{*\}$. We are careful to differentiate it from the terminal object in the category of simplicial sets, which we denote Δ^0, and the terminal object in the category of manifolds, which we denote $\bullet$. For example, the underlying topological space of $\bullet$ is $\{*\}$. We will use the letter t to denote maps to the terminal object in any category that has a terminal object $*$.

We often denote points in a space or elements of a set, not using the typical $\in$ symbol, but using a morphism from the terminal object. In other words, we might write "let $p : \{*\} \to M$ be a point in M" rather than "let $p \in M$ be a point".

If $\mathcal{C}$ and $\mathcal{D}$ are categories, there is a constant functor $c_* : \mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ which takes an object $X : [0] \to \mathcal{C}$ to the composite

$$\mathcal{D} \xrightarrow{t} [0] \xrightarrow{X} \mathcal{C},$$

where $t : \mathcal{D} \to [0]$ is the terminal map. We may instead choose to denote $c_* X$ by $\underline{X} : \mathcal{D} \to \mathcal{C}$.

The notation

$$L : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

will always denote an adjoint pair, in which L is the left adjoint and R is the right adjoint. That is, there are natural (but unspecified) isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(L(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, R(Y))$$

for every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. If $g : L(X) \to Y$ is a morphism, we may instead denote it by $g^\flat$ and its adjoint by $g^\sharp : X \to R(Y)$. We would reserve $g : X \to Y$ (which exists in the join category $\mathcal{C} * \mathcal{D}$) for the unspecified adjoint.

A *diagram* in a category $\mathcal{C}$ is a functor $X : I \to \mathcal{C}$, where I is a small category. We denote the category of functors from I to $\mathcal{C}$ and natural transformations between them by $\mathcal{C}^I$. If $F : \mathcal{C} \to \mathcal{D}$ is a functor, then it induces a functor $F^I : \mathcal{C}^I \to \mathcal{D}^I$, which we also denote simply by F. Note that if $I = [0]$ or $I = [1]$, this is already considered standard notation; we simply extend the notation to include all diagrams. We generally say that F induces F^I *object-wise*. However if I is a category of open sets, we will use the term *open-wise*; if I is the category of combinatorial simplices, we will use the term *level-wise*. These will help us when dealing with objects such as presheaves of simplicial functors.

We assume a lot of familiarity with limits and colimits in categories. For example, if $L : I \to \mathcal{C}$ is an I-shaped diagram in a category $\mathcal{C}$ which has a colimit and/or a limit in $\mathcal{C}$ and C is any object of $\mathcal{C}$, then

$$\mathrm{Hom}_{\mathcal{C}}(\mathrm{colim}\, L, C) \cong \lim(\mathrm{Hom}(L, C))$$

etc. We will state such facts without hesitation, as well as facts concerning the interaction between limits and adjoints and the interaction between finite limits and filtered colimits. Both [21] or [1] are good references.

Some authors call functors of the form $\mathrm{Hom}(M, -)$ "corepresentable," but we just call them representable.

We denote Cartesian and coCartesian squares by placing a corner symbol ($\ulcorner$, etc.) near the limit or colimit of the diagram, and we denote homotopy Cartesian and homotopy coCartesian squares by placing a diamond symbol ($\diamond$) near the homotopy limit or homotopy colimit of the diagram. We always denote limits by lim and colimits by colim (i.e. we do not resort to using under-arrows like $\varprojlim$ or $\varinjlim$).

If $\mathcal{M}$ is a simplicial model category, we will denote by $\mathcal{M}^\circ$ the full subcategory of cofibrant-fibrant objects. A (simplicial) functor $F : \mathcal{M} \to \mathcal{N}$ between simplicial model categories induces a functor $F^\circ : \mathcal{M}^\circ \to \mathcal{N}^\circ$ by using the functorial cofibrant-fibrant replacement which is part of the model structure of $\mathcal{N}$.

Our version of homotopy colimits and homotopy limits in simplicial model categories differs from that in [13]. His version does not give an invariant notion in the sense that two diagrams can be object-wise weakly equivalent and yet have non-weakly equivalent homotopy colimits or limits. However, if one first replaces all objects by cofibrant-fibrants, then the homotopy colimit or limit is independent of the choice. In this work, we take the homotopy colimit or limit of a diagram by first replacing it with a weakly equivalent diagram of cofibrant-fibrants, then performing Hirschhorn's homotopy colimit or limit construction. Finally, we apply the functorial cofibrant-fibrant replacement to this colimit or limit. In other words, homotopy colimits and homotopy limits are assumed cofibrant-fibrant. This convention improves the readability of many proofs.

If S is a simplicial set, we shall abuse notation and write $x \in S$ if and only if x is an element of the set $\amalg_{n\in\mathbb{N}} S_n$. If $\mathcal{C}$ is any category, then the notation $s\mathcal{C}$ will always mean the category $\mathcal{C}^{\Delta^{\mathrm{op}}}$ of simplicial objects in $\mathcal{C}$. If S is a simplicial set with non-degenerate simplices concentrated in degree 0, we will call S a *discrete* or *constant* simplicial set. Because there is a fully faithful left adjoint **Sets** $\to$ **sSets**, we sometimes disregard the difference between discrete simplicial sets and sets.

Functors between simplicial model categories are assumed to preserve the simplicial structure unless otherwise stated. Thus, if (F, G) are an adjoint pair of functors between simplicial model categories then it follows that

$$\mathrm{Map}(FX, Y) \cong \mathrm{Map}(X, GY).$$

We will usually denote manifolds by symbols M, N, P, etc., and denote more general topological objects by X, Y, Z, etc.

Let **Man** denote the category of manifolds. Note that it is equivalent to a small category, namely the category of submanifolds of $\mathbb{R}^\infty$.

We will never have occasion to work with non-commutative rings. We use the term "ring" to mean commutative ring.

Chapter 1

Motivation

1.1 Local models

The theory of schemes and the theory of manifolds are both based on the concept of local models. One would like to say that a scheme is "locally Spec(R)", but this does not really make sense without a discussion of rings, ringed spaces, local ringed spaces, and affine schemes. One would like to say that manifolds and schemes are somehow analogous, but it is difficult to find the *precise* analogy without a bit of thought. For example, what in the theory surrounding **Man** is analogous to a local-ringed space from algebraic geometry?

In both cases we proceed as follows. Begin with a category $\mathcal{R}$ of local models (such as Spec(R) or $\mathbb{R}^n$); define a notion of $\mathcal{R}$-rings (e.g. rings or C^∞-rings); define $\mathcal{R}$-ringed spaces and local $\mathcal{R}$-ringed spaces in such a way that objects in $\mathcal{R}$ can be given the structure of a local $\mathcal{R}$-ringed space; and finally define $\mathcal{R}$-schemes.

In this work, we will build up derived manifolds using local models in exactly the same way. The current chapter is not necessary for understanding the rest of the work; it is only here to motivate our definition of smooth rings, local smooth-ringed spaces, etc.

Let **Top** denote the Grothendieck site whose objects are topological spaces, whose morphisms are continuous maps, with the following topology: A set $\{f_i : V_i \to V\}$ is a covering if and only if each f_i is the inclusion of an open subset, and every point $x \in V$ is in the image of some f_i. If X is a topological space, we denote the category of open sets of X by $\mathrm{Op}(X)$.

Remark 1.1.1. In this chapter, we work out a theory of local models for a small Grothendieck site $\mathcal{R}$ that is equipped with underlying topological spaces. This extra structure is not at all necessary; however, without it, we would need to go through a lot of category-theoretic preliminaries. Many constructions in this chapter are neither as general nor as elegant as they could otherwise be, but they have the benefit of being relatively low-tech. Work in progress will generalize these results, obviate many of the somewhat arbitrary assumptions which follow, and give full proofs of many unproven statements in the following text.

Again, this dissertation does not in any way rely on the statements of this motivational chapter.

Definition 1.1.2. Let $\mathcal{R}$ be a small Grothendieck site and let $\mathbf{U} : \mathcal{R} \to \mathbf{Top}$ be a morphism of Grothendieck sites . We say that $\mathbf{U}$ is *a basis for its image* if, for every $R \in \mathcal{R}$ and every covering $\{T_i \to \mathbf{U}(R)\}_{i \in I}$ of $\mathbf{U}(R) \in \mathbf{Top}$, there exists a refinement of $\{T_i\}$ by open sets in the image of $\mathbf{U}$. That is, for each i there is a set of objects $R_i^j \in \mathcal{R}$ and a cover $\{\mathbf{U}(R_i^j) \to T_i\}$.

Suppose that $\mathcal{L}$ is a set of diagrams $L : I_L \to \mathcal{R}$. Let $\mathcal{S}$ be a full subcategory of $\mathcal{R}$. We say that $\mathcal{S}$ is *closed under taking $\mathcal{L}$-limits* if every diagram $L : I_L \to \mathcal{R}$ in $\mathcal{L}$ which factors through $\mathcal{S}$ has a limit in $\mathcal{S}$. We say that $\mathcal{S}$ is *closed under gluing* if, whenever $\{V_i \to V\}$ is a cover in $\mathcal{R}$ and each V_i is in $\mathcal{S}$, so is V. We say that a set of objects $G \in \mathcal{R}$ *generates* $\mathcal{S}$ if $\mathcal{S}$ is the smallest subcategory of $\mathcal{R}$ which contains G and is closed under taking $\mathcal{L}$-limits and gluing.

If $\mathcal{R}$ is a small Grothendieck site which is closed under taking $\mathcal{L}$-limits, $\mathbf{U} : \mathcal{R} \to \mathbf{Top}$ is a basis for its image, and there exists an object $\mathbb{A}$ which generates all of $\mathcal{R}$ under $\mathcal{L}$-limits and gluing, then we call the quadruple $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ a *category of local models* and we call $\mathbb{A}$ the *affine line*. We refer to the objects in $\mathcal{R}$ as *affines* and morphisms in $\mathcal{R}$ as *maps of affines*. We refer to a morphism $R_i \to R$ in $\mathcal{R}$ as *the inclusion of a principal open affine* if it is an element of some covering $\{R_j \to R\}_{j \in J}$.

Example 1.1.3. Let **Rings** denote the category of (commutative) rings, and let $\mathbf{Aff} = \mathbf{Rings}^{\mathrm{op}}$ denote the category of affine schemes. We tend not to differentiate between an affine scheme and its associated ring. **Aff** is a Grothendieck site under the Zariski topology, and there exists a functor $\mathbf{U} : \mathbf{Aff} \to \mathbf{Top}$ which takes an affine scheme to its underlying topological space. The principal affine schemes form a basis for the Zariski topology, so $\mathbf{U}$ is a basis for its image. All small limits exist in **Aff**, and we take $\mathcal{L}$ be the set of all (small) diagrams in **Aff**.

Let R be any ring. There exists sets I and J and a morphism between free $\mathbb{Z}$-algebras $r : \mathbb{Z}[x^I] \to \mathbb{Z}[x^J]$ such that R is the colimit of a diagram

$$\begin{tikzcd}
\mathbb{Z}[x^I] \arrow[r, "0"] \arrow[d, "r"'] & \mathbb{Z} \arrow[d] \\
\mathbb{Z}[x^J] \arrow[r, "g"'] \arrow[ur, phantom, "\lrcorner"] & R.
\end{tikzcd}$$

Here $\{g_j\}_{j \in J}$ are generators, and $\{r_i\}_{i \in I}$ are relations for R. Switching from **Rings** to **Aff**, it is clear that $\mathbb{Z}[x]$ generates **Aff**. We refer to the sequence $\overline{\mathbf{Aff}} = (\mathbf{Aff}, \mathbf{U}, \mathcal{L}, \mathbb{Z}[x])$ as the *algebraic category of local models* and to $\mathbb{Z}[x]$ as the affine line for $\overline{\mathbf{Aff}}$.

Example 1.1.4. Let $\mathcal{E}$ denote the category whose objects are finite dimensional Euclidean spaces $\mathbb{R}^n, n \geq 0$ with their usual smooth structure, and whose morphisms are smooth (C^∞) maps. Each $\mathbb{R}^n$ has an underlying topological space

$\mathbf{U}(\mathbb{R}^n) := \mathbb{R}^n$, and any open covering of $\mathbb{R}^n$ can be refined to an open covering by Euclidean spaces. Thus, $\mathbf{U}$ is a basis for its image. Let $\mathcal{L}$ denote the set of product diagrams, i.e. functors $L : I \to \mathcal{E}$ for which I is a (possibly empty) finite set. Clearly, $\mathbb{R}$ generates $\mathcal{E}$ under $\mathcal{L}$-limits. We refer to the sequence $\overline{\mathcal{E}} = (\mathcal{E}, \mathbf{U}, \mathcal{L}, \mathbb{R})$ as the *Euclidean category of local models* and to $\mathbb{R}$ as the affine line for $\overline{\mathcal{E}}$.

Remark 1.1.5. In Definition 1.1.2, we do not really need the assumption that there exists an affine line $\mathbb{A}$; we could just look at $\mathcal{L}$-limit preserving functors $F : \mathcal{R} \to \mathbf{Sets}$. The existence of $\mathbb{A}$ guarantees that this theory is comprehensible: a functor F is determined, up to limits of sets, by one piece of data, the set $F(\mathbb{A})$.

Definition 1.1.6. Let $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ be a category of local models. An *$\overline{\mathcal{R}}$-ring* is a covariant functor $F : \mathcal{R} \to \mathbf{Sets}$ that preserves limits for diagrams $L : I \to \mathcal{R}$ in $\mathcal{L}$. A morphism of $\overline{\mathcal{R}}$-rings is a natural transformation of functors.

If F is an $\overline{\mathcal{R}}$-ring, we refer to $F(\mathbb{A})$ as *the underlying set of* F and sometimes denote it by $|F|$.

Example 1.1.7. Let $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ be a category of local models. For any affine $V \in \mathcal{R}$, the representable functor

$$H_V := \mathrm{Hom}_{\mathcal{R}}(V, -) : \mathcal{R} \to \mathbf{Sets}$$

is an $\overline{\mathcal{R}}$-ring, because it preserves all limits in $\mathcal{R}$ (including those in $\mathcal{L}$).

Example 1.1.8. Let $\overline{\mathcal{E}} = (\mathcal{E}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ denote the Euclidean category of local models. An $\overline{\mathcal{E}}$-ring X is also known as a C^∞-ring (see [17] or [22]). One thinks about the underlying set $|X|$ of X as "a set which has interpretations of all smooth functions," as we now explain.

Consider the function $+ : \mathbb{R}^2 \to \mathbb{R}, (a, b) \mapsto a + b$. Since X is a product-preserving functor, one applies X to obtain

$$+_X : |X| \times |X| \to |X|,$$

which is in some sense an "interpretation" of $+$ in X. Similarly, X has interpretations $\times_X, -_X, 0_X, 1_X$ of multiplication, subtraction, nullity, and unity. Since the distributivity diagram

$$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "{(a,b+c)}"] \arrow[d, "{(ab,ac)}"'] & \mathbb{R}^2 \arrow[d, "a(b+c)"] \\
\mathbb{R}^2 \arrow[r, "ab+ac"'] & \mathbb{R}
\end{tikzcd}$$

commutes in $\mathcal{E}$, the corresponding diagram commutes after applying the functor X; thus $|X|$ satisfies the distributive law. Likewise, $|X|$ satisfies all the axioms that define a commutative ring.

However $|X|$ is called a C^∞-ring because it is more than just a ring: it has interpretations of all smooth functions. In other words, if $f : \mathbb{R}^n \to \mathbb{R}$ is any smooth map, then there is a map of sets $f_X : |X|^n \to |X|$, which we are calling the "interpretation of f," and which satisfies all appropriate commutative diagrams. For example, whatever functions $\cos_X, \sin_X : |X| \to |X|$ may be, we know that $\cos_X^2 + \sin_X^2 = 1_X$.

Example 1.1.9. If $\overline{\mathbf{Aff}} = (\mathbf{Aff}, \mathbf{U}, \mathcal{L}, \mathbb{Z}[x])$ is the algebraic category of local models, we refer to $\overline{\mathbf{Aff}}$-rings as *algebraic rings* and denote the category of algebraic rings by $\mathbf{AR}$. We will now explore this category.

Proposition 1.1.10. *The underlying set functor* $|\cdot| : \mathbf{AR} \to \mathbf{Sets}$ *factors through* $\mathbf{Rings}$.

Proof. Let $F \in \mathbf{AR}$ be an algebraic ring, which for the purposes of this proof, we shall consider as a contravariant functor from **Rings** to **Sets** which takes tensor products (colimits) of rings to limits of sets. In particular, F takes tensor products over $\mathbb{Z}$ to products of sets. For example, $F(\mathbb{Z}[x_1, \ldots, x_n]) \cong |F|^n$, and $F(\mathbb{Z}) = \{*\}$. The proposition then follows from the fact that $\mathbb{Z}[x]$ is a ring object in the category of rings. This may be a bit too slick, so we give a more detailed argument below.

Let $f_0, f_1 : \mathbb{Z}[x] \to \mathbb{Z}$ be the maps that send x to the integers zero and one, respectively. We denote by 0 and 1 the corresponding elements $F(f_0), F(f_1) : \{*\} \to |F|$. Let

$$f_+, f_-, f_\times : \mathbb{Z}[z] \to \mathbb{Z}[x, y] = \mathbb{Z}[x] \otimes_\mathbb{Z} \mathbb{Z}[y]$$

be the maps that send z to $x + y$, to $y - z$, and to yz, respectively. We denote by +,-, and $\times$ the corresponding maps

$$F(f_+), F(f_-), F(f_\times) : |F|^2 \to |F|.$$

We claim that with these definitions, the set $|F|$ admits the structure of a commutative ring.

To check this, one must show that the associative property, the commutative property, the distributive property, etc. hold for the operations $+, \times, -, 0, 1$ on $|F|$. Each of these properties can be checked via the commutativity of a corresponding diagram. For example + is an associative operation on $|F|$ if and only if the obvious diagram

$$\begin{tikzcd}
|F|^3 \arrow[r] \arrow[d] & |F|^2 \arrow[d] \\
|F|^2 \arrow[r] & |F|
\end{tikzcd}$$

commutes. That it does so follows from the fact that the diagram of rings

$$\begin{tikzcd}
\mathbb{Z}[x_1,x_2,x_3] & \mathbb{Z}[x_{12},x_3] \arrow[l, "(x_1+x_2,x_3)"'] \\
\mathbb{Z}[x_1,x_{23}] \arrow[u, "(x_1,x_2+x_3)"] & \mathbb{Z}[x_{123}] \arrow[l, "x_1+x_{23}"] \arrow[u, "x_{12}+x_3"']
\end{tikzcd}$$

commutes. Likewise, we obtain diagrams which validate the other ring axioms by applying F to corresponding diagrams of free $\mathbb{Z}$-algebras.

□

Remark 1.1.11. Proposition 1.1.10 showed that for an algebraic ring F, the underlying set $|F|$ has the structure of a ring, which we call *the underlying ring of* F. We shall abuse notation and write $|F|$ to denote that ring, as opposed to just its underlying set.

The following proposition says that the category of rings is *equivalent* to the category of algebraic rings, hence justifying the name "algebraic ring". Note that in Example 1.1.9, we defined an algebraic ring as a limit-preserving covariant functor from **Aff** to **Sets**. For the purposes of convenience in the next proof, we consider an algebraic ring as a contravariant functor from **Rings** to **Sets** which sends colimits to limits.

Proposition 1.1.12. *Let* $H : \mathbf{Rings} \to \mathbf{AR}$ *denote the functor which sends a ring* R *to the algebraic ring*

$$H_R : S \mapsto \mathrm{Hom}_{\mathbf{Rings}}(S, R).$$

Then H *and* $|\cdot|$ *are inverse equivalences of categories between* **Rings** *and* **AR**.

Proof. Note that if R is a ring then H_R is an algebraic ring because $H_R = \mathrm{Hom}_{\mathbf{Rings}}(-, R)$ takes colimits of rings to limits of sets.

It is clear that for any ring R,

$$|H_R| = \mathrm{Hom}_{\mathbf{Rings}}(\mathbb{Z}[x], R) \cong R$$

(see Remark 1.1.11). It suffices to show that for any algebraic ring $F : \mathbf{Rings}^{\mathrm{op}} \to \mathbf{Sets}$, we have $H_{|F|} \cong F$. Let S be a ring, and write it as a colimit

$$\begin{tikzcd}
\mathbb{Z}[x^I] \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\
\mathbb{Z}[x^J] \arrow[r] \arrow[ur, phantom, "\lrcorner"] & S.
\end{tikzcd}$$

Note that since F takes colimits to limits, $F(\mathbb{Z}[x^I]) = |F|^I$ and $F(\mathbb{Z}) = \{*\}$. Both $F(S)$ and $H_{|F|}(S) = \mathrm{Hom}_{\mathbf{Rings}}(S, |F|)$ are naturally isomorphic to the limit of the diagram

$$\begin{tikzcd} & \{*\} \arrow[d] \\ |F|^J \arrow[r] & |F|^I. \end{tikzcd}$$

Therefore they are naturally isomorphic to each other.

$\square$

Let $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ be a category of local models, and let $\mathcal{C}$ denote the category of $\overline{\mathcal{R}}$-rings. We will now discuss presheaves and sheaves of $\overline{\mathcal{R}}$-rings on a topological space X. This may be confusing at first, because an $\overline{\mathcal{R}}$-ring is a covariant functor, so a presheaf of $\overline{\mathcal{R}}$-rings is a functor $\mathcal{F} : \mathrm{Op}(X)^{\mathrm{op}} \to \mathcal{C}$, or more explicitly, a bifunctor

$$\mathcal{F} : \mathrm{Op}(X)^{\mathrm{op}} \times \mathcal{R} \to \mathbf{Sets};$$

i.e. we put the contravariant variable first and the covariant variable second. In particular, note that for any open set $U \subset X$ the covariant functor $\mathcal{F}(U, -)$ is an $\overline{\mathcal{R}}$-ring, and for any affine $V \in \mathcal{R}$, the contravariant functor $\mathcal{F}(-, V)$ is a sheaf of sets on X.

If $X : I \to \mathcal{C}$ is a diagram of $\overline{\mathcal{R}}$-rings then its limit exists as a functor $(\lim X) : \mathcal{R} \to \mathbf{Sets}$; it is taken object-wise on $\mathcal{R}$. Moreover, $\lim X$ is again an $\overline{\mathcal{R}}$-ring because the limit of functors which preserve $\mathcal{L}$-limits preserves $\mathcal{L}$-limits.

In other words, $\mathcal{C}$ is a complete category. In particular, it has equalizers. If X is a topological space and $\mathcal{F}$ is a presheaf on X with values in $\mathcal{C}$, we say that $\mathcal{F}$ is a *sheaf* if the diagram of $\overline{\mathcal{R}}$-rings

$$\mathcal{F}(U, -) \to \prod_i \mathcal{F}(U_i, -) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \cap U_j, -)$$

is an equalizer for any open covering $\{U_i \to U\}_{i \in I}$.

Definition 1.1.13. Let $\overline{\mathcal{R}}$ be a category of local models. An *$\overline{\mathcal{R}}$-ringed space* is a pair $\mathcal{X} = (X, \mathcal{O}_X)$ for which X is a topological space and $\mathcal{O}_X$ is a sheaf of $\overline{\mathcal{R}}$-rings on X. We refer to $\mathcal{O}_X$ as *the structure sheaf of $\mathcal{X}$*.

A morphism of $\overline{\mathcal{R}}$-ringed spaces is a pair $(f, f^\flat)$ such that $f : X \to Y$ is a map of topological spaces and $f^\flat : f^*\mathcal{O}_Y \to \mathcal{O}_X$ is a morphism of sheaves on X.

Let $(X, \mathcal{O}_X)$ be an $\overline{\mathcal{R}}$-ringed space. If $U \subset X$ is an open set and $R \in \mathcal{R}$ is an affine, it will be useful for the reader to picture $\mathcal{O}_X(U, R)$ somehow as the set of maps $U \to R$. It is contravariant in U and covariant in R, and recall that R has an underlying topological space, so we already see that this conception has some merits. We will see later (Equation 1.1 at the end of this section) that it is quite justified in most cases of interest.

Recall that a morphism $\mathcal{A} \to \mathcal{B}$ of sheaves of sets on a topological space X is called a surjection if, for every point $x \in X$ there exists a neighborhood $U \ni x$ of x such that $\mathcal{A}(U) \to \mathcal{B}(U)$ is a surjection.

Definition 1.1.14. Let $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ be a category of local models, let X be a topological space, and let $\mathcal{F}$ be a sheaf of $\overline{\mathcal{R}}$-rings on X. We say that $\mathcal{F}$ is *a sheaf of local* $\overline{\mathcal{R}}$*-rings* on X if it satisfies the following condition.

- Suppose that $V \in \mathcal{R}$ is an affine and that $\{V_i \to V\}_{i \in I}$ is a covering. Then the sheafification of the morphism

$$\coprod_{i \in I} \mathcal{F}(-, V_i) \to \mathcal{F}(-, V)$$

  is a surjection of sheaves of sets.

In this case we refer to the pair $(X, \mathcal{F})$ as a *local* $\overline{\mathcal{R}}$*-ringed space.*

A morphism of sheaves $g : \mathcal{F} \to \mathcal{G}$ on X is called *an* $\overline{\mathcal{R}}$*-local morphism* if it satisfies the following condition.

- If $V_i \to V \in \mathrm{Mor}(\mathcal{R})$ is the inclusion of a principal open affine, then the diagram

$$\begin{tikzcd} \mathcal{F}(-, V_i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{G}(-, V_i) \arrow[d] \\ \mathcal{F}(-, V) \arrow[r] & \mathcal{G}(-, V) \end{tikzcd}$$

  is a fiber product of sheaves of sets.

A morphism of $\overline{\mathcal{R}}$-ringed spaces $(f, f^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called *a morphism of local* $\overline{\mathcal{R}}$*-ringed spaces* if $f^\flat$ is an $\mathcal{R}$-local morphism.

Let $\mathbf{LRS}_{\overline{\mathcal{R}}}$ denote the category of local $\overline{\mathcal{R}}$-ringed spaces and local morphisms.

In general, if U and R are objects in some category $\mathcal{T}$ of topological spaces with extra structure, then for every element $f \in \mathrm{Hom}_{\mathcal{T}}(U, R)$ and every covering $\{R_i \to R\}$, the preimages $f^{-1}(R_i)$ form a cover of U. If instead, $U \in \mathrm{Op}(X)$ and $R \in \mathcal{R}$, then replacing $\mathrm{Hom}_{\mathcal{T}}$ with $\mathcal{O}_X$, one recovers the locality condition. Thus the locality condition simply states that pullbacks of covers are covers. This is made more explicit in section 3.1.

The point is that the locality condition on $\mathcal{O}_X$ enforces a strong connection between the sheaf $\mathcal{O}_X$ and the topological structure of X.

For the next two propositions, one should keep in mind that if $\overline{\mathbf{Aff}}$ is the algebraic category of local models, then by Proposition 1.1.12, an $\overline{\mathbf{Aff}}$-ring can be thought of as a ring (and vice versa). Recall that a ringed space is called a local ringed space (in the usual sense) if all of its stalks are local rings (i.e. each has a unique maximal ideal). Recall also that a morphism of local ringed spaces is called a local morphism (in the usual sense) if, for each point, the maximal ideal "pulls back" along the morphism to the maximal ideal. See [11] for details. These two ideas are algebraic in nature, whereas our definition of $\overline{\mathbf{Aff}}$-local rings and morphisms is geometric in nature. We show below that the our definitions agree with the usual ones.

Proposition 1.1.15. *Let X be a topological space and let $\mathcal{F}$ be a sheaf of $\overline{\mathbf{Aff}}$-rings on X. Then $\mathcal{F}$ is a sheaf of local rings (in the usual sense) if and only if it is a sheaf of $\overline{\mathbf{Aff}}$-local rings.*

Proof. Let $R \in \mathbf{Aff}$ be an affine scheme and let $\{\alpha_i : R_i \to R\}_{i \in i}$ be a covering by principal open affines. As a ring, each R_i is a localization of R. The morphism

$$a : \coprod_{i \in I} \mathcal{F}(-, R_i) \to \mathcal{F}(-, R)$$

is a surjection of sheaves of sets if and only if for every point $p : \{*\} \to X$, the morphism $p^*(a)$ on stalks is surjective. Pick a point p. By Proposition 1.1.12, we may consider the stalk of $\mathcal{F}$ at p as a ring F, and rephrase the locality condition by saying that $\mathcal{F}$ is $\overline{\mathbf{Aff}}$-local if and only if

$$\coprod_i \mathrm{Hom}_{\mathbf{Rings}}(R_i, F) \to \mathrm{Hom}_{\mathbf{Rings}}(R, F)$$

is a surjection for any open covering by localizations of rings $\{R \to R_i\}$. It is a basic fact of commutative ring theory that this condition is equivalent to F being a local ring.

□

Proposition 1.1.16. *Let X be a topological space, let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\overline{\mathbf{Aff}}$-local rings on X, and let $f : \mathcal{F} \to \mathcal{G}$ be a morphism between them. Then f is a local morphism (in the usual sense) if and only if it is an $\overline{\mathbf{Aff}}$-local morphism of sheaves.*

Proof. Note that since taking stalks is performed by taking a filtered colimit, it commutes with fiber products of sheaves. In other words, it suffices to show that for any point $p : \{*\} \to X$, f is local morphism if and only if it is $\overline{\mathbf{Aff}}$-local. So let F and G be the stalks at p of $\mathcal{F}$ and $\mathcal{G}$; both are local rings. We need to show that f is local (i.e. that f^{-1} takes the maximal ideal of G to the maximal ideal of F) if and only if the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Rings}}(R[x^{-1}], F) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{Rings}}(R[x^{-1}], G) \arrow[d] \\
\mathrm{Hom}_{\mathbf{Rings}}(R, F) \arrow[r] & \mathrm{Hom}_{\mathbf{Rings}}(R, G)
\end{tikzcd}$$

is a fiber product of sets for every element $x \in R$.

Giving a morphism of rings $a : R[x^{-1}] \to G$ is equivalent to giving a morphism $a : R \to G$ such that x is sent to a unit. Suppose that a is an arbitrary such morphism and that $b : R \to F$ is an arbitrary morphism such that $f \circ r = a$.

Then f is local if and only if $b(x)$ is a unit, which is true if and only if b factors through a map $R[x^{-1}] \to F$. This proves the result.

$\square$

The previous propositions show that our definition of local $\overline{\mathcal{R}}$-ringed spaces generalizes the usual definition of local ringed spaces. We now move on to discuss affine $\overline{\mathcal{R}}$-schemes and general $\overline{\mathcal{R}}$-schemes, which will again generalize the usual definitions.

Example 1.1.17. Let $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ be a category of local models and let $R \in \mathcal{R}$ be an affine. We can define a sheaf $\mathcal{H}_R$ of $\overline{\mathcal{R}}$-rings on the topological space $\mathbf{U}(R)$ by specifying it on a basis of $\mathbf{U}(R)$. Since $\mathbf{U}$ is a basis for its image, we must only specify, for each principal open $R_i \to R$, an $\overline{\mathcal{R}}$-ring $\mathcal{H}_R(R_i)$. Take

$$\mathcal{H}_R(R_i) := H_{R_i}$$

(see Example 1.1.7).

In the case of the Euclidean category of local models, if $\mathbb{R}^n$ is a Euclidean space then $\mathcal{H}_{\mathbb{R}^n}$ is the sheaf of smooth functions on $\mathbb{R}^n$. In the case of the algebraic category of local models, if R is a ring then $\mathcal{H}_R$ is the structure sheaf on $\mathrm{Spec}(R)$ (see Proposition 1.1.12).

Proposition 1.1.18. *Let $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ be a category of local models, and let $R \in \mathcal{R}$ be an affine. Let $\mathcal{H}_R$ be the sheaf on $\mathbf{U}(R)$ that was specified in Example 1.1.17. Then $\mathcal{H}_R$ is a sheaf of local $\overline{\mathcal{R}}$-rings. Thus $(\mathbf{U}(R), \mathcal{H}_R)$ is a local $\overline{\mathcal{R}}$-ringed space.*

The functor $a : R \mapsto (\mathbf{U}(R), \mathcal{H}_R)$ from $\mathcal{R}$ to the category $\mathbf{LRS}_{\overline{\mathcal{R}}}$ of local $\overline{\mathcal{R}}$-ringed spaces is a fully faithful inclusion of categories.

Proof. That $\mathcal{H}_R$ satisfies the locality condition is implied by the definition of $\mathcal{R}$ being a Grothendieck site (because the preimage of a covering is a covering).

If $f : R \to S$ is a map of affines, we must show that $f^\flat : f^*\mathcal{H}_S \to \mathcal{H}_R$ is local; i.e. that the diagram

$$\begin{tikzcd}
f^*\mathcal{H}_S(-, M_\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathcal{H}_R(-, M_\alpha) \arrow[d] \\
f^*\mathcal{H}_S(-, M) \arrow[r] & \mathcal{H}_R(-, M)
\end{tikzcd}$$

is a fiber product of sheaves of sets for any open $M_\alpha \subset M$. Let $p \in R$ be an arbitrary point, and let $q = f(p)$. Taking the stalk at p, this reduces to the following simple claim. If U is an open neighborhood of $q \in S$ and $g : U \to M$ is a morphism such that $gf : f^{-1}(U) \to M$ factors through M_α, then there exists an open $V \subset U$ containing p, and a map $g' : V \to M_\alpha$ such that g' is the restriction of g.

Finally, we must show that a is fully faithful. Let R, S be objects in $\mathcal{R}$. The functor a sends a morphism $f : R \to S$ in $\mathcal{R}$ to the morphism $(f, f^\flat)$:

$(R, \mathcal{H}_R) \to (S, \mathcal{H}_S)$, where $f^\flat : f^*\mathcal{H}_S \to \mathcal{H}_R$ is a local morphism. The fact that a is fully faithful then follows from Yoneda's lemma.

□

Definition 1.1.19. Let $\overline{\mathcal{R}}$ be a category of local models. A local $\overline{\mathcal{R}}$-ringed space $\mathcal{X}$ is called an *affine $\overline{\mathcal{R}}$-scheme* if it is of the form $(\mathbf{U}(R), \mathcal{H}_R)$ for some affine $R \in \mathcal{R}$.

A local $\overline{\mathcal{R}}$-ringed space $\mathcal{Y}$ is called an *$\overline{\mathcal{R}}$-scheme* if it has an open covering by affine $\overline{\mathcal{R}}$-schemes.

Example 1.1.20. Let $\overline{\mathcal{E}}$ denote the Euclidean category of local models and let $\overline{\mathbf{Aff}}$ denote the algebraic category of local models. A Euclidean scheme (i.e. an $\overline{\mathcal{E}}$-scheme) is usually referred to as a smooth manifold; an algebraic scheme (i.e. an $\overline{\mathbf{Aff}}$-scheme) is usually referred to as a scheme. One may also realize smooth schemes (Example 1.2.3), manifolds with boundary (Example 1.2.4), etc. in this way.

1.2 How derived manifolds fit in.

At this point, we hope the reader is convinced that the concept of $\overline{\mathcal{R}}$-rings is useful, as it generalizes the above four classical notions. We shall use a variant of it in what follows when we introduce derived manifolds. The biggest difference is that we need to work homotopically, so we replace **Sets** with **sSets**, limits with homotopy limits, sheaves with homotopy sheaves, etc. This is done so as to retain deformation-theoretic information, in the form of higher homotopy, when intersecting non-transverse submanifolds.

There is another difference as well, which could be confusing if not mentioned. We begin with a lemma.

Lemma 1.2.1. *Let $\overline{\mathcal{R}}$ be a category of local models. The category of $\overline{\mathcal{R}}$-ringed spaces is complete.*

Proof. Let $\mathcal{C}$ denote the category of $\overline{\mathcal{R}}$-ringed spaces. If $X : I \to \mathcal{C}$ is a diagram, then its limit is computed by taking the limit of the underlying diagram of topological spaces and the colimit of the underlying diagram of sheaves. The colimit of a diagram of sheaves is the sheafification of the colimit of the corresponding diagram of presheaves. The colimit of a diagram of presheaves is taken open-wise.

Thus, we are reduced to proving that colimits exist in the category of $\overline{\mathcal{R}}$-rings, which is proven in Corollary A.2.8.

□

Suppose that $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ is a category of local models, and let $\mathcal{C}$ denote the category of $\overline{\mathcal{R}}$-ringed spaces; recall the imbedding $\mathcal{R} \to \mathcal{C}$ from Proposition 1.1.18. Let $\mathcal{L}'$ be a set of diagrams in $\mathcal{R}$. The limits of these diagrams exist in $\mathcal{C}$ by Lemma 1.2.1. Let $\mathcal{R}' = \mathcal{R}[\lim \mathcal{L}']$ denote the full subcategory of $\mathcal{C}$ consisting of the objects in $\mathcal{R}$ and the limits of diagrams in $\mathcal{L}'$. It is still a small Grothendieck site in a natural way. Furthermore, $\mathbf{U}$ can be extended to a morphism of sites $\mathcal{R}' \to \mathbf{Top}$ (taking the new-formed limits to limits of topological spaces), and the affine line $\mathbb{A}$ generates $\mathcal{R}'$ under $\mathcal{L} \cup \mathcal{L}'$ and gluing. Therefore,

$$\overline{\mathcal{R}'} = \overline{\mathcal{R}}[\lim \mathcal{L}'] := (\mathcal{R}', \mathbf{U}, \mathcal{L} \cup \mathcal{L}', \mathbb{A})$$

is a category of local models.

Definition 1.2.2. Let $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ be a category of local models and let $\mathcal{L}'$ be a set of diagrams in $\mathcal{R}$. We refer to the category of local models $\overline{\mathcal{R}}[\lim \mathcal{L}']$ as *the $\mathcal{L}'$ extension of* $\overline{\mathcal{R}}$.

Our situation for derived manifolds is an example of this idea. We use the category of manifolds as our category $\mathcal{R}$ of local models. There is an obvious functor $\mathbf{U} : \mathbf{Man} \to \mathbf{Top}$ which is a basis for its image. We take $\mathcal{L}$ to be the category of "submersion pullbacks"; that is diagrams of the form

$$\begin{tikzcd} & M \arrow[d, "s"] \\ N \arrow[r, "a"] & P, \end{tikzcd}$$

where M, N and P are manifolds, s is a submersion, and a is any smooth map. Any manifold is locally $\mathbb{R}^n$, so $\mathbb{R}$ generates $\mathbf{Man}$ under $\mathcal{L}$-limits and gluing. Therefore $\overline{\mathbf{Man}} = (\mathbf{Man}, \mathbf{U}, \mathcal{L}, \mathbb{R})$ is a category of local models. We will refer to $\overline{\mathbf{Man}}$-rings as *smooth rings*.

However, our category of "derived manifolds" is not the category of $\overline{\mathbf{Man}}$-schemes, which is actually equivalent to $\mathbf{Man}$ itself, or even its homotopical counterpart. We want our affines to include *all* finite limits of manifolds, not just those which already make sense in the category of manifolds. To say it another way, we want to extend $\mathbf{Man}$ to include arbitrary fiber products of manifolds.

Let $\mathcal{L}'$ be the set of all diagrams of the form

$$\begin{tikzcd} & M \arrow[d] \\ N \arrow[r] & P \end{tikzcd}$$

in the category of manifolds and let $\overline{\mathbf{Man}'} = \overline{\mathbf{Man}}[\lim \mathcal{L}']$ be the $\mathcal{L}'$ extension of $\overline{\mathbf{Man}}$. The notion of $\overline{\mathbf{Man}}$-rings and the notion of local $\overline{\mathbf{Man}}$-ringed spaces is the same as the corresponding notions for $\overline{\mathbf{Man}'}$. But the notion of $\overline{\mathbf{Man}'}$-schemes is different than that of $\overline{\mathbf{Man}}$-schemes; an $\overline{\mathbf{Man}}$-scheme is just a manifold but an $\overline{\mathbf{Man}'}$-scheme is much more general. See Remark 4.1.5. We may once again look to algebraic geometry for an analogous example.

Example 1.2.3. We will work over $\mathbb{Z}$, but any other base scheme would work equally well. Let $\mathcal{S}$ be the category of smooth schemes in the usual sense of algebraic geometry. Let $U : \mathcal{S} \to \mathbf{Top}$ be the underlying space functor, let $\mathbb{A} = \mathbb{Z}[x]$ be the affine line. We take $\mathcal{L}$ to be the set of diagrams of the form

$$\begin{tikzcd} & X \arrow[d, "s"] \\ Y \arrow[r, "a"] & Z, \end{tikzcd}$$

in which $s : X \to Z$ is a smooth morphism and $a : Y \to Z$ is any morphism. One can easily show that $\mathbb{A}$ generates $\mathcal{S}$ under limits in $\mathcal{L}$. Thus $\overline{\mathcal{S}} = (\mathcal{S}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ is a category of local models, which one might call the smooth algebraic category of local models. One can show that the notion of $\mathcal{S}$-rings is the same as the notion of **Aff**-rings (a.k.a algebraic rings, a.k.a. rings), and the category of $\mathcal{S}$-ringed spaces is the same as the ordinary category of ringed spaces. However, the category of smooth algebraic schemes differs from the category of algebraic schemes in exactly the way one would think: the former consists of smooth schemes over $\mathbb{Z}$ (in the usual sense), and the latter consists of all schemes over $\mathbb{Z}$.

Nevertheless, one could start with the smooth algebraic category of local models $\mathcal{S}$ and obtain from it the category of all schemes, by the process of extending affines. That is, if we let $\mathcal{L}'$ denote the set of all small diagrams of rings and let $\overline{\mathcal{S}'} = \overline{\mathcal{S}}[\lim \mathcal{L}']$, then $\overline{\mathcal{S}'}$-schemes are just the same as (algebraic) schemes.

Example 1.2.4. Let $\overline{\mathcal{E}}$ denote the Euclidean category of local models. Let $\beta : \mathbb{R} \to \mathbb{R}$ denote a (smooth) bump function, such that $\beta(x)$ is positive for $x < 0$ and 0 for $x \geq 0$. Let $\mathcal{L}'$ denote the single diagram

$$\mathbb{R} \xrightarrow{\beta} \mathbb{R} \xleftarrow{0} \bullet.$$

Then one can show that the category of $\overline{\mathcal{E}}[\lim \mathcal{L}']$-schemes is the category of manifolds with boundary.

Let us mention one more fact, which was hinted at above. First we need a definition.

Definition 1.2.5. Let $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ be a category of local models. We will say that the set $\mathcal{L}$ *has enough monomorphisms* if there exists a subcategory $i : \mathcal{M} \in \mathcal{R}$ such that

- for every morphism $f : A \to B$ in $\mathcal{M}$, $i(f)$ is a monomorphism in $\mathcal{R}$,
- the restriction of $\mathbf{U}$ to $\mathcal{M}$ is a basis for $\mathbf{U}(\mathcal{R})$, and
- for each $f : A \to B$ in $\mathcal{F}$, the diagram $A \xrightarrow{f} B \xleftarrow{f} A$ is in $\mathcal{L}$.

Note that a morphism $f : A \to B$ is a monomorphism in a category if and only if A is the limit of $A \xrightarrow{f} B \xleftarrow{f} A$. If $F : \mathcal{R} \to \mathbf{Sets}$ is a $\mathcal{L}$-limit preserving functor, then it preserves all monomorphisms in $\mathcal{L}$.

Example 1.2.6. Note that the algebraic category of local models, $\overline{\mathbf{Aff}}$, has enough monomorphisms, and so does the smooth category of local models, $\overline{Man}$. However the Euclidean category $\overline{\mathcal{E}}$ does not.

Let $\overline{\mathcal{R}} = (\mathcal{R}, \mathbf{U}, \mathcal{L}, \mathbb{A})$ be a category of local models such that $\mathcal{L}$ has enough monomorphisms, and suppose that $\mathbf{U}(R)$ is a T_0 space for all $R \in \mathcal{R}$. If $\mathcal{X} = (X, \mathcal{O}_X)$ is a local $\overline{\mathcal{R}}$-ringed space, then its structure sheaf "knows" functions from $\mathcal{X}$ to affine $\overline{\mathcal{R}}$-schemes. That is, suppose R is an affine, and let R also denote the corresponding affine $\overline{\mathcal{R}}$-scheme $(\mathbf{U}(R), \mathcal{H}_R)$. If we let $\mathrm{Hom}(\mathcal{X}, R)$ denote the set of morphisms of local ringed spaces from $\mathcal{X}$ to R, then there is a natural isomorphism

$$\mathcal{O}_X(X, R) \cong \mathrm{Hom}(\mathcal{X}, R). \tag{1.1}$$

Of course a similar isomorphism holds over any open subset $U \subset X$. (We will not prove this result here, though it is not hard. See Theorem 3.3.3 for the analogous result in the setting of derived manifolds.)

So simply being an $\overline{\mathcal{R}}$-local sheaf on X forces $\mathcal{O}_X$ to store a lot of mapping data. For instance, in algebraic geometry if $(X, \mathcal{O}_X)$ is a local-ringed space (or scheme), and $\mathrm{Spec}(R)$ is an affine scheme, then it is an exercise in [11] to show that

$$\mathrm{Hom}_{\mathbf{Rings}}(R, \mathcal{O}_X(X)) \cong \mathrm{Hom}(X, \mathrm{Spec}(R)).$$

Equation 1.1 generalizes this fact about the local ringed spaces to any category of local $\overline{\mathcal{R}}$-ringed spaces.

Chapter 2

Smooth Rings

2.1 Smooth Rings

In Chapters 2, 3, and 4, we will follow the basic outline prescribed in the motivation Chapter 1. That is, we will introduce a category of local models (the category of smooth manifolds), and we will proceed to define smooth rings, smooth-ringed spaces, local smooth-ringed spaces, and finally derived manifolds. As mentioned in the previous chapter, the biggest difference is that in order for our fundamental theorem to hold, we need to work homotopically. Thus, we replace sets with simplicial sets, colimits with homotopy colimits, etc.

Let **Man** denote the category of smooth manifolds without boundary and smooth maps between them. This category is equivalent to a small category, namely the category of smooth manifolds with an imbedding into $\mathbb{R}^\infty$, and maps between them *that do not respect this imbedding.* Because **Man** is equivalent to a small category, constructions such as functor categories (e.g. $\mathbf{sSets}^{\mathbf{Man}}$) are well defined.

The category $\mathbf{sSets}^{\mathbf{Man}}$ of covariant functors from manifolds to simplicial sets has a projective and an injective model structure, and the two are Quillen equivalent [20, A.3.3.2]. The injective model structure is given by object-wise weak equivalences and object-wise cofibrations. The projective model structure is given by object-wise weak equivalences and object-wise fibrations. Both model structures are proper and cofibrantly generated ([20, A.3.3.3, A.3.3.5]). For any simplicial set S, there is an associated constant functor $\underline{S} : \mathbf{Man} \to \mathbf{sSets}$; this makes $\mathbf{sSets}^{\mathbf{Man}}$ tensored over **sSets**. Both model structures are simplicial, with $(F \otimes \Delta^i)(M) = F(M) \otimes \Delta^i$ and $\mathrm{Map}(F, G)_i = \mathrm{Hom}(F \otimes \Delta^i, G)$.

We will not have occasion to use the projective structure in this work.

Definition 2.1.1. We refer to the model category of functors $\mathbf{Man} \to \mathbf{sSets}$ with the injective model structure as *the model category of pre-rings* and denote it **PR**.

A pre-ring $F : \mathbf{Man} \to \mathbf{sSets}$ is called *discrete* if for every manifold M, the simplicial set $F(M)$ is discrete (i.e. its only non-degenerate simplices have degree 0).

If M is a manifold, let H_M denote the discrete pre-ring given by the formula

$$H_M(N) = \mathrm{Hom}_{\mathbf{Man}}(M, N).$$

The terminal manifold, $\mathbb{R}^0$, will be denoted $\bullet$.

Note that if M is a manifold and F is a pre-ring, then

$$\mathrm{Map}(H_M, F) = F(M).$$

Lemma 2.1.2. *Suppose that $F : \mathbf{Man} \to \mathbf{sSets}$ is a discrete pre-ring. Then F is a fibrant pre-ring.*

Proof. If $A \to B$ is an acyclic cofibration of pre-rings, then in particular $\pi_0(A(M)) \to \pi_0(B(M))$ is an isomorphism for all manifolds M. The result follows from the fact that π_0 is left adjoint to the constant simplicial set functor $c_* : \mathbf{Sets} \to \mathbf{sSets}$, of which F is in the image.

□

Let $\mathcal{M}$ be a proper cellular simplicial model category and let Ψ be a set of morphisms in $\mathcal{M}$. Recall (see [13]) that the localization $\mathcal{M}'$ of $\mathcal{M}$ with respect to Ψ is defined as follows. As a simplicial category, $\mathcal{M}' = \mathcal{M}$. The cofibrations of $\mathcal{M}'$ are the same as the cofibrations of $\mathcal{M}$. The fibrant objects of $\mathcal{M}'$ are those objects $F \in \mathcal{M}$ such that

1. F is fibrant in $\mathcal{M}$, and
2. for all morphisms $\psi : A \to B$ in Ψ, F is *ψ-local*. That is, the morphism of simplicial sets
$$\mathrm{Map}(B, F) \to \mathrm{Map}(A, F)$$
is a weak equivalence.

The weak equivalences in $\mathcal{M}'$ are all maps $f : C \to D$ such that for every fibrant object F in $\mathcal{M}'$, the map $\mathrm{Map}(D, F) \to \mathrm{Map}(C, F)$ is a weak equivalence of simplicial sets. The fibrations in $\mathcal{M}'$ are given by the right lifting property with respect to acyclic cofibrations.

The localization $\mathcal{M}'$ of $\mathcal{M}$ at Ψ has the universal property that

1. there exists a left Quillen functor $j : \mathcal{M} \to \mathcal{M}'$,
2. the total left derived functor of $Lj : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M}')$ sends the images in $\mathrm{Ho}(\mathcal{M})$ of morphisms in Ψ to isomorphisms in $\mathrm{Ho}(\mathcal{M}')$, and
3. if $k : \mathcal{M} \to \mathcal{N}$ is another category over $\mathcal{M}$ that satisfies these properties then there exists a unique left Quillen functor $a : \mathcal{M}' \to \mathcal{N}$ such that $k = aj$.

We will define the model category of smooth rings to be a localization of the model category of pre-rings at a set Ψ of morphisms. Suppose that $s: M \to N$ is a submersion of manifolds, and $a: P \to N$ is any map of manifolds. The fiber product $Q = M \times_N P$ exists in the category of manifolds. Let G denote the homotopy colimit in the diagram of pre-rings

$$\begin{tikzcd} H_N \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & H_P \arrow[d] \\ H_M \arrow[r] & G \end{tikzcd} .$$

The functoriality of H induces a map $\psi_{s,a}: G \to H_Q$. Let Ψ' denote the set

$$\Psi' = \{\psi_{s,a} | \ s \text{ is a submersion and } a \text{ is any map of manifolds.}\}$$

We must include the empty limit diagram; so let $\emptyset^{\mathbf{Man}}: \mathbf{Man} \to \mathbf{sSets}$ (sending $M \mapsto \emptyset$) denote the initial pre-ring and let $\psi_\emptyset : \emptyset^{\mathbf{Man}} \to H_\bullet$ denote the unique map. Let Ψ denote the union of Ψ' and $\{\psi_\emptyset\}$.

Definition 2.1.3. The model category of *smooth rings*, denoted **SR** is defined to be the localization of the model category $\mathbf{sSets}^{\mathbf{Man}}$ of pre-rings at the set Ψ.

Remark 2.1.4. Fibrant smooth rings are "submersion-pullback preserving" in the following sense. Suppose that $s: M \to N$ is a submersion and $a: P \to N$ is any smooth map of manifolds, so that $\psi_{s,a} : H_M \amalg_{H_N} H_P \to H_{M\times_N P}$. A fibrant smooth ring F is Ψ-local and in particular $\psi_{s,a}$-local. Therefore, it must satisfy the equivalence

$$\mathrm{Map}((H_M \amalg_{H_N} H_P), F) \simeq \mathrm{Map}(H_{M\times_N P}, F).$$

By Yoneda's lemma, the square

$$\begin{tikzcd} F(M \times_N P) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & F(M) \arrow[d] \\ F(P) \arrow[r] & F(N) \end{tikzcd}$$

is a homotopy pullback square.

In particular, since any product of manifolds is a pullback over the terminal manifold $\bullet$, and since any map to $\bullet$ is a submersion, fibrant smooth rings also preserve products of manifolds.

Lemma 2.1.5. *1. The functor $H_\bullet$: **Man** $\to$ **sSets** is a homotopy initial object in the category of smooth rings.*

2. If F is a fibrant smooth ring, $s: M \to N$ is a submersion, and $a: P \to N$ is any map of manifolds, then the natural map

$$F(M \times_N P) \xrightarrow{\simeq} F(M) \times_{F(N)} F(P)$$

is a weak equivalence.

3. *If M is a manifold then H_M is a fibrant smooth ring.*

4. *If M is a manifold then H_M is a cofibrant smooth ring.*

5. *If M is a manifold then H_M is small: If I is a cofiltered category and $X : I \to \mathbf{SR}$ is a diagram of smooth rings, then the natural map*

$$\operatorname*{colim}_I \operatorname{Map}(H_M, X) \to \operatorname{Map}(H_M, \operatorname*{colim}_I X)$$

   is an isomorphism.

6. *$H_\emptyset : M \mapsto \operatorname{Hom}(\emptyset, M) = \{*\}$ is the final object in the category of smooth rings.*

7. *$\emptyset^{\mathbf{Man}} : M \mapsto \emptyset$ is the initial object in the category of smooth rings, but is not fibrant (and will never be discussed again).*

Proof. For any fibrant smooth ring F, there is a weak equivalence

$$\operatorname{Map}(H_\bullet, F) \xrightarrow{\simeq} \operatorname{Map}(\emptyset^{\mathbf{Man}}, F) = \Delta^0,$$

since $\emptyset^{\mathbf{Man}}$ is the initial functor; this proves the first assertion. The second assertion is proved similarly. The fourth assertion follows from the fact that $H_M(N)$ is constant, hence cofibrant, for all N. The sixth assertion is obvious. The fifth assertion follows from the fact that the localization functor $\mathbf{sSets}^{\mathbf{Man}} \to \mathbf{SR}$ does not change the categorical structure, only the model structure. Therefore, since H_M is small in $\mathbf{sSets}^{\mathbf{Man}}$, it is also small in $\mathbf{SR}$. In the seventh assertion, it is obvious that $\emptyset^{\mathbf{Man}}$ is the initial object. It is not fibrant because it is not local with respect to $\psi_\emptyset$.

We now come to the third assertion. The simplicial set $H_M(N)$ is constant for all manifolds N. Suppose that $\mathcal{F} \to \mathcal{G}$ is an acyclic cofibration of functors for which the diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & H_M \arrow[d] \\ \mathcal{G} \arrow[r] & H_\emptyset \end{tikzcd}$$

commutes. This is equivalent to the commutativity of the solid arrow diagram

$$\begin{tikzcd} \pi_0\mathcal{F} \arrow[r] \arrow[d] & H_M \arrow[d] \\ \pi_0\mathcal{G} \arrow[r] \arrow[ur, dashed] & H_\emptyset. \end{tikzcd}$$

The left hand map is an isomorphism of functors, hence the dotted arrow also exists. In other words, H_M is fibrant as a pre-ring. Thus it suffices to show that H_M is Ψ-local. However, this is obvious since $H_M(-) = \operatorname{Hom}_{\mathbf{Man}}(M, -)$ preserves all limits, including those which constitute Ψ.

$\square$

The following Proposition is necessary in order for us to eventually be able to localize the category of presheaves of smooth rings, which in turn is necessary in order to obtain a category of sheaves of smooth rings, which are central to the theory.

Proposition 2.1.6. *The model category of smooth rings is a cofibrantly generated left proper simplicial model category.*

Proof. The model category of smooth rings is a localization of a cofibrantly generated proper simplicial model category, so **SR** is cofibrantly generated and left proper as well by [13, 4.1.1].

□

Remark 2.1.7. In this work, we shall mainly be using smooth rings which are cofibrant-fibrant. Every pre-ring is cofibrant, hence every smooth ring is cofibrant. Recall that a smooth ring is fibrant if it is fibrant as a pre-ring and is Ψ-local.

A pre-ring $F : \mathbf{Man} \to \mathbf{sSets}$ is fibrant if it has the right lifting property with respect to natural transformations $\phi : A \to B$ of functors which are objectwise acyclic cofibrations. In particular, in order for F to be fibrant it is necessary (but not sufficient) that for every smooth map $f : M \to N$, the induced map $F(M) \to F(N)$ is a fibration of simplicial sets.

One way to look at the fibrancy condition is to instead look at the fibrant replacement functor. From this angle we see that the fibrancy condition for pre-rings is relatively benign: if $F \to \widehat{F}$ is a fibrant replacement then for all manifolds M, the map $F(M) \to \widehat{F}(M)$ is a weak equivalence of simplicial sets, because weak equivalences in the model category of pre-rings are simply objectwise weak equivalences.

This is not so with regards to the fibrant replacement functor for smooth rings, as we will see in Example 2.1.10 below. That is, the process of replacing a functor $F : \mathbf{Man} \to \mathbf{sSets}$ by one that is submersion-pullback preserving (i.e. Ψ-local) will usually change the homotopy type of $F(M)$. So in some sense, this is the more significant job of the fibrant replacement functor for smooth rings. This is the reasoning behind the following definition.

Definition 2.1.8. We say that a smooth ring F is *almost fibrant* if, for every $\psi \in \Psi$, $F(\psi)$ is a weak equivalence.

Thus a smooth ring is fibrant if and only if it is fibrant as a pre-ring and almost fibrant. If F is an almost fibrant smooth ring, $\widehat{F}$ is its fibrant replacement, and M is a manifold then $F(M) \simeq \widehat{F}$.

Example 2.1.9. The simplicial set Δ^n is not a fibrant simplicial set; let D^n denote a fibrant replacement of Δ^n. The constant functor, which we denote $\underline{D^n} : \mathbf{Man} \to \mathbf{sSets}$ is not a fibrant pre-ring (and hence not a fibrant smooth ring), essentially because the colimit of a diagram of acyclic cofibrations need not be an acyclic cofibration. Let $\widehat{D^n}$ denote the fibrant replacement of $\underline{D^n}$ in the model category of pre-rings. Since weak equivalences are taken objectwise,

$\widehat{D^n}(M) \simeq D^n(M) = D^n$ is contractible for any manifold M. Thus $\widehat{D^n}$ is almost fibrant, hence fibrant as a smooth ring.

Example 2.1.10. We explore the fibrant replacement functor for smooth rings in the simplest case.

Let M and N be manifolds and let F be the coproduct $H_M \amalg H_N$, *taken in the category of pre-rings*, which is just an object-wise coproduct of functors. (Usually we reserve the notation $A \amalg_B C$ for homotopy colimits taken in the category of smooth rings). Now $H_M \amalg H_N$ is a cofibrant-fibrant pre-ring by Lemma 2.1.2. However it is not a fibrant smooth ring because it does not respect submersion pullbacks. For example, let A and B be manifolds. Then $F(A \times B)$ is the discrete simplicial set

$$\mathrm{Hom}(M, A \times B) \amalg \mathrm{Hom}(N, A \times B)$$

which is certainly not weakly equivalent to

$$(\mathrm{Hom}(M, A) \amalg \mathrm{Hom}(N, A)) \times (\mathrm{Hom}(M, B) \amalg \mathrm{Hom}(N, B)).$$

Such a weak equivalence is necessary in order for F to be a fibrant smooth ring, since the diagram

$$\begin{tikzcd} M \times N \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & M \arrow[d] \\ N \arrow[r] & \bullet \end{tikzcd}$$

is a submersion-pullback.

Consider the map $\psi : H_M \amalg H_N \to H_{M \times N}$ given by the projections $M \times N \to M$ and $M \times N \to N$. The map ψ is in fact a weak equivalence in the model category of smooth rings because it is an element of the localizing set Ψ. Moreover, $H_{M \times N}$ is a cofibrant-fibrant smooth ring by Lemma 2.1.5. Therefore ψ is a fibrant replacement.

Example 2.1.11. Along the same lines, the constant functor $\emptyset^{\mathbf{Man}} : M \mapsto \emptyset$ is not a fibrant smooth ring, because it is not local with respect to the map $\psi_\emptyset \in \Psi$.

Note, by the way, that $\emptyset^{\mathbf{Man}}$ is *not* the same thing as $H_\emptyset$. The first is the initial object and the second is the terminal object in $\mathbf{sSets}^{\mathbf{Man}}$.

Example 2.1.12. Let $F : \mathbf{Man} \to \mathbf{sSets}$ be the functor which takes a manifold M to the constant simplicial set $\amalg_p T_p(M)$ whose elements are pairs (m, α), where $m \in M$ is a point and $\alpha \in T_m(M)$ is a tangent vector at m. This is an almost fibrant smooth ring. Indeed, let $s : M \to N$ be a submersion, $a : P \to N$ any map, and let X be the fiber product manifold in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner" very near start] & M \arrow[d, "s"] \\ P \arrow[r, "a"'] & N. \end{tikzcd}$$

Then for any point $x \in X$, $T_x(X)$ is the fiber product of $T_x(M)$ and $T_x(P)$ over $T_x(N)$ (where we have written x for its image in M, N, and P). This is enough to show that F is almost fibrant, as we shall show in Example 2.3.22.

In fact, for any $k \in \mathbb{N}$, the functor which takes a manifold M to its set of k-jets is an almost fibrant smooth ring.

Example 2.1.13. We will be returning to the following example throughout this paper. Let $p : \bullet \to \mathbb{R}$ be the point corresponding to the origin in $\mathbb{R}$, and let $z : H_{\mathbb{R}} \to H_{\bullet}$ be the corresponding map of representable smooth rings. Let $t : \mathbb{R} \to \bullet$ be the terminal map out of $\mathbb{R}$. Let G be the smooth ring obtained as the homotopy colimit in the diagram

$$\begin{tikzcd} H_{\mathbb{R}} \arrow[r, "z"] \arrow[d, "z"'] \arrow[dr, phantom, "\diamond"] & H_{\bullet} \arrow[d] \\ H_{\bullet} \arrow[r] & G. \end{tikzcd}$$

It is not hard to see that G corresponds to the intersection of the origin with itself in the real line.

To compute G, we need to replace our diagram with a cofibrant one, and since **SR** is left proper, it suffices to replace one instance of z with a cofibration. Consider the smooth ring C whose nondegenerate simplices are concentrated in degrees 0 and 1, and are given by

$$H_{\mathbb{R}} \rightrightarrows H_{\mathbb{R}},$$

in which the top map is identity and the bottom map sends $f : \mathbb{R} \to M$ to $(f \circ p \circ t) : \mathbb{R} \to M$, for any manifold M. The map $H_{\mathbb{R}} \to C$ is a cofibration because it is a monomorphism.

We show that $C \to H_{\bullet}$ is a weak equivalence: it is enough to show that for each manifold M, the map $C(M) \to H_{\bullet}(M)$ is a weak equivalence. For each M, $C(M)$ is a "star-shaped" simplicial set, i.e. a simplicial set which has no non-degenerate simplices in dimensions $n > 1$, and for which there is a vertex v such that every other vertex is connected by a unique path to v. In this case, every function $f : \mathbb{R} \to M$ is connected by a unique path to the constant function $f(x) = f(0) \in M$. Thus $C(M)$ is homotopy equivalent to the (constant simplicial) set of points $m \in M$, which is precisely $H_{\bullet}(M)$.

Thus G is weakly equivalent to the colimit of the diagram

$$\begin{tikzcd} H_{\mathbb{R}} \arrow[r] \arrow[d] & C \\ H_{\bullet}, & \end{tikzcd}$$

which is the almost fibrant smooth ring whose nondegenerate part is

$$H_{\mathbb{R}} \rightrightarrows H_{\bullet},$$

where both face maps are z.

This is analogous to the situation in homological algebra, in which one wants the derived tensor product of a ring k with itself over $k[x]$. The result is the chain complex

$$0 \to k\langle x\rangle \xrightarrow{x\mapsto 0} k \to 0$$

concentrated in degrees 0 and 1, where $k\langle x\rangle$ denotes the free 1-dimensional k-module spanned by x.

Proposition 2.1.14. *Let M be a manifold. The functor $-(M) : \mathbf{SR} \to \mathbf{sSets}$, given by $F \mapsto F(M)$, is a right Quillen functor. Its left adjoint is $- \otimes H_M : X \mapsto X \otimes H_M$. Thus the adjoint functors are Quillen.*

Proof. The functors are adjoint because

$$\mathrm{Map}(X \otimes H_M, F) \cong \mathrm{Map}(X, \mathrm{Map}(H_M, F)) \cong \mathrm{Map}(X, F(M)).$$

If $X \to Y$ is a cofibration (resp. acyclic cofibration) of simplicial sets, then $X \otimes H_M \to Y \otimes H_M$ is a cofibration (resp. acyclic cofibration) of smooth rings by [13, 9.3.7].

□

Lemma 2.1.15. *Let F be a fibrant smooth ring, and let $i : M_\alpha \subset M$ be an open inclusion of manifolds. Then the homotopy fibers of the map $F(i) : F(M_\alpha) \to F(M)$ are either empty or contractible.*

Proof. Since i is a monomorphism, the following is a pullback diagram of manifolds:

$$\begin{tikzcd} M_\alpha \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & M_\alpha \arrow[d] \\ M_\alpha \arrow[r] & M. \end{tikzcd}$$

Since i is a submersion, it remains a (homotopy) pullback diagram after applying F. Pick a point $x : \Delta^0 \to F(M)$. If the homotopy fiber of x in $F(M_\alpha)$ is nonempty, then there exists a lift $x' : \Delta^0 \to F(M)$ of x (up to homotopy). Both the left and right squares in the diagram

$$\begin{tikzcd} \Delta^0 \arrow[r, "x'"] \arrow[d] \arrow[dr, phantom, "\diamond", very near start] & F(M_\alpha) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond", very near start] & F(M_\alpha) \arrow[d] \\ \Delta^0 \arrow[r, "x'"] & F(M_\alpha) \arrow[r, "F(i)"] & F(M) \end{tikzcd}$$

are homotopy pullbacks, so the big rectangle is too. The homotopy fiber of $F(i)\circ x'$ is weakly equivalent to the homotopy fiber of x, and the above argument shows it is contractible.

□

2.2 Presheaves of smooth rings

We now discuss presheaves of smooth rings on a space. For a topological space X, let $Op(X)$ denote the category of open sets and open inclusions. We denote by $\mathbf{Pre}(X, \mathbf{SR})$ the category of presheaves of smooth rings on $\mathrm{Op}(X)$. If F is a smooth ring, we may also use F to denote the corresponding presheaf of smooth rings on the terminal space $\{*\}$.

The category $\mathbf{Pre}(X, \mathbf{SR})$ has the structure of a left proper, cofibrantly generated, simplicial model category using the injective model structure (i.e. open-wise weak equivalences and open-wise cofibrations). The simplicial mapping spaces are defined in the usual way (see [13, 11.7.2]). We refer to this model structure on $\mathbf{SR}^{\mathrm{Op}(X)^{\mathrm{op}}}$ as *the objectwise model structure*. This is to distinguish it from another model structure on $\mathbf{SR}^{\mathrm{Op}(X)^{\mathrm{op}}}$, which we will define later and call the "local model structure".

At this point, there is an annoying bit of notation that must be cleared up. If $\mathcal{F}$ is a presheaf of smooth rings on X, then it takes in two variables, an open set U in X on which $\mathcal{F}$ is contravariant, and a manifold M on which $\mathcal{F}$ is covariant. This is reminiscent of the Hom bifunctor in a category (or more accurately of a correspondence between categories). Indeed, the two are closely related, as we will see later. We therefore let the first position in the bifunctor $\mathcal{F}(-,-)$ be the variable corresponding to the open subset of X, and we let the second position be the variable corresponding to the manifold. In other words, for an open set U, the covariant functor $\mathcal{F}(U,-) : \mathbf{Man} \to \mathbf{sSets}$ is a smooth ring, and for a manifold M, the contravariant functor $\mathcal{F}(-,M) : \mathrm{Op}(X)^{\mathrm{op}} \to \mathbf{sSets}$ is a presheaf of simplicial sets on X.

Notation 2.2.1. Sometimes (e.g. in Lemma 2.2.6), the above notation can be cumbersome. In such situations, we may abbreviate $\mathcal{F}(U,-)$ by $\mathcal{F}(U)$ and $\mathcal{F}(-,M)$ by $\mathcal{F}[M]$.

Let M be a manifold. Let $\mathcal{H}_M$ denote the presheaf of smooth rings on M given on an open submanifold $U \subset M$ by

$$U \mapsto H_U = \mathrm{Hom}_{\mathbf{Man}}(U,-).$$

This is called the *structure presheaf* on M, and it contains all the data that makes up M as a manifold. In particular, $\mathcal{H}(-,\mathbb{R})$ is the usual C^∞-structure sheaf.

Definition 2.2.2. Let X be a topological space, and let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X with values in a simplicial model category $\mathcal{M}$. Define $\mathbf{Map}_{\mathcal{M}}(\mathcal{F},\mathcal{G})$ to be the presheaf of simplicial sets

$$U \mapsto \mathrm{Map}_{\mathcal{M}}(\mathcal{F}|_U, \mathcal{G}|_U), U \in \mathrm{Op}(X).$$

We refer to it as *the presheaf of maps* from $\mathcal{F}$ to $\mathcal{G}$.

Lemma 2.2.3. *Let $\mathcal{E}, \mathcal{F}$, and $\mathcal{G}$ be presheaves of smooth rings on a space X. Then one has a canonical isomorphism*

$$\mathbf{Map}(\mathcal{E} \otimes \mathcal{F}, \mathcal{G}) \cong \mathbf{Map}(\mathcal{E}, \mathbf{Map}(\mathcal{F}, \mathcal{G})).$$

Proof. For each open set $U \subset X$, one has a natural isomorphism

$$\mathrm{Map}(\mathcal{E}(U) \otimes \mathcal{F}(U), \mathcal{G}(U) \xrightarrow{\cong} \mathrm{Map}(\mathcal{E}(U), \mathrm{Map}(\mathcal{F}(U), \mathcal{G}(U)),$$

and the lemma follows from this naturality.

□

Notation 2.2.4. If F is a smooth ring and X is a topological space, we use the symbol $\underline{F}$ to denote the constant presheaf t^*F, where $t : X \to \{*\}$ is the terminal map out of X.

Note that if M is a manifold then the sheaves $\mathcal{H}_M$ and $\underline{H_M}$ on M agree on global sections but are very different presheaves.

Let $\mathbf{Pre}(X, \mathbf{sSets})$ denote the category of presheaves of simplicial sets on a topological space X. We consider $\mathbf{Pre}(X, \mathbf{sSets})$ as a model category with the injective model structure (i.e. with objectwise weak equivalences and cofibrations), and refer to it as the *objectwise structure* on $\mathbf{sSets}^{\mathrm{Op}(X)^{\mathrm{op}}}$.

Remark 2.2.5. The objectwise model category $\mathbf{Pre}(X, \mathbf{sSets})$ should not be confused with Jardine's model category of simplicial presheaves (see [16] and [6]), which is a localization of $\mathbf{Pre}(X, \mathbf{sSets})$. We will have occasion to use the latter model category soon enough, and we will refer to it as "the local model structure" (on $\mathbf{sSets}^{\mathrm{Op}(X)^{\mathrm{op}}}$) in keeping with standard terminology and shall denote it as $\mathbf{Shv}(X, \mathbf{sSets})$.

Lemma 2.2.6. *Let X be a topological space and let M be a manifold. The functor $-[M] : \mathbf{Pre}(X, \mathbf{SR}) \to \mathbf{Pre}(X, \mathbf{sSets})$, given by $\mathcal{F} \mapsto \mathcal{F}[M]$, is a right Quillen functor (see Notation 2.2.1). Its left adjoint is*

$$- \otimes \underline{H_M} : \mathcal{G} \mapsto \mathcal{G} \otimes \underline{H_M}.$$

Proof. The functor $-[M] : \mathbf{Pre}(X, \mathbf{SR}) \to \mathbf{Pre}(X, \mathbf{sSets})$ is the composition of a right Quillen functor (adjoint to the localization $\mathbf{PR} \to \mathbf{SR}$) and the evaluation functor

$$-[M] : \mathbf{Pre}(X, \mathbf{PR}) \to \mathbf{Pre}(X, \mathbf{sSets}).$$

Thus, it suffices to show that the latter is a right Quillen functor. To see that it is a right adjoint, notice that the functor $\mathcal{F} \mapsto \mathcal{F}[M]$ is the same as the functor $\mathcal{F} \mapsto \mathbf{Map}(\underline{H_M}, \mathcal{F})$. Then apply Lemmas 2.1.14 and 2.2.3.

It suffices to show that $\mathcal{G} \mapsto \mathcal{G} \otimes \underline{H_M}$ preserves cofibrations and acyclic cofibrations in $\mathcal{G}$. In the injective model structure, these conditions are checked on open sets, where it follows from [13, 9.3.7].

□

Definition 2.2.7. Let X and Y be topological spaces, let $f : X \to Y$ be continuous, let $\mathcal{C}$ be a category, and let $\mathcal{F}$ be a presheaf on X with values in $\mathcal{C}$. The *direct image presheaf of $\mathcal{F}$ under f* is the presheaf $U \mapsto \mathcal{F}(f^{-1}(U)); U \in \mathrm{Op}(Y)$.

Lemma 2.2.8. *Let $f : X \to Y$ be a map of topological spaces. The direct image functor $f_* : \mathbf{Pre}(X, \mathbf{SR}) \to \mathbf{Pre}(Y, \mathbf{SR})$ has a left adjoint f^*, and (f^*, f_*) is a Quillen pair.*

Proof. The map f induces a functor $\mathrm{Op}(Y)^{\mathrm{op}} \to \mathrm{Op}(X)^{\mathrm{op}}$, and the functors f^* and f_*, given by left Kan extension and by restriction of functors, as in Lemma A.3.10, are adjoint.

We need to check that f^* preserves object-wise cofibrations and object-wise acyclic cofibrations. Let $\phi : A \to B$ be an object-wise cofibration (resp. object-wise acyclic cofibration) of presheaves on Y. Let $U \subset X$ be an open set. The map $f^*(\phi)(U) : (f^*A)(U) \to (f^*B)(U)$ is given by the filtered colimit

$$\operatorname*{colim}_{f^{-1}(V) \supset U} (A(V) \to B(V)).$$

Each $A(V) \to B(V)$ represented in the colimit is a monomorphism of functors, and hence so is the resulting diagram. Since filtered colimits preserve finite limits, they preserve monomorphisms. Hence $f^*(\phi)$ is a monomorphism of functors. That is, it is an object-wise cofibration on $\mathrm{Op}(X)^{\mathrm{op}} \times \mathbf{Man}$, and thus a cofibration in the injective model structure.

□

Lemma 2.2.9. *Let X be a topological space and $i : U \subset X$ an open subset. The functor $i^* : \mathbf{Pre}(X, \mathbf{SR}) \to \mathbf{Pre}(U, \mathbf{SR})$ has a right adjoint $i_! : \mathbf{Pre}(U, \mathbf{SR}) \to \mathbf{Pre}(X, \mathbf{SR})$, given as follows. For a presheaf $\mathcal{F}$ on U, define $i_!\mathcal{F}$ on an open set $V \subset X$ to be $\mathcal{F}(V)$ if $V \subset U$ and $\emptyset^{\mathbf{Man}}$ otherwise.*

Proof. The inclusion of an open subset is an open map. That is, i induces a map $\iota : \mathrm{Op}(U) \to \mathrm{Op}(X)$, and i^* is just the corresponding restriction of functors. By Lemma A.3.10, i^* has a left adjoint $i_!$ defined by the formula

$$i_!\mathcal{F}(V) = \operatorname*{colim}_{(V \to \iota(A)) \in (V \downarrow \iota)} \mathcal{F}(A).$$

If $V \subset U$ then V is the initial object of $(V \downarrow \iota)$, so $i_!\mathcal{F}(V) = \mathcal{F}(V)$. If $V \not\subset U$ then $(V \downarrow \iota)$ is empty, and the colimit is the initial object $\emptyset^{\mathbf{Man}} \in \mathbf{sSets}^{\mathbf{Man}}$.

□

Definition 2.2.10. If $i : U \subset X$ is the inclusion of an open subset, we call $i_!$ the *extension by zero* functor.

Definition 2.2.11. Let X be a topological space, $i : U \subset X$ an open set, and let F be a smooth ring. Let $t : X \to \{*\}$ be the terminal map out of X. Let $r_F U$ denote the presheaf of smooth rings on X,

$$r_F U := i_! i^* t^* F$$

so that $r_F U$ is F if $V \subset U$ and is $\emptyset^{\mathbf{Man}}$ if $V \not\subset U$.

If M is a manifold, we denote $r_{H_M} U$ simply by $r_M U$, and we denote $r_\emptyset U$ simply by rU. We will say that a presheaf $\mathcal{F}$ is *representable (over M)* if it is isomorphic to $r_M U$.

Remark 2.2.12. Let X be a topological space, $i : U \subset X$ an open set, and let M be a manifold. The definition of the presheaf $r_M U$ may look complicated, but the presheaf itself is easy to understand. First, it is the extension by zero of the constant sheaf on U with value H_M. In particular, it is a discrete presheaf in the sense that for every open set V and manifold N, the simplicial set $r_M U(V, N)$ is a discrete simplicial set.

More importantly, $r_M U$ represents a nice functor. By adjointness (see lemmas 2.2.8 and 2.2.9), if $\mathcal{G}$ is a presheaf of smooth rings then

$$\mathrm{Map}(r_M U, \mathcal{G}) \cong \mathcal{G}(U, M).$$

For example, $\mathrm{Map}(r_M U, r_N V)$ is isomorphic to the discrete simplicial set $\mathrm{Hom}_{\mathbf{Man}}(N, M)$ if $U \subset V$, and to $\emptyset$ otherwise.

The functor $H_\emptyset$ is a constant smooth ring (with value Δ^0.) Thus one could think of $rU = r_{H_\emptyset} U$ as just a presheaf of simplicial sets or even a presheaf of sets. As such, the notation the same as that in [6]: rU is the representable presheaf of sets on X given by $\mathrm{Hom}_{\mathrm{Op}(X)}(-, V)$. Moreover, one may equate $r_M V$ with $rV \otimes \underline{H_M}$.

Remark 2.2.13. One may realize the model category $\mathbf{Pre}(X, \mathbf{SR})$ as the localization of the injective model structure on $\mathcal{C} = \mathbf{Pre}(X, \mathbf{PR})$. To see that, let Ψ be the localizing set for $\mathbf{SR}$. For every $\psi : F \to G \in \Psi$ and every open set $U \subset X$, let $\psi_U : r_F U \to r_G U$ be the induced map. The localization of $\mathcal{C}$ at the set $\{\psi_U | U \in \mathrm{Op}(X), \psi \in \Psi\}$ is Quillen equivalent to $\mathbf{Pre}(X, \mathbf{SR})$.

Lemma 2.2.14. *Let M be a manifold. The presheaf $\mathcal{H}_M$ is a cofibrant-fibrant presheaf of smooth rings.*

Proof. It follows directly from lemma 2.1.5 that $\mathcal{H}_M$ is cofibrant, and it follows from reasoning similar to that in the proof of the third assertion in lemma 2.1.5 that $\mathcal{H}_M$ is fibrant.

□

Lemma 2.2.15. *Let X be a topological space and let $i : U \subset X$ be an open subset. The functors $(i_!, i^*)$, defined in lemma 2.2.9 are a Quillen pair in either the injective or the projective model structures.*

Proof. Clearly i^* takes object-wise fibrations (resp. object-wise acyclic fibrations) to object-wise fibrations (resp. object-wise acyclic fibrations). Since $i_!$ is extension by zero, it is also easy to see that it takes object-wise cofibrations to object-wise cofibrations.

□

2.3 Sheaves of smooth rings

The next step is to define the local model structure on $\mathbf{Pre}(X, \mathbf{SR})$. To do so we must define hypercovers for our context. Hypercovers can be defined for arbitrary Grothendieck sites in a manner which is fairly easy to write down. However the situation in which X is a topological space is much simpler conceptually because $\mathrm{Op}(X)$ is a Verdier site and a simple one at that. By [6, Theorem 9.6] (which we closely follow here), we may restrict our attention to the "basal" hypercovers, which we refer to simply as hypercovers. Before giving the definition, we give a brief outline.

Let X be a topological space and V an open subset. If one covers V with open subsets, then the subsets, their two-fold intersections, their three-fold intersections, etc. form a simplicial complex, called the Čech complex (see example 2.3.2). This complex, and its augmentation map to V, is the most basic example of a hypercover. A general hypercover is less restricted in that at each stage, one may refine the intersections of the previous stage (whereas the Čech hypercover is a trivial refinement at each stage). To make this precise, one does not work with the open sets V_i themselves, but with the presheaves of sets rV_i which they represent.

Finally, before giving the definition, note that if $A = \amalg_i A_i$ and $B = \amalg_j B_j$ are coproducts of representable presheaves, then any map $A \to B$ is defined by giving for each index i a prescribed value of j and a map $A_i \to B_j$.

Definition 2.3.1. Let X be a topological space and $V \subset X$ an open subset. A *hypercover* of V is an augmented simplicial presheaf $\mathcal{U} \to rV$, with the following properties:

1. For each n, $\mathcal{U}_n$ is the coproduct $\coprod_i rU_n^i$ of representable presheaves (for open subsets $U_n^i \subset V$),
2. for each n, the nth matching object $M_n\mathcal{U}$ is the coproduct $\coprod_j rM_n^j$ of representable presheaves (for open subsets $M_n^j \subset V$),
3. the various maps $U_n^i \to M_n^j$ are inclusions of open sets, and
4. if we fix a j_0, the collection of maps $\{U_n^i \to M_n^{j_0}\}$ forms an open covering of $M_n^{j_0}$.

In fact the second item, which says that each $M_n\mathcal{U}$ must be a coproduct of representables, follows from the other items by [6, Lemma 9.4]. Although it was more convenient in the definition to use the notation $\mathcal{U} \to rV$ to denote the hypercover, we will generally use the notation $U_\bullet \to rV$, as it better connotes a simplicial object.

As mentioned above, the most basic hypercovers are the so-called Čech hypercovers.

Example 2.3.2. Let I be a set, X a topological space, and let $U : I \to \mathrm{Op}(X)$ be a functor that associates to each $i \in I$ an open set $U_i \subset X$. Suppose that the U_i cover X, i.e. $\cup_{i\in I} U_i = X$. For any finite subset $J \subset I$, define $U_J = \cap_{j\in J} U_j$

(e.g. $U_{ij} = U_i \cap U_j$); if $J = \emptyset$ then of course $U_J = X$. For every natural number $n \in \mathbb{N}$, let

$$U_n = \coprod_{|J|=n+1} rU_J$$

It is easy to show that $U_\bullet$ is a simplicial presheaf, and it has a natural augmentation map to $U_{-1} = rX$. We call $U_\bullet$ the *Čech hypercover* associated to the open covering U.

Definition 2.3.3. Let X be a topological space. Let $\mathbf{Shv}(X, \mathbf{sSets})$ denote the Jardine model structure on the category of presheaves of simplicial sets on X, as defined in [16] or [6]. All objects in this category are cofibrant. We refer to the fibrant objects in $\mathbf{Shv}(X, \mathbf{sSets})$ as *sheaves of simplicial sets on* X.

By [7], $\mathbf{Shv}(X, \mathbf{sSets})$ is the localization of $\mathbf{Pre}(X, \mathbf{sSets})$ at the set of hypercovers, as defined above.

Recall that for any manifold M, the functor $-[M] : \mathbf{Pre}(X, \mathbf{SR}) \to \mathbf{Pre}(X, \mathbf{sSets})$ is a right Quillen functor. Its left adjoint is $- \otimes H_M$. There is a (left) localization $\mathbf{Pre}(X, \mathbf{sSets}) \to \mathbf{Shv}(X, \mathbf{sSets})$. We wish to find the initial left localization $\mathbf{Shv}(X, \mathbf{SR})$ of $\mathbf{Pre}(X, \mathbf{SR})$ such that all the induced functors $-[M] : \mathbf{Shv}(X, \mathbf{SR}) \to \mathbf{Shv}(X, \mathbf{sSets})$ are right Quillen functors. By [13, 3.3.20], we simply tensor the hypercovers with H_M for each M.

Definition 2.3.4. Let X be a topological space and $V \subset X$ an open subset. For each manifold P, let $\mathrm{hyp}_P(X)$ denote the set

$$\mathrm{hyp}_P = \{\mathrm{hocolim}(U_\bullet \otimes H_P) \to (rV \otimes H_P) |\, U_\bullet \to rV \text{ is a hypercover}\}.$$

We define the *local model structure* on $\mathbf{SR}^{\mathrm{Op}(X)^{\mathrm{op}}}$ to be the localization of the objectwise model structure on $\mathbf{Pre}(X, \mathbf{SR})$ at the set

$$\mathrm{hyp}(X) = \bigcup_{P \in \mathbf{Man}} \mathrm{hyp}_P$$

of all hypercovers. We write $\mathbf{Shv}(X, \mathbf{SR})$ to denote the local model structure on the category of presheaves.

All objects are cofibrant. We call the fibrant objects in this model structure *sheaves of smooth rings* on X. We denote the full subcategory of sheaves of smooth rings on X by $\mathbf{Shv}^\circ(X, \mathbf{SR})$. We will refer to general objects in $\mathbf{Shv}(X, \mathbf{SR})$ only as *presheaves of smooth rings (in the local model structure)*, since they do not necessarily satisfy descent for hypercovers. The simplicial mapping spaces in $\mathbf{Shv}(X, \mathbf{SR})$ are the same as those in $\mathbf{Pre}(X, \mathbf{SR})$.

Lemma 2.3.5. *Let X be a topological space. If $\mathcal{F} \in \mathbf{Shv}(X, \mathbf{SR})$ is a sheaf, then it satisfies*

$$\mathcal{F}(V, P) \simeq \mathrm{holim}(\mathcal{F}(U_\bullet, P))$$

for any hypercover $U_\bullet \to rV$.

Proof. This follows directly from the definition.

□

Proposition 2.3.6. *Let X be a topological space and $\mathcal{F}$ a sheaf of smooth rings on X. Suppose that $s : M \to N$ is a submersion, $P \to N$ is any map, and $Q = M \times_N P$ is their fiber product. Then the diagram*

$$\begin{tikzcd} \mathcal{F}(-,Q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{F}(-,M) \arrow[d] \\ \mathcal{F}(-,P) \arrow[r] & \mathcal{F}(-,N) \end{tikzcd}$$

is a fiber product of sheaves of simplicial sets.

Proof. This follows from the fact that the functor from sheaves on X to presheaves on X is a right Quillen functor, and $\mathcal{F}$ is fibrant.

□

Proposition 2.3.7. *Let X be a topological space and M a manifold. The functor*

$$-[M] : \mathbf{Shv}(X, \mathbf{SR}) \to \mathbf{Shv}(X, \mathbf{sSets})$$

is a right Quillen functor. Moreover, an object $\mathcal{F} \in \mathbf{Shv}(X, \mathbf{SR})$ is fibrant if and only if $\mathcal{F}[M]$ is fibrant in $\mathbf{Shv}(X, \mathbf{sSets})$ for each manifold M.

Proof. First note that representable presheaves are cofibrant (in either model structure), and so is the homotopy colimit of a diagram of coproducts of representable presheaves. That is, both the source and target of any hypercover are cofibrant. Now, the first claim follows from [13, 3.3.20] and lemma 2.2.6. The second claim follows from the first and adjointness of tensor and $-[M]$.

□

Lemma 2.3.8. *Let X be a topological space, and let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of smooth rings on X. Then ϕ is a weak equivalence if and only if for each open set $U \subset X$ and each manifold M, $\phi(U, M) : \mathcal{F}(U, M) \to \mathcal{G}(U, M)$ is a weak equivalence of simplicial sets.*

Proof. Recall (see [13, 3.2.13]) that localizing a model category does not change the weak equivalences between local objects. Since sheaves of smooth rings are local objects (i.e. they are defined to be fibrant in the localization), we check weak equivalences between them objectwise.

□

Example 2.3.9. Suppose that $U_\bullet \to rX$ is a hypercover on a topological space X, that $\mathcal{F}$ is a sheaf on X, and that M is a manifold. Then $\mathcal{F}(U_\bullet, M)$ is a bisimplicial set. It satisfies the formula

$$\mathcal{F}(X, M) \simeq \operatorname{holim}(\mathcal{F}(U_\bullet, M)).$$

Furthermore, we have

$$\operatorname{holim}(\mathcal{F}(U_\bullet, M)) \simeq \operatorname{Map}(H_M, \operatorname{holim}(\mathcal{F}(U_\bullet))) \simeq \operatorname{holim}(\mathcal{F}(U_\bullet))(M),$$

so in fact

$$\mathcal{F}(X, -) \simeq \operatorname{holim}(\mathcal{F}(U_\bullet))$$

is a weak equivalence of smooth rings.

Proposition 2.3.10. *Let M be a manifold, and let $\mathcal{H}_M$ be the structure presheaf of smooth rings on M. Then $\mathcal{H}_M$ is in fact a sheaf.*

Proof. The cofibrations in $\mathbf{Shv}(X, \mathbf{SR})$ are the same as those in $\mathbf{Pre}(X, \mathbf{SR})$, so $\mathcal{H}_M$ is cofibrant. To show that $\mathcal{H}_M$ is fibrant in $\mathbf{Shv}(X, \mathbf{SR})$, it suffices to show that $\mathcal{H}_M[P]$ is fibrant in $\mathbf{Shv}(X, \mathbf{sSets})$ for any manifold P, by Proposition 2.3.7.

Let $M' \subset M$ be an open set, and let $U_\bullet \to rM'$ be a hypercover. We must show that the map

$$\operatorname{Hom}(M', P) \longrightarrow \operatorname{holim}\Big(\prod_i \operatorname{Hom}(U_i, P) \rightrightarrows \prod_{i,j} \operatorname{Hom}(U_{ij}, P) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

is a weak equivalence. Each map in the homotopy limit diagram is a map between finite sets, hence a fibration. Thus the homotopy limit can be calculated simply as the limit, which in turn can be calculated as the equalizer of the truncated 2-term diagram. The result follows from the fact that one can glue together maps of submanifolds that agree on (the covers of) the pairwise intersections.

∎

Let $a : \mathbf{Pre}(X, \mathbf{SR}) \to \mathbf{Pre}(Y, \mathbf{SR})$ be a left Quillen functor. If a carries hypercovers on X to hypercovers on Y, then a induces a left Quillen functor, which we shall also denote by $a : \mathbf{Shv}(X, \mathbf{SR}) \to \mathbf{Shv}(Y, \mathbf{SR})$, by [13, 3.3.20]. This is not much of an abuse of notation, since **Pre** and **Shv** are equal as categories (with different model structures) and the two versions of a are equal as functors.

Proposition 2.3.11.

1. *Let $f : X \to Y$ be a morphism of topological spaces. Then $f^* : \mathbf{Pre}(Y, \mathbf{SR}) \to \mathbf{Pre}(X, \mathbf{SR})$ extends to a left Quillen functor*

$$f^* : \mathbf{Shv}(Y, \mathbf{SR}) \to \mathbf{Shv}(X, \mathbf{SR}).$$

2. *Let $i : U \subset X$ be the inclusion of an open subset in a topological space. Then $i_! : \mathbf{Pre}(U, \mathbf{SR}) \to \mathbf{Pre}(X, \mathbf{SR})$ extends to a left Quillen functor*

$$i_! : \mathbf{Shv}(U, \mathbf{SR}) \to \mathbf{Shv}(X, \mathbf{SR}),$$

hence

3. *$i^* : \mathbf{Shv}(X, \mathbf{SR}) \to \mathbf{Shv}(U, \mathbf{SR})$ is both a left and a right Quillen functor.*

Proof. Both f^* and $i_!$ take hypercovers to hypercovers. The result follows from lemmas 2.2.8 and 2.2.9.

$\square$

Corollary 2.3.12. *Let X be a topological space and $i : U \subset X$ an open subset. If $\mathcal{F}$ is a sheaf of smooth rings on X then the restriction $i^*\mathcal{F}$ is a sheaf of smooth rings on U.*

Proof. The functor $i^* : \mathbf{Shv}(X, \mathbf{SR}) \to \mathbf{Shv}(U, \mathbf{SR})$ is a right Quillen functor by Proposition 2.3.11. Therefore, fibrant objects in $\mathbf{Shv}(X, \mathbf{SR})$ are sent under i^* to fibrant objects in $\mathbf{Shv}(U, \mathbf{SR})$.

$\square$

Definition 2.3.13. We call a pair $\mathcal{X} = (X, \mathcal{O}_X)$ a *smooth-ringed space*, if X is a T_1 topological space (i.e. points are closed in X) and $\mathcal{O}_X$ is a sheaf of smooth rings on X. We call $\mathcal{O}_X$ the *structure sheaf* of $\mathcal{X}$.

Let $\mathcal{X} = (X, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, \mathcal{O}_Y)$ be smooth-ringed spaces. We define the *mapping space* $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ as

$$\mathrm{Map}(\mathcal{X}, \mathcal{Y}) = \coprod_{f : X \to Y} \mathrm{Map}(f^*\mathcal{O}_Y, \mathcal{O}_X).$$

A pair $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is called a *morphism of smooth-ringed spaces* if it is a vertex in the simplicial set $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$.

Example 2.3.14. There is an isomorphism of categories

$$\mathbf{Pre}(\{*\}, \mathbf{SR}) \cong \mathbf{SR}$$

between the category of presheaves of smooth rings on a point $\{*\}$ and the category of smooth rings; we tend not to distinguish between them.

If X is a topological space and $t : X \to \{*\}$ is the terminal map out of X, then for a cofibrant-fibrant smooth ring F, t^*F is a sheaf of smooth rings on X, and (X, t^*F) is an example of a smooth-ringed space.

Example 2.3.15. Let M be a manifold, and let $\mathcal{H}_M$ be the structure sheaf of smooth rings on M (see proposition 2.3.10). Then $(M, \mathcal{H}_M)$ is the fundamental example of a smooth-ringed space. In fact, we often abbreviate $(M, \mathcal{H}_M)$ simply by M.

So far, we have chosen to work with $\mathbf{Shv}(X, \mathbf{SR})$ from the point of view of [6], i.e. as a localization of the injective model structure on $\mathbf{Pre}(X, \mathbf{SR})$ at the set of hypercovers. However, Jardine's original work on simplicial presheaves did not use this presentation. Instead, a map $f : \mathcal{F} \to \mathcal{G}$ of presheaves of simplicial sets was defined to be a weak equivalence if it induces isomorphisms on all the homotopy sets. This version of the story will have its advantages later, so we recall the definition of homotopy sets for a presheaf.

Definition 2.3.16. Let X be a topological space and let $\mathcal{F}$ be a sheaf of simplicial sets on X. Let $\pi_0(\mathcal{F})$ denote the sheafification (in the usual sense) of the presheaf of sets $U \mapsto \pi_0(\mathcal{F}(U))$. Let $x : \Delta^0 \to \mathcal{F}(X)$ be a point in the global sections of $\mathcal{F}$. For $i > 0$, define $\pi_i(\mathcal{F}, x)$ to be the sheafification (in the usual sense) of the presheaf of sets $U \mapsto \pi_i(\mathcal{F}(U), x|_U)$.

Remark 2.3.17. The fact that for $i > 0$, $\pi_i(\mathcal{F}, x)$ is a group will not concern us.

Remark 2.3.18. We will see in the next chapter that for nice sheaves $\mathcal{F}$ of smooth rings (i.e. sheaves that satisfy that "locality condition"), there are no maps from the terminal sheaf $H_\emptyset$ into $\mathcal{F}$. That is why we do not define "global" higher homotopy sets $\pi_i, i > 0$ above. The most we will need is homotopy groups for the various $\mathcal{F}[M]$, which are defined in definition 2.3.16.

Theorem 2.3.19. *Let X be a topological space, and let $\phi : \mathcal{F} \to \mathcal{G}$ be a map of sheaves of simplicial sets on X. Then ϕ is a weak equivalence in* $\mathbf{Shv}(X, \mathbf{sSets})$ *if and only if*

$$\pi_0(\phi) : \pi_0(\mathcal{F}) \to \pi_0(\mathcal{G})$$

is a bijection and, for all $i \geq 0$ and all $x : \Delta^0 \to \mathcal{F}(X)$, the maps

$$\pi_i(\phi) : \pi_i(\mathcal{F}, x) \to \pi_i(\mathcal{G}, x)$$

are bijective.

Proof. This is [6, Theorem 6.2]. □

Definition 2.3.20. Suppose that I is a small category, let $\mathcal{C}$ denote the simplicial category of smooth-ringed spaces, and let $\mathcal{D} : I \to \mathcal{C}$ be a diagram. Suppose that $\mathcal{X}$ is a smooth-ringed space equipped with a natural transformation $\underline{\mathcal{X}} \to \mathcal{D}$, where $\underline{\mathcal{X}} : I \to \mathcal{C}$ is the constant diagram. We call $\mathcal{X}$ a *homotopy limit* of $\mathcal{D}$ if for all smooth-ringed spaces $\mathcal{Y}$, the induced map

$$\mathrm{Map}(\mathcal{Y}, \mathcal{X}) \to \operatorname*{holim}_{i \in I} \mathrm{Map}(\mathcal{Y}, \mathcal{D}_i)$$

is a weak equivalence.

Proposition 2.3.21. *Finite homotopy limits exist in the simplicial category of smooth-ringed spaces.*

Proof. Let I be a finite category and $\mathcal{D}$ an I-shaped diagram of smooth-ringed spaces. This is obvious when I is the empty category: the limit is $(\bullet, H_\bullet)$. Thus we may assume that $I = \Lambda^2_2$ is the category with three objects and two non-identity morphisms, depicted as $(\cdot \to \cdot \leftarrow \cdot)$.

Limits exist in the category of topological spaces, and homotopy colimits exist in the category of sheaves of smooth rings on a space. For each $\mathcal{D}_i$, let D_i be the underlying topological space and let $\mathcal{O}_{D_i}$ be the structure sheaf on D_i. Let $X = \lim_i D_i$ with structure maps $p_i : X \to D_i$, and let $\mathcal{O}_X = \operatorname{hocolim}_i p_i^* \mathcal{O}_{D_i}$.

We need to show that the map

$$\coprod_{f:Y\to\lim_i D_i} \operatorname{holim}_i(\mathrm{Map}(f^*p_i^*\mathcal{O}_{D_i},\mathcal{O}_Y) \to \operatorname{holim}_i \coprod_{f_i:Y\to D_i} \mathrm{Map}(f_i^*\mathcal{O}_{D_i},\mathcal{O}_Y)$$

is a weak equivalence. One can see that the mapping spaces in the diagram are fibrant (and so are the coproducts of such), so the homotopy limits are just limits. The limit of a diagram of simplicial sets is taken levelwise, so we may assume that the mapping spaces are sets. In the following, A_i will denote $\mathrm{Hom}(Y, D_i)$ and $M(i, f_i)$ will denote $\mathrm{Map}(f_i^*\mathcal{O}_{D_i},\mathcal{O}_Y)$.

The problem then reduces to the following. Let $A : I \to \mathbf{Sets}$ be a diagram and let LA be the Grothendieck construction. That is, LA is a category whose set of objects is

$$\mathrm{Ob}(LA) = \{(i,a)|i \in I, a \in A(i)\}$$

and where $\mathrm{Hom}((i,a),(j,b)) = \{x : i \to j|A(x) : a \mapsto b\}$. Let $B : LA \to \mathbf{Sets}$ be another diagram. Finally, let $p : I \times \lim_{i\in I} A_i \to LA$ be the functor which sends a system to its ith component. We must show that the map of sets

$$\coprod_{f\in\lim_j A_j} \lim_{i\in I} Bp(i,f) \longrightarrow \lim_{i\in I} \coprod_{f_i\in A_i} B(i,f_i)$$

is an isomorphism. This is proven in lemma 2.3.23 below.

□

Example 2.3.22. Above, in example 2.1.12, we discussed the smooth ring F : $\mathbf{Man} \to \mathbf{sSets}$ given by taking a manifold M to its set of tangent vectors $\{m, \alpha|m \in M, \alpha \in T_mM\}$. To conclude that F is almost fibrant, we need the following isomorphism:

$$\coprod_{x\in X}(T_xM \times_{T_xN} T_x(P)) \cong \left(\coprod_{m\in M} T_mM\right) \times_{(\coprod_{n\in N} T_nN)} \left(\coprod_{p\in P} T_pP\right).$$

This follows from lemma 2.3.23, with $I = \Lambda_2^2$ the category depicted as

$$\begin{tikzcd} & 0 \arrow[d] \\ 1 \arrow[r] & 2, \end{tikzcd}$$

and with $A : I \to \mathbf{Sets}$ given by letting A_0 be the set of points of M, A_1 the set of points of P, and A_2 the set of points of N. Finally, $B : LA \to \mathbf{Sets}$ takes a point in M, N, or P to its set of tangent vectors.

Let $\mathcal{C}$ be a category, let $\mathbf{Cat}$ be the category of small categories, and let $F : \mathcal{C} \to \mathbf{Cat}$ be a functor. Recall that the Grothendieck construction of F is the category LF whose objects are pairs (c, a), where $c \in \mathcal{C}$ and $a \in F(c)$ are objects, and whose morphisms are given by letting $\mathrm{Hom}_{LF}((c, g),(d, h))$ be the set of pairs (α, ϕ) where $\alpha : c \to d$ is a morphism of $\mathcal{C}$ and $\phi : F(\alpha)(g) \to h$ is a morphism in the category $F(d)$.

In particular, if F factors through the inclusion **Sets** $\to$ **Cat**, then LF takes an easier form because there are no non-identity morphisms in **Sets**. Thus for objects (c, g) and (d, h) in LF, one may identify $\mathrm{Hom}((c, g), (d, h))$ with the subset of $\mathrm{Hom}_{\mathcal{C}}(c, d)$ consisting of morphisms $f : c \to d$ for which $F(f)(g) = h$.

Lemma 2.3.23. *Let Λ^2_2 be the category with three objects and two non-identity arrows, depicted as $(\cdot \to \cdot \leftarrow \cdot)$. Suppose that $I = \Lambda^2_2$, or that I is the empty category, or that $I = [0]$. Let $A : I \to$ **Sets** be a diagram, let LA be the Grothendieck construction of A, and let $B : LA \to$ **Sets** be another diagram. Finally, let $p : I \times \lim_{i\in I} A_i \to LA$ be the functor which sends a system to its ith component. Then the map of sets*

$$\coprod_{f\in \lim_j A_j} \lim_{i\in I} Bp(i, f) \longrightarrow \lim_{i\in I} \coprod_{f_i\in A_i} B(i, f_i)$$

is an isomorphism.

Proof. It is easy to see that the lemma holds when I is the empty category and when $I = [0]$.

The usual distributive for sets implies that we may assume $A_0 = \{x_0\}$ and $A_1 = \{x_1\}$ are singleton sets. Let y_0 be the image of x_0 and let y_1 be the image of x_1 in A_2. We have reduced to showing that the set

$$B_{x_0} \times_{\amalg_{y\in A_2} B_y} B_{x_1}$$

is isomorphic to $B_{x_0} \times_{B_{y_0}} B_{x_1}$ if $y_0 = y_1$ and is the empty set if $y_0 \neq y_1$. This follows from the fact that the maps $B_{x_0} \to \coprod_{y\in A_2} B_y$ takes the B_{x_0} into only one component of the coproduct, and similarly for B_{x_1}.

□

Example 2.3.24. Lemma 2.3.23 is called the generalized distribuitive law because it generalizes the fact that for sets X, Y, and Z, the map

$$(X \times Y) \amalg (X \times Z) \to X \times (Y \amalg Z)$$

is an isomorphism. Indeed, take $A_0 = \{x\}, A_1 = \{y, z\}$, and $A_2 = \{a\}$; and take $B_{0,x} = X, B_{1,y} = Y, B_{1,z} = Z$, and $B_{2,a} = \{*\}$.

Notation 2.3.25. Suppose that $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ are smooth-ringed spaces, and $\mathcal{X}$ is the (homotopy) fiber product in a diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "e"] \arrow[d, "f"'] \arrow[dr, phantom, "\diamond"] & \mathcal{A} \arrow[d, "g"] \\ \mathcal{B} \arrow[r, "h"'] & \mathcal{C}. \end{tikzcd}$$

Then $\mathcal{O}_X$ is the homotopy colimit in the diagram

$$\begin{tikzcd} e^*g^*\mathcal{O}_C \arrow[r, "e^*g^\flat"] \arrow[d, "f^*h^\flat"'] & e^*\mathcal{O}_A \arrow[d, "e^\flat"] \arrow[dl, phantom, "\diamond"] \\ f^*\mathcal{O}_B \arrow[r, "f^\flat"'] & \mathcal{O}_X . \end{tikzcd}$$

However, for ease of notation, we suppress the inverse-image functors and simply write

$$\begin{tikzcd} \mathcal{O}_C \arrow[r] \arrow[d] & \mathcal{O}_A \arrow[d] \arrow[dl, phantom, "\diamond"] \\ \mathcal{O}_B \arrow[r] & \mathcal{O}_X \end{tikzcd}$$

to denote this homotopy colimit.

Our notion of smooth-ringed space is analogous to the notion of ringed space from algebraic geometry. In the next chapter, we will introduce the analogue of local-ringed spaces. We conclude this chapter by returning to our running example, first discussed in Example 2.1.13.

Example 2.3.26. Let $\mathcal{G}$ be the homotopy limit of smooth ringed spaces in the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "p"'] & \mathbb{R}, \end{tikzcd}$$

where $p : \bullet \to \mathbb{R}$ is the point corresponding to the origin in $\mathbb{R}$. We call $\mathcal{G}$ the squared origin. (Note that $\mathcal{G}$ should *not* be thought of in the way that one thinks of $\mathrm{Spec}(k[\epsilon]/\epsilon^2)$, i.e. as "the origin plus a tangent vector." In fact, $\mathcal{G}$ is contained as a codimension 1 closed subspace of the point. See example 4.1.12.) The underlying space G of $\mathcal{G}$ is $\{*\}$. The structure sheaf $\mathcal{O}_G$ on $\mathcal{G}$ is the homotopy colimit in the diagram

$$\begin{tikzcd} p^*\mathcal{H}_{\mathbb{R}} \arrow[r] \arrow[d] & H_\bullet \arrow[d] \arrow[dl, phantom, "\diamond"] \\ H_\bullet \arrow[r] & \mathcal{O}_G . \end{tikzcd}$$

Note that p^* commutes with homotopy colimits and that p^*p_* is isomorphic to the identity functor for sheaves on $\{*\}$. Therefore, $\mathcal{O}_G$ is isomorphic to $p^*(G)$, where G is the smooth ring from example 2.1.13.

Chapter 3

Local Smooth-Ringed Spaces

In this chapter, we define a category **LRS** of local smooth-ringed spaces, and prove several theorems which will be useful later. Of greatest importance is the "structure theorem," (Theorem 3.3.3) which relates the sheaf of smooth rings $\mathcal{O}_X$ on a local smooth-ringed space $\mathcal{X}$ to the maps out of $\mathcal{X}$. Precisely, the structure theorem states that for any local smooth-ringed space $\mathcal{X}$ and any manifold M, one has a weak equivalence of simplicial sets

$$\mathrm{Map}_{\mathbf{LRS}}(\mathcal{X}, M) \simeq \mathcal{O}_X(X, M).$$

3.1 Definition of local smooth-ringed space

The structure theorem provides a strong connection between the sheaf $\mathcal{O}_X$ on a local smooth-ringed space $\mathcal{X} = (X, \mathcal{O}_X)$ and the topology of X. A necessary condition for the structure theorem to hold is that for any manifold M, open cover $\cup_\alpha M_\alpha = M$, and morphism $f : \mathcal{X} \to M$, the preimages $\mathcal{X}_\alpha = f^{-1}M_\alpha$ must cover $\mathcal{X}$. We take a variant of this condition to be the defining condition for local smooth-ringed spaces (see Remark 3.1.9).

Definition 3.1.1. Let X be a topological space and let $\mathcal{F}$ be a sheaf of smooth rings on X. We say that $\mathcal{F}$ is *a local sheaf on* X if, for every open cover $\cup_\alpha M_\alpha = M$ of a smooth manifold M, the natural map of sheaves of sets

$$\pi_0 \left(\coprod_\alpha \mathcal{F}(-, M_\alpha) \right) \longrightarrow \pi_0 \mathcal{F}(-, M)$$

is a surjection.

Let $\mathcal{G}$ be a sheaf of smooth rings on X, and let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. We say that ϕ is *a local morphism of sheaves on* X if, for any open subset $i : M_\alpha \subset M$ the diagram

$$\begin{tikzcd}
\pi_0\mathcal{F}(-,M_\alpha) \arrow[r, "\phi"] \arrow[d, "i"'] \arrow[dr, phantom, "\ulcorner", very near start] & \pi_0\mathcal{G}(-,M_\alpha) \arrow[d, "i"] \\
\pi_0\mathcal{F}(-,M) \arrow[r, "\phi"'] & \pi_0\mathcal{G}(-,M)
\end{tikzcd}$$

is a fiber product of sheaves of sets. Let $\mathrm{Map}_{\mathrm{loc}}(\mathcal{F},\mathcal{G})$ denote the subcomplex of $\mathrm{Map}(\mathcal{F},\mathcal{G})$ consisting of those simplices whose vertices are local morphisms.

We say that a pair $\mathcal{X} = (X, \mathcal{O}_X)$ is a *local smooth-ringed space* if $\mathcal{O}_X$ is a local sheaf of smooth rings on X. Let $\mathcal{Y} = (Y, \mathcal{O}_Y)$. We say that a morphism of smooth-ringed spaces $(\phi, \phi^\flat) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a *local morphism* if $\phi^\flat : \phi^*\mathcal{O}_Y \to \mathcal{O}_X$ is a local morphism of sheaves on X. Let

$$\mathrm{Map}_{\mathbf{LRS}}(\mathcal{X},\mathcal{Y}) = \coprod_{\phi: X\to Y} \mathrm{Map}_{\mathrm{loc}}(\phi^*\mathcal{O}_Y, \mathcal{O}_X);$$

we refer to it as the *space of local morphisms* from $\mathcal{X}$ to $\mathcal{Y}$.

We define the *category of local smooth-ringed spaces*, denoted **LRS**, to be the category whose objects are local smooth-ringed spaces and whose morphism spaces are spaces of local morphisms.

Remark 3.1.2. Note that the locality condition in Definition 3.1.1 includes the case in which M is empty. Since the empty set can be written as the empty union, a local sheaf of smooth rings $\mathcal{F}$ is required to satisfy the equation

$$\mathcal{F}(V, \emptyset) = \emptyset$$

for any open set V.

The following definition should be read with the structure theorem in mind. It says that preimages of an open cover form an open cover. See Remark 3.1.9.

Lemma 3.1.3. *Let X be a topological space, and suppose that $\mathcal{F}$ is a sheaf of smooth rings on X. Then $\mathcal{F}$ is local if and only if the following condition is satisfied for all manifolds M, open sets $V \subset X$, and elements $f \in \pi_0\mathcal{F}(V, M)$:*

- *If A is a set and $\{i_\alpha : M_\alpha \subset M\}_{\alpha\in A}$ is a cover (i.e. $\cup_{\alpha\in A} M_\alpha = M$),*
- *then there exist open subsets $\rho_\alpha : V_\alpha \subset V$ with $\cup_{\alpha\in A} V_\alpha = V$ and elements $f^\alpha \in \pi_0\mathcal{F}(V_\alpha, M_\alpha)$ such that the diagram*

$$\begin{tikzcd}
\{*\} \arrow[r, "f"] \arrow[d, "f^\alpha"'] & \pi_0\mathcal{F}(V,M) \arrow[d, "\rho_\alpha"] \\
\pi_0\mathcal{F}(V_\alpha, M_\alpha) \arrow[r, "i_\alpha"'] & \pi_0\mathcal{F}(V_\alpha, M)
\end{tikzcd}$$

commutes.

Proof. This follows by definition.

□

Definition 3.1.4. With notation as given in Lemma 3.1.3, we refer to an element $f^\alpha \in \pi_0\mathcal{F}(V_\alpha, M_\alpha)$ which makes the diagram commute as *a restriction of f to V_α over M_α*.

Lemma 3.1.5. *Let X be a topological space, $V \subset U \subset X$ open subsets, $\mathcal{F}$ a sheaf of smooth rings on X, $i : M_\alpha \subset M$ the inclusion of an open submanifold, and $f \in \pi_0\mathcal{F}(U, M)$. Then there exists a restriction of f to V over M_α if and only if, for any representative $f_0 \in \mathcal{F}(U, M)_0$ of f, the homotopy fiber of $\mathcal{F}(V, M_\alpha) \to \mathcal{F}(V, M)$ over f_0 is contractible.*

Proof. Let Res denote the homotopy fiber in the diagram

$$\begin{tikzcd}
Res \arrow[r] \arrow[dd] & \Delta^0 \arrow[d, "f_0"] \\
 & \mathcal{F}(U, M) \arrow[d] \\
\mathcal{F}(V, N) \arrow[r] & \mathcal{F}(V, M).
\end{tikzcd}$$

It is either empty or contractible by Lemma 2.1.15. It is clear that a restriction of f to V exists over M_α if and only if the restriction space Res is nonempty.

□

Lemma 3.1.6. *Let X be a topological space, $U \subset X$ an open subset, $\mathcal{F}$ a sheaf on X, and $i : M_\alpha \subset M$ the open inclusion of a submanifold. If $f \in \pi_0\mathcal{F}(U, M)$ is an element and $\{V_j \subset U \mid j \in J\}$ is a set of open subsets for which restrictions $f_j^\alpha \in \mathcal{F}(V_j, M_\alpha)$ of f exist, then there exists a restriction f^α of f to the union $V = \cup_j V_j$ over M_α.*

Proof. Let $f_0 \in \mathcal{F}(U, M)$ be a representative of f. For each $j \in J$, the homotopy fiber of $\mathcal{F}(V_j, M_\alpha) \to \mathcal{F}(V_j, M)$ over f_0 is contractible. Let $V_\bullet$ denote the Čech nerve of the cover $\cup_j V_j = V$. The homotopy fiber of $\mathcal{F}(V_\bullet, M_\alpha) \to \mathcal{F}(V_\bullet, M)$ is contractible, so by the sheaf condition the homotopy fiber of $\mathcal{F}(V, M_\alpha) \to \mathcal{F}(V, M)$ is also contractible. Hence a restriction $f^\alpha \in \mathcal{F}(V, M_\alpha)$ exists.

□

Remark 3.1.7. Note that Lemma 3.1.6 cannot be proven by simply gluing together sections of the sheaf $\pi_0\mathcal{F}(-, M_\alpha)$, because $\pi_0(\mathcal{F}(V, M) \neq (\pi_0\mathcal{F}(-, M))(V)$. There is sheafification involved in the definition of π_0.

Definition 3.1.8. Let X be a topological space, $U \subset X$ an open subset, $\mathcal{F}$ a sheaf of smooth rings on X, and let $i_\alpha : M_\alpha \subset M$ be the inclusion of an open submanifold. For any component $f \in \pi_0\mathcal{F}(U, M)$, there exists a maximal subset $V \subset U$ (possibly empty) on which a restriction f^α of f exists over M_α

(see Lemma 3.1.6). We call V *the preimage of* M_α *under* f, and we write $V = f^{-1}(N)$.

Remark 3.1.9. Thus, Lemma 3.1.3 says that $\mathcal{F}$ is local if and only if for any $f \in \mathcal{F}(U, M)$ and any open cover $\cup_\alpha M_\alpha$, the preimages $f^{-1}(M_\alpha)$ cover U.

Definition 3.1.10. Suppose that F is a smooth ring. We say that F is a *local smooth ring* if it is local as a sheaf on the one point topological space $\{*\}$.

Lemma 3.1.11. *A smooth ring F is local if and only if the following condition is satisfied. Let M be a manifold, $\cup_\alpha M_\alpha = M$ an open cover, and let $i_\alpha : M_\alpha \subset M$ denote the inclusions. Then for any element $x \in \pi_0 F(M)$, there exists an α and an element $x' \in \pi_0 F(M_\alpha)$ such that $F(i_\alpha)(x') = x$.*

Proof. This follows by definition.

□

Proposition 3.1.12. *Let X be a topological space, $\mathcal{F}$ and $\mathcal{G}$ sheaves of smooth rings on X, and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves. Let $M_1, M_2 \subset M$ be open submanifolds, and let $g \in \pi_0 \mathcal{F}(X, M)$ be an element. Then*

1. $g^{-1}(M_1 \cap M_2) = g^{-1}(M_1) \cap g^{-1}(M_2)$.

2. *If $M_1 \subset M_2$ then $g^{-1}(M_1) \subset g^{-1}(M_2)$.*

3. *If $\rho : U \subset X$ is open then*

$$g^{-1}(M_1) \cap U = \rho(g)^{-1}(M_1),$$

   where $\rho(g) \in \mathcal{F}(U, M)$ is the presheaf restriction.

4. *If M_1 and M_2 do not intersect, then the preimages $g^{-1}(M_1)$ and $g^{-1}(M_2)$ do not intersect.*

5. *ϕ is local if and only if for every $f \in \pi_0 \mathcal{F}(U, M)$ and every open subset $M_\alpha \subset M$, the preimage $f^{-1}(M_\alpha)$ is equal to the preimage $h^{-1}(M_\alpha)$, where $h = \phi(f) \in \pi_0 \mathcal{G}(U, M)$.*

6. *If $\mathcal{G}$ is a local sheaf and ϕ is a local map then $\mathcal{F}$ is a local sheaf.*

7. *$\mathcal{F}$ is local if and only if for each point $x : \{*\} \to X$, the stalk $x^*\mathcal{F}$ is a local smooth ring.*

8. *ϕ is local if and only if for each point $x : \{*\} \to X$, the induced map $x^*\phi : x^*\mathcal{F} \to x^*\mathcal{G}$ is local.*

9. *If Y is a topological space, $a : Y \to X$ is continuous, and $\mathcal{F}$ is local on X, then $a^*\mathcal{F}$ is local on Y. If ϕ is local on X then $a^*\phi$ is local on Y.*

10. *ϕ is a weak equivalence if and only if for each point $x : \{*\} \to X$, the map $x^*(\phi) : x^*\mathcal{F} \to x^*\mathcal{G}$ is a weak equivalence of smooth rings.*

Proof. 1. The containment $g^{-1}(M_1 \cap M_2) \subset g^{-1}(M_1) \cap g^{-1}(M_2)$ is easy: a restriction over $M_1 \cap M_2$ gives a restriction over M_1 and a restriction over M_2.

The other containment follows from the fibrancy of $\mathcal{F}$, which implies that the square

$$\begin{tikzcd}
\mathcal{F}(V, M_1 \cap M_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{F}(V, M_1) \arrow[d] \\
\mathcal{F}(V, M_2) \arrow[r] & \mathcal{F}(V, M)
\end{tikzcd}$$

is homotopy Cartesian, for any $V \subset X$. Let $V_1 = g^{-1}(M_1)$ and $V_2 = g^{-1}(M_2)$. There exists a homotopy commutative diagram

$$\begin{tikzcd}
\Delta^0 \arrow[r] \arrow[d] & \mathcal{F}(V_1 \cap V_2, M_1) \arrow[d] \\
\mathcal{F}(V_1 \cap V_2, M_2) \arrow[r] & \mathcal{F}(V_1 \cap V_2, M),
\end{tikzcd}$$

which induces a map $\Delta^0 \to \mathcal{F}(V_1 \cap V_2, M_1 \cap M_2)$, as desired.

2. This assertion follows from the previous one.

3. This assertion follows easily from the definition of preimage and the sheaf condition on $\mathcal{F}$.

4. Since M_1 and M_2 do not intersect, the diagram

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & M_1 \arrow[d] \\
M_2 \arrow[r] & M
\end{tikzcd}$$

is a fiber product of manifolds. Since $\mathcal{F}$ is fibrant, the diagram

$$\begin{tikzcd}
\mathcal{F}(V, \emptyset) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{F}(V, M_1) \arrow[d] \\
\mathcal{F}(V, M_2) \arrow[r] & \mathcal{F}(V, M)
\end{tikzcd}$$

is a homotopy pullback of simplicial sets for any $V \subset X$. By Remark 3.1.2, $\mathcal{F}(V, \emptyset) = \emptyset$, which implies that the preimages do not intersect.

5. This follows from Proposition 2.3.6 and Lemma 3.1.5.

6. This assertion follows from the previous and Remark 3.1.9.

7. The surjectivity of a morphism of sheaves of sets can be checked on stalks.

8. The functor x^* is a left adjoint that commutes with π_0 and preserves finite limits; this proves one implication. For the other implication, note that a diagram of sheaves of sets is a fiber product diagram if and only if its image in each stalk is a fiber product diagram of sets.

9. This assertion follows from the previous two.

10. By Lemma 2.3.8, the map ϕ is a weak equivalence if and only if for each manifold M, $\phi(-,M) : \mathcal{F}(-,M) \to \mathcal{F}(-,N)$ is a weak equivalence. This is so if and only if it induces a bijection on homotopy sets

$$\pi_i(\phi(-,M),p) : \pi_i(\mathcal{F}(-,M),p) \longrightarrow \pi_i(\mathcal{G}(-,M),p)$$

for all base points $p \in \mathcal{F}(X,M)_0$. This is something that can be checked on stalks, and the result follows.

□

3.2 Manifolds and other examples

Lemma 3.2.1. *Let M be a manifold and $U \subset M$ an open submanifold, and let $Q_\alpha \subset Q$ also be the inclusion of an open submanifold. An element $f \in \pi_0\mathcal{H}_M(U,Q)$ can be identified with a smooth map $f : U \to Q$. The preimage of Q_α under f, as defined in Definition 3.1.8, is the same as the manifold-theoretic preimage of Q_α under f.*

Proof. We may identify $\mathcal{H}_M$ with $\pi_0\mathcal{H}_M$, since it is a discrete sheaf of smooth rings. Recall that the preimage of Q_α under f is defined to be the largest subset $V \subset U$ for which there exists an $f^\alpha \in \pi_0\mathcal{H}_M(V,Q_\alpha)$ and a commutative diagram of the form

$$\begin{tikzcd} \{*\} \arrow[r, "f"] \arrow[d, "f^\alpha"'] & \mathrm{Hom}_{\mathbf{Man}}(U,Q) \arrow[d] \\ \mathrm{Hom}_{\mathbf{Man}}(V,Q_\alpha) \arrow[r] & \mathrm{Hom}_{\mathbf{Man}}(V,Q). \end{tikzcd}$$

Clearly, V must be the fiber product in the diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Q_\alpha \arrow[d, "i"] \\ U \arrow[r, "f"'] & Q; \end{tikzcd}$$

i.e. it is the preimage of Q_α under f in the category of manifolds.

□

Proposition 3.2.2. *Let M be a manifold. Then $(M, \mathcal{H}_M)$ is a local smooth-ringed space.*

Proof. By Proposition 2.3.10, $\mathcal{H}_M$ is a sheaf of smooth rings on M; we must show it satisfies the locality condition.

If $f : M \to Q$ is any morphism of manifolds, then the manifold-theoretic preimage of a cover of Q is a cover of M. The result then follows from Lemma 3.2.1 and Remark 3.1.9.

□

Proposition 3.2.3. *Let M and N be manifolds, and let $g : M \to N$ be a smooth map. There is a natural map*

$$g^\flat : g^*\mathcal{H}_N \to \mathcal{H}_M,$$

given by composition with g, such that $(g, g^\flat) : (M, \mathcal{H}_M) \to (N, \mathcal{H}_N)$ is a morphism of local smooth-ringed spaces. Moreover, we have

$$\mathrm{Map}((M, \mathcal{H}_M), (N, \mathcal{H}_N)) = \mathrm{Map}_{\mathbf{Man}}(M, N),$$

where $\mathrm{Map}_{\mathbf{Man}}(M, N)$ *is the constant simplicial set of smooth maps $M \to N$. In other words, the inclusion i : **Man** $\to$ **LRS** is a fully faithful functor of simplicial categories.*

Proof. Let $U \subset M$ be an open subset; let P be a manifold. For each open $V \subset N$ such that $U \subset g^{-1}(V)$, there is a map

$$\mathrm{Hom}(V, P) \to \mathrm{Hom}(U, P)$$

given by composition with g. This defines $g^\flat$ on $g^*\mathcal{H}_N(U)$. We must show that $g^\flat$ is local.

Let $P \subset Q$ be the inclusion of an open submanifold. An element $f \in g^*\mathcal{H}_N(M, Q)$ is represented by an open subset $W \subset N$ with $g^{-1}(W) = M$, and a map $f : W \to Q$. By Lemma 3.2.1, the locality of $g^\flat$ follows from the fact that $g^{-1}(f^{-1}(P)) = (f \circ g)^{-1}(P)$ and Proposition 3.1.12.

We now show that i is fully faithful. It suffices to show that for any map of manifolds $g : M \to N$, the space $M = \mathrm{Map}_{\mathrm{loc}}(g^*\mathcal{H}_N \to \mathcal{H}_M)$ is contractible. It is clearly discrete, so we must show that it consists of exactly one point. Choose a local map $\phi : g^*\mathcal{H}_N \to \mathcal{H}_M$, and let f denote the induced map $M \to N$ given by taking global sections of ϕ and applying Yoneda's lemma; we must show that $f = g$.

The maps $f, g : M \to N$ are equal if and only if, for every $p \in N$, the preimages $f^{-1}(p)$ and $g^{-1}(p)$ are equal as subsets of M. It suffices to show that for every neighborhood $V_i \ni p$, the preimages $f^{-1}(V_i)$ and $g^{-1}(V_i)$ are equal. But this follows from Proposition 3.1.12.

□

Proposition 3.2.4. *There is a pair of adjoint functors*

$$L : \mathbf{Top} \rightleftarrows \mathbf{LRS} : U,$$

where U is the functor which takes a local smooth-ringed space $(X, \mathcal{O}_X)$ to its underlying topological space X.

Proof. If X is a topological space, define a sheaf $\mathcal{O}_X$ on open sets U by

$$\mathcal{O}_X(U, M) = \mathrm{Hom}_{\mathbf{Top}}(U, M).$$

Define $L(X) = (X, \mathcal{O}_X)$. It is easy to check that $\mathcal{O}_X$ is a sheaf of smooth rings and that it satisfies the locality condition.

Suppose that $(Y, \mathcal{O}_Y)$ is a local smooth-ringed space. To give a map on topological spaces $f : X \to Y$ uniquely determines a local morphism $f^*\mathcal{O}_Y \to \mathcal{O}_X$, by the same argument as in Proposition 3.2.3 (this requires that X is a T_1 topological space). Therefore (L, U) is an adjoint pair.

□

Remark 3.2.5. If M is a manifold note that $L(M)$, as defined in Proposition 3.2.4, has a similar flavor to but is different than $(M, \mathcal{H}_M)$. The structure sheaf of the former involves maps between manifolds *as topological spaces*, whereas the structure sheaf of the latter involves only smooth maps between manifolds.

Proposition 3.2.6. *There is a pair of adjoint functors*

$$U : \mathbf{LRS} \rightleftarrows \mathbf{Top} : R,$$

where U is the functor which takes a local smooth-ringed space $(X, \mathcal{O}_X)$ to its underlying topological space X. The functor R is fully faithful.

Proof. For a topological space X, let $t : X \to \{*\}$ be the terminal map out of X. Define $\mathcal{O}_X = t^*H_\bullet$ and $\mathcal{X} = (X, \mathcal{O}_X)$. Then for any local smooth-ringed space $\mathcal{Y} = (Y, \mathcal{O}_Y)$ and map $f : Y \to X$, the corresponding component of $\mathrm{Map}(\mathcal{Y}, \mathcal{X})$ is

$$\mathrm{Map}(f^*t^*H_\bullet, \mathcal{O}_Y) = \mathcal{O}_Y(Y, \bullet) = \{*\}.$$

It is clear that R is fully faithful.

□

Remark 3.2.7. Suppose X is a topological space. We sometimes speak of it as a local smooth-ringed space, by way of the fully faithful functor $R : \mathbf{Top} \to \mathbf{LRS}$, defined in Proposition 3.2.6.

3.3 The structure theorem

Recall (Proposition 3.2.3) that if $(M, \mathcal{H}_M)$ is a manifold, then for any open set $U \subset M$,

$$\mathcal{H}_M(U, P) = \mathrm{Hom}_{\mathbf{Man}}(U, P) = \mathrm{Map}_{\mathbf{LRS}}(U, P).$$

In other words, the structure sheaf on M classifies maps from M to other manifolds. The structure theorem says that the same is true for any local smooth-ringed space $(X, \mathcal{O}_X)$. It may be that there is an easy proof of this fact, but the one we present is quite technical and relies heavily on a straightening theorem of [20]. In order to use it, we will need a few facts about the Joyal model structure on the category of simplicial sets. See Section A.1 for the definition of this model structure, and a few basic results about it. We also need to use the language of ∞-categories.

Let X be a topological space, and let $\mathcal{U} = \mathrm{Op}(X)^{\mathrm{op}}$. We define a category $\mathcal{P}$ as follows. Objects of $\mathcal{P}$ are inclusions of open subsets $(U \subset V)$. A morphism $(U \subset V) \to (U' \subset V')$ is a pair of inclusions $U' \subset U$, $V \subset V'$. That is, the mapping space between any two objects of $\mathcal{P}$ is either the empty or the one-point set.

For presheaves $\mathcal{F}$, $\mathcal{G}$ on X with values in a simplicial model category $\mathcal{M}$, let $T : \mathcal{P} \to \mathbf{sSets}$ be the functor

$$T(U \subset V) = \mathrm{Map}(\mathcal{F}(V), \mathcal{G}(U)).$$

The obvious map $\mathrm{Map}(\mathcal{F}, \mathcal{G}) \to \lim(T)$ is an isomorphism of simplicial sets. One of the technical issues we encounter when proving the structure theorem is the following. Suppose $\mathcal{F}$ and $\mathcal{G}$ are cofibrant-fibrant presheaves in the injective model structure, and that each $T(U \subset V)$ is contractible. We want to conclude that $\mathrm{Map}(\mathcal{F}, \mathcal{G})$ is contractible. This is indeed the case, because $\mathrm{Map}(\mathcal{F}, \mathcal{G})$ is the *homotopy* limit of a certain diagram of T's, as we show in the proof below.

Proposition 3.3.1. *Let X be a space, let $\mathcal{M}$ be a simplicial model category, and let $\mathcal{U}$ and $\mathcal{P}$ be as above. Let $\mathcal{F}, \mathcal{G} : \mathcal{U} \to \mathcal{M}$ be cofibrant-fibrant presheaves on X, and let $T : \mathcal{P} \to$ **sSets** be as above. If $T(U \subset V)$ is contractible for all $U \subset V$, then $\mathrm{Map}(\mathcal{F}, \mathcal{G})$ is contractible.*

Proof. The underlying ∞-category of $\mathcal{M}$ is $N(\mathcal{M}^\circ)$, the nerve of the subcategory of cofibrant fibrants. By [20, A.3.6.1] there is a categorical equivalence of ∞-categories

$$N((\mathcal{M}^{\mathcal{U}})^\circ) \to \mathrm{Fun}(N(\mathcal{U}), N(M^\circ)).$$

That is, any strict presheaf can be replaced by an equivalent "weak" presheaf. In particular, the simplicial mapping space between objects on the left is weakly equivalent to the simplicial mapping space between their images on the right.

For any map of ∞-categories $i : \mathcal{S} \to N(\mathcal{U})$, define

$$R(i) := \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}, N(M^\circ))}(\mathcal{F} \circ i, \mathcal{G} \circ i).$$

In particular, $R(\mathrm{id}_{N(\mathcal{U})})$ is weakly equivalent to $\mathrm{Map}(\mathcal{F}, \mathcal{G})$. By Lemma A.1.6, R takes homotopy colimits in the Joyal model structure to homotopy limits. By Corollary A.1.5, it suffices to check that $R(i)$ is contractible when $\mathcal{S}$ is Δ^0 or Δ^1.

A map $i : \Delta^0 \to N(\mathcal{U})$ corresponds to an open set $U \subset X$, and we have

$$R(i) = \mathrm{Map}(\mathcal{F}(U), \mathcal{G}(U)) = T(U \subset U),$$

which is contractible by assumption. A map $i : \Delta^1 \to N(\mathcal{U})$ corresponds to an open inclusion $U \subset V$, and $R(i)$ is the homotopy limit in the diagram

$$\begin{tikzcd} R(i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & T(U \subset U) \arrow[d] \\ T(V \subset V) \arrow[r] & T(U \subset V). \end{tikzcd}$$

Since each of the T's is contractible by assumption, so is $R(i)$.

□

We actually need a slight variant on the above proposition. Let $h : \mathcal{F}(X) \to \mathcal{G}(X)$ be a map on global sections, and for $U \subset V \subset X$, define $\mathrm{Map}_h(\mathcal{F}(V), \mathcal{G}(U))$ to be the full subcomplex of $\mathrm{Map}(\mathcal{F}(V), \mathcal{G}(U))$ spanned by restrictions of h. Similarly, define $\mathrm{Map}_h(\mathcal{F}, \mathcal{G})$ to be the subcomplex of $\mathrm{Map}(\mathcal{F}, \mathcal{G})$ consisting of morphisms whose image in $\mathrm{Map}(\mathcal{F}(X), \mathcal{G}(X))$ is h. We must show that if, for all $U \subset V \subset X$, the simplicial set $\mathrm{Map}_h(\mathcal{F}(V), \mathcal{G}(U))$ is contractible, then $\mathrm{Map}_h(\mathcal{F}, \mathcal{G})$ is also contractible.

Proposition 3.3.2. *Let X be a space, let $\mathcal{M}$ be a simplicial model category, and let $\mathcal{U}$ and $\mathcal{P}$ be as above. Let $\mathcal{F}, \mathcal{G} : \mathcal{U} \to \mathcal{M}$ be cofibrant-fibrant presheaves on X. Let $h : \Delta^0 \to \mathrm{Map}(\mathcal{F}(X), \mathcal{G}(X))$ be a map on global sections. Define $T'(U \subset V) : \mathcal{P} \to$ **sSets** as the homotopy limit in the diagram*

$$\begin{tikzcd} T'(U \subset V) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \Delta^0 \arrow[d, "\rho \circ h"] \\ \mathrm{Map}(\mathcal{F}(V), \mathcal{G}(U)) \arrow[r] & \mathrm{Map}(\mathcal{F}(X), \mathcal{G}(U)). \end{tikzcd}$$

If $T'(U \subset V)$ is contractible for all $U \subset V$, then the homotopy limit in the diagram

$$\begin{tikzcd} \mathrm{Map}_h(\mathcal{F}, \mathcal{G}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \Delta^0 \arrow[d, "h"] \\ \mathrm{Map}(\mathcal{F}, \mathcal{G}) \arrow[r] & \mathrm{Map}(\mathcal{F}(X), \mathcal{G}(X)) \end{tikzcd}$$

is contractible.

Proof. The underlying ∞-category of $\mathcal{M}$ is $N(\mathcal{M}^\circ)$, the nerve of the subcategory of cofibrant fibrants. By [20, A.3.6.1] there is a categorical equivalence of ∞-categories

$$N((\mathcal{M}^{\mathcal{U}})^\circ) \to \mathrm{Fun}(N(\mathcal{U}), N(M^\circ)).$$

That is, any strict presheaf can be replaced by a "weak" presheaf. In particular, the equivalence preserves simplicial mapping spaces and homotopy limits.

For any map of ∞-categories $i : \mathcal{S} \to N(\mathcal{U})$, define

$$R(i) = \mathrm{Map}_{\mathrm{Fun}(\mathcal{S}, N(M^\circ))}(\mathcal{F} \circ i, \mathcal{G} \circ i).$$

Let $\underline{\mathcal{F}(X)}$ denote the constant presheaf on $N(\mathcal{U})$ with value $\mathcal{F}(X)$. The map $\underline{\mathcal{F}(X)} \to \mathcal{F}$ induces a map $R(i) \to \mathrm{Map}(\underline{\mathcal{F}(X)}, \mathcal{G} \circ i)$. Define $R'(i)$ as the homotopy limit in the diagram

$$\begin{tikzcd}
R'(i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \Delta^0 \arrow[d, "\rho\circ h"] \\
R(i) \arrow[r] & \mathrm{Map}(\underline{\mathcal{F}(X)}, \mathcal{G} \circ i).
\end{tikzcd}$$

In particular, $R'(\mathrm{id}_{N(\mathcal{U})})$ is weakly equivalent to $\mathrm{Map}_h(\mathcal{F}, \mathcal{G})$. By Lemma A.1.6, R takes homotopy colimits in the Joyal model structure to homotopy limits and hence so does R'. By Corollary A.1.5, it suffices to check that $R(i)$ is contractible when $\mathcal{S}$ is Δ^0 or Δ^1.

A map $i : \Delta^0 \to N(\mathcal{U})$ corresponds to an open set $U \subset X$, and

$$R'(i) = \mathrm{Map}_h(\mathcal{F}(U), \mathcal{G}(U)) = T'(U \subset U),$$

which is contractible by assumption. A map $i : \Delta^1 \to N(\mathcal{U})$ corresponds to an open inclusion $U \subset V$, and $R'(i)$ is the homotopy limit in the diagram

$$\begin{tikzcd}
R'(i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T'(U \subset U) \arrow[d] \\
T'(V \subset V) \arrow[r] & T'(U \subset V).
\end{tikzcd}$$

Since each of the $T'(- \subset -)$'s is contractible by assumption, so is $R'(i)$. $\square$

Theorem 3.3.3. *Let $\mathcal{X} = (X, \mathcal{O}_X)$ be a local smooth-ringed space and let $\mathcal{M} = (M, \mathcal{H}_M)$ a manifold. There is a homotopy equivalence of simplicial sets*

$$\mathrm{Map}_{\mathbf{LRS}}(\mathcal{X}, \mathcal{M}) \simeq \mathcal{O}_X(X, M).$$

Proof. We first need to define morphisms

$$K : \mathrm{Map}_{\mathbf{LRS}}(\mathcal{X}, \mathcal{M}) \to \mathcal{O}_X(X, M)$$
$$L : \mathcal{O}_X(X, M) \to \mathrm{Map}_{\mathbf{LRS}}(\mathcal{X}, \mathcal{M}).$$

We will then show that they are mutually homotopy inverse. For the readers convenience, we recall the definition

$$\mathrm{Map}_{\mathbf{LRS}}(\mathcal{X}, \mathcal{M}) = \coprod_{f: X \to M} \mathrm{Map}_{\mathrm{loc}}(f^*\mathcal{H}_M, \mathcal{O}_X).$$

The map K is fairly easy and can be defined without use of the locality condition. Suppose that $\phi : X \to M$ is a map of topological spaces. The restriction of K to the corresponding components of $\mathrm{Map}(\mathcal{X}, \mathcal{M})$ is given by taking global sections

$$\begin{aligned}\mathrm{Map}_{\mathrm{loc}}(\phi^*\mathcal{H}_M, \mathcal{O}_X) \to \mathrm{Map}(\mathcal{H}_M(M, -), \mathcal{O}_X(X, -)) &= \mathrm{Map}(H_M, \mathcal{O}_X(X, -)) \\ &= \mathcal{O}_X(X, M).\end{aligned}$$

To define L is a bit harder, and depends heavily on the assumption that $\mathcal{O}_X$ is a local sheaf on X. First, given an n-simplex $g \in \mathcal{O}_X(X, M)_n$, we need to define a map of topological spaces $G = L(g) : X \to M$. For each point $m \in M$, let $\{M_{m,i}\}_{i \in I}$ be a sequence of neighborhoods of m whose intersection is $\{m\}$; for each pair (m, i), let $X_{m,i} = g^{-1}(M_{m,i}) \subset X$ be the preimage (see Definition 3.1.8). For each $j \in I$, the set $\{M_{m,j}\}_{m \in M}$ covers M, so by the locality condition the set $\{X_{m,j}\}_{m \in M}$ covers X. By the local compactness of M, every point $x \in X$ is in $\cap_i X_{m,i}$ for some $m \in M$. Suppose that $m, n \in M$ are distinct points, and pick an i so that $M_{m,i}$ and $M_{n,i}$ are disjoint neighborhoods. Proposition 3.1.12 implies that $X_{m,i}$ and $X_{n,i}$ are disjoint.

We can therefore define G to be the map of sets that sends a point $x \in X$ to the unique point $m \in M$ such that $x \in \cap_i X_{m,i}$. To see that G is continuous, it suffices to show that if $M_\alpha \subset M$ is an open inclusion, then the open set $g^{-1}(M_\alpha) \in X$ is in fact the set-theoretic preimage of G; i.e.

$$g^{-1}(M_\alpha) = G^{-1}(M_\alpha).$$

This is clear: both inclusions follow from Proposition 3.1.12.

Now, we need to define a map of presheaves $G^\flat : G^*\mathcal{H}_M \to (\mathcal{O}_X)_n$, which we do by defining its adjoint

$$G^\sharp : \mathcal{H}_M \to G_*(\mathcal{O}_X)_n.$$

Since $\mathrm{Map}(\mathcal{H}_M(M, -), \mathcal{O}_X(X, -)) \cong \mathcal{O}_X(X, M)$, we have a canonical map on global sections $g : \mathcal{H}_M(M, -) \to (G_*\mathcal{O}_X)(M, -)_n$.

For any open inclusion $M_\alpha \subset M_\beta \subset M$, let $X_\beta = G^{-1}(M_\beta)$; note that $(G_*\mathcal{O}_X)(M_\beta, -) = \mathcal{O}_X(X_\beta, -)$. The simplicial set $T(M_\alpha \subset M_\beta)$ obtained as the homotopy fiber in the diagram

$$\begin{tikzcd}
T(M_\alpha \subset M_\beta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \Delta^0 \arrow[d, "g"] \\
\mathrm{Map}_{\mathbf{SR}}(\mathcal{H}_M(M_\alpha, -), \mathcal{O}_X(X_\beta, -)) \arrow[r] & \mathrm{Map}_{\mathbf{SR}}(\mathcal{H}_M(M, -), \mathcal{O}_X(X_\beta, -))
\end{tikzcd}$$

is contractible by Lemma 2.1.15. Thus, by Proposition 3.3.2, the space $\mathrm{Map}_g(\mathcal{H}_M, G_*\mathcal{O}_X)$ of all maps $\mathcal{H}_M \to G_*\mathcal{O}_X$ extending g is contractible. We choose $G^\sharp$ as any element of this contractible set, and $G^\flat$ as its adjoint.

It is easy to check that the composition $K \circ L$ is isomorphic to the identity. It follows from the contractibility of $\mathrm{Map}_g(\mathcal{H}_M, G_*\mathcal{O}_X)$ that the composition $L \circ K$ is homotopic to the identity, so K and L are mutually inverse.

The only thing that we have not shown is that $G^\flat : G^*\mathcal{H}_M \to \mathcal{O}_X$ is local on X. Let $f : M \to N$ be a smooth map, $N_\alpha \subset N$ an open submanifold, $M_\alpha = f^{-1}(N_\alpha)$, and $g = G^\flat(f)$. Define $X_\alpha = G^{-1}(M_\alpha)$ and $X'_\alpha = g^{-1}(N_\alpha)$; we must show that the restriction map $G'_\alpha = G|_{X'_\alpha} : X'_\alpha \to M$ factors through M_α. There is a commutative square

$$\begin{tikzcd}
\Delta^0 \arrow[r, "G'_\alpha"] \arrow[d] & \mathcal{O}_X(X'_\alpha, M) \arrow[d] \\
\mathcal{O}_X(X'_\alpha, N) \arrow[r, "g"'] & \mathcal{O}_X(X'_\alpha, N_\alpha).
\end{tikzcd}$$

Since M_α is the fiber product of the diagram $M \to N \leftarrow N_\alpha$, the fibrancy of the smooth ring $\mathcal{O}_X(X'_\alpha, -)$ implies that G'_α factors through M_α. This completes the proof.

□

This theorem justifies the term "preimage" as defined in Definition 3.1.8: If $g \in \mathcal{O}_X(U, M)$ is a simplex, then it induces a map $G : X \to M$, and the preimage $g^{-1}(M_\alpha)$ of $M_\alpha \subset X$, which was defined solely in terms of the structure sheaf, is equal to the topological preimage $G^{-1}(M_\alpha)$.

If $\mathcal{X} = (X, \mathcal{O}_X)$ is a local smooth-ringed space and M is a manifold, we will tend not to differentiate between $\mathrm{Map}(\mathcal{X}, M)$ and $\mathcal{O}_X(X, M)$.

Definition 3.3.4. Suppose that $\mathcal{M}$ is a simplicial model category, and let $f : X \to Y$ be a map in $\mathcal{M}$. The Čech nerve $\check{\mathrm{C}}(f)$ of f is the simplicial object

$$\check{\mathrm{C}}(f) = \Big(\cdots \Rrightarrow X \times_Y X \times_Y X \Rrightarrow X \times_Y X \rightrightarrows X \Big) \in s\mathcal{M}.$$

Lemma 3.3.5. *Suppose that $f : X \to Y$ is a map of simplicial sets that is surjective in every degree, and $\check{\mathrm{C}}(f)$ is the Čech nerve of f. Then there is a weak equivalence*

$$\operatorname{hocolim} \check{\mathrm{C}}(f) \simeq Y.$$

Proof. We first consider the case in which f is a surjection of discrete simplicial sets. Then we may assume that Y is a point, and we want to show that the simplicial set

$$\check{\mathrm{C}}(f) = \Big(\cdots \Rrightarrow X \times X \times X \Rrightarrow X \times X \rightrightarrows X \Big)$$

is contractible. To do so, we will show that the map $\check{\mathrm{C}}(f) \to \Delta^0$ is an acyclic fibration.

A map $\partial\Delta^n \to \check{\mathrm{C}}(f)$ can be identified with a sequence of $(n+1)$ row vectors of length n,

$$\begin{array}{llllll}
x_0 & = (x_{0,0} & x_{0,1} & \cdots & x_{0,n-1}) \\
x_1 & = (x_{1,0} & x_{1,1} & \cdots & x_{1,n-1}) \\
\vdots & \quad \vdots & \vdots & \ddots & \quad \vdots \\
x_n & = (x_{n,0} & x_{n,1} & \cdots & x_{n,n-1}),
\end{array}$$

in which the ith row is the element of X^n corresponding to the ith face of $\partial\Delta^n$. These rows must satisfy a compatibility condition corresponding to the fact that the faces of $\partial\Delta^n$ are not disjoint. For each row vector $x_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,n-1})$, and each $0 \leq j \leq n-1$, let

$$d_j(x_i) = (x_{i,0}, \ldots, \widehat{x_{i,j}}, \ldots, x_{i,n-1})$$

be the result of deleting the jth entry. Then the simplicial identities imply, in particular, that

$$\begin{aligned} d_i(x_{i+1}) &= d_i(x_i), \quad 0 \leq i \leq n-1, \text{ and} \\ d_0(x_n) &= d_{n-1}(x_0). \end{aligned}$$

This guarantees a lift $\Delta^n \to \check{\mathrm{C}}(f)$ given by $(x_{n,0}, x_{0,0}, x_{0,1}, \ldots, x_{n,n-1})$. Thus $\check{\mathrm{C}}(f)$ is contractible.

Now we consider the case for a general f satisfying the hypotheses of the lemma. Consider the simplicial resolution

$$\begin{array}{ccccccc}
\ddots & & \vdots & & \vdots & & \vdots \\
 & & \downdownarrows\downdownarrows & & \downdownarrows\downdownarrows & & \downdownarrows\downdownarrows \\
\cdots \Rrightarrow & X_2 \times_{Y_2} & X_2 & \times_{Y_2} X_2 \Rrightarrow & X_2 \times_{Y_2} X_2 & \rightrightarrows & X_2 \\
 & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\cdots \Rrightarrow & X_1 \times_{Y_1} & X_1 & \times_{Y_1} X_1 \Rrightarrow & X_1 \times_{Y_1} X_1 & \rightrightarrows & X_1 \\
 & & \downdownarrows & & \downdownarrows & & \downdownarrows \\
\cdots \Rrightarrow & X_0 \times_{Y_0} & X_0 & \times_{Y_0} X_0 \Rrightarrow & X_0 \times_{Y_0} X_0 & \rightrightarrows & X_0.
\end{array}$$

Each object in the diagram is a constant simplicial set. The homotopy colimit of this diagram can be computed in two ways, either vertically followed by horizontally or vice-versa. If we compute vertically followed by the horizontally, we obtain $\operatorname{hocolim} \check{\mathrm{C}}(f)$. If we compute horizontally first, we find that the homotopy colimit of the ith row is weakly equivalent to Y_i, by the case studied above. Then the homotopy colimit of the resulting vertical simplicial diagram of Y_i is weakly equivalent to Y.

$\square$

One of the most important consequences of the locality condition is the following:

Theorem 3.3.6. *Suppose that X is a space and that $\mathcal{F}$ is a sheaf of smooth rings on X. The following are equivalent:*

1. *The sheaf $\mathcal{F}$ is local.*

2. *For every manifold M and open cover $\cup_i M_i = M$, the natural map*

$$\operatorname{hocolim} \mathcal{F}(-, M_\bullet) \to \mathcal{F}(-, M)$$

is a weak equivalence of simplicial sheaves, where $M_\bullet = \check{\mathrm{C}}(\amalg_i M_i \to M)$ denotes the Čech nerve of the cover.

3. *For every manifold M and open cover $\cup_i M_i = M$ by Euclidean spaces $M_i \cong \mathbb{R}^{n_i}$, the natural map*

$$\operatorname{hocolim} \mathcal{F}(-, M_\bullet) \to \mathcal{F}(-, M)$$

is a weak equivalence of simplicial sheaves, where $M_\bullet = \check{\mathrm{C}}(\amalg_i M_i \to M)$.

Proof. It is clear that (2) is equivalent to (3). We prove that (1) is equivalent to (2). By looking at stalks, we may assume that X is a point and $F = \mathcal{F}(X)$ is a smooth ring. Assume first that (1) holds; i.e. F is local. It follows from Theorem 3.3.3 that $F(\amalg_i M_i) \simeq \amalg_i F(M_i)$ is a weak equivalence of simplicial sets. The map $H_M \to \amalg_i H_{M_i}$ is an injective map of functors because any two distinct maps from M to a manifold N restrict to distinct maps on at least one element of the cover. Hence the map

$$F(\amalg M_i)_n = \operatorname{Hom}(\amalg H_{M_i} \otimes \Delta^n, F) \to \operatorname{Hom}(H_M \otimes \Delta^n, F) = F(M)_n$$

is a surjection of sets for all n, and we apply Lemma 3.3.5.

Now suppose that (2) holds; i.e. the map $\operatorname{hocolim} F(M_\bullet) \to F(M)$ is a weak equivalence. The map

$$\pi_0 \coprod_i F(M_i) \to \pi_0 \operatorname{hocolim} F(M_\bullet)$$

is surjective because π_0 takes homotopy colimits to coequalizers and if $A \rightrightarrows B \to C$ is a coequalizer of sets, then $B \to C$ is a surjection. Thus the composition map

$$\pi_0 \coprod_i F(M_i) \twoheadrightarrow \pi_0 \operatorname{hocolim} F(M_\bullet) \xrightarrow{\cong} \pi_0 F(M)$$

is a surjection.

□

Corollary 3.3.7. *Suppose that $f : \mathcal{F} \to \mathcal{G}$ is a map of sheaves of local smooth rings on a space X, and that there is a weak equivalence $\mathcal{F}(-, \mathbb{R}) \xrightarrow{\simeq} \mathcal{G}(-, \mathbb{R})$ of underlying sheaves of simplicial sets. Then f is itself a weak equivalence.*

Proof. By Proposition 3.1.12, we may suppose that X is a point, and we write $F = \mathcal{F}(X), G = \mathcal{G}(X)$ for the corresponding local smooth rings. Let M be a manifold of dimension n. Note that $F(\mathbb{R}^n) \simeq G(\mathbb{R}^n)$. By [3], we can cover M by open subsets M_i such that each M_i, each two-fold intersection M_{ij}, each three-fold-intersection M_{ijk}, etc., is diffeomorphic to either $\mathbb{R}^n$ or $\emptyset$ (i.e. there exists a "good cover" of M). Then by Theorem 3.3.6, we see that

$$F(M) \simeq \operatorname{hocolim} F(M_\bullet) \simeq \operatorname{hocolim} G(M_\bullet) \simeq G(M).$$

□

3.4 C^∞-rings

Moerdijk and Reyes [22] present the following notion of C^∞-ring. A C^∞-ring is a product preserving functor from the Euclidean category to sets. By the Euclidean category, we mean the category $\mathcal{E}$ whose objects are Euclidean spaces $\mathbb{R}^n$, and whose morphisms are all smooth maps. We sometimes identify a C^∞-ring with its value on the affine line $\mathbb{R}$.

Clearly, every smooth ring gives rise to a simplicial C^∞-ring by restriction along the inclusion $\mathcal{E} \to \mathbf{Man}$. We will only be concerned with connected components. Hence, we define

$$C^\infty : \mathbf{SR} \longrightarrow C^\infty\text{-rings},$$

by $C^\infty(F) = \pi_0(F|_{\mathcal{E}})$. If $F \cong H_M$ for some manifold M, we denote $C^\infty(F)$ simply by $C^\infty(M)$. Note that C^∞ is a covariant functor on smooth rings, but contravariant when considered as a functor on manifolds.

Moerdijk and Reyes refer to a C^∞-ring as local if its underlying ring is local. This is *not* a priori consistent with our terminology, but we will see that the two notions in fact turn out to be the same (Theorem 3.4.5).

By Corollary 3.3.7, very little is lost by applying C^∞.

Definition 3.4.1. Let M be a manifold and $U \subset M$ an open subset. Let $\mathbb{R} - \{0\} \to \mathbb{R}$ denote the open inclusion. A smooth function $f : M \to \mathbb{R}$ is called a *characteristic function* for U if the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{R} - \{0\} \arrow[d] \\ M \arrow[r, "f"'] & \mathbb{R} \end{tikzcd}$$

is a pullback in the category of manifolds.

Lemma 3.4.2. *Let M be a manifold. Every open subset $U \subset M$ has a characteristic function.*

Proof. Using a partition of unity, it suffices to consider the case $M \cong \mathbb{R}^n$. This case is proven in [22, 1.4].

□

Lemma 3.4.3. *Suppose that $\cup_i U_i$ is an open cover of $\mathbb{R}$ such that U_i has characteristic function u_i. Then every function $A : C^\infty(\mathbb{R}) \to C^\infty(\bullet)$ factors through $C^\infty(\mathbb{R})[u_i^{-1}]$ for some i.*

Proof. Let $\mathbb{E}$ be the category of locally closed subspaces of $\mathbb{R}$ and smooth maps. By [22, 1.5], the contravariant functor from $\mathbb{E}$ into finitely generated C^∞-rings

$$X \mapsto C^\infty(X)$$

is fully faithful. Thus there is some map $a : \bullet \to \mathbb{R}$ such that $A = C^\infty(a)$. By definition a factors through $U_i \subset \mathbb{R}$ for some i. By [22, 1.6], $C^\infty(U_i) \cong C^\infty(\mathbb{R})[u^{-1}]$. Therefore A factors through $C^\infty(\mathbb{R})[u_i^{-1}]$.

□

Definition 3.4.4. A smooth ring F is *finitely presented* if it is of the form

$$\begin{tikzcd} H_{\mathbb{R}^n} \arrow[r, "0"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near end] & H_\bullet \arrow[d] \\ H_{\mathbb{R}}^m \arrow[r] & F. \end{tikzcd}$$

Theorem 3.4.5. *Let F be a finitely presented smooth ring, and let $|F|$ denote its underlying simplicial commutative $\mathbb{R}$-algebra (see Definition 3.5.5). Then F is a local smooth ring if and only if $\pi_0|F|$ is a local (algebraic) ring.*

Proof. First, suppose that F is a local smooth ring. Let $U = (0, \infty)$ and $V = (-\infty, \frac{1}{2})$ be open intervals in $\mathbb{R}$. Let $u : \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \to \mathbb{R}$ be the smooth maps sending $\mathbb{R}$ isomorphically onto U and V respectively. By Lemma 3.1.10, any element $x : \Delta^0 = F(\mathbb{R}^0) \to F(\mathbb{R})$ factors through either u or v. And if x factors through v then $1 - x$ factors through u. It is easy to show that any element of $F(\mathbb{R})_0$ which factors through u is invertible in $\pi_0|F|$. Therefore, $\pi_0|F|$ is a local ring.

Now suppose that $\pi_0|F|$ is a local ring. Let M be a manifold and $\cup_{\alpha \in A} M_\alpha = M$ an open cover. We must show that the map

$$\pi_0 \coprod_{\alpha \in A} F(M_\alpha) \longrightarrow \pi_0 F(M)$$

is a surjection. By the Whitney imbedding theorem, there exists a closed immersion $j : M \to \mathbb{R}^N$. There exists an open neighborhood of M in $\mathbb{R}^N$ that retracts onto M (since M is an ANR or by [10, pp. 69-70]). Thus there exists an open cover $\cup_{i \in I} P_i = \mathbb{R}^N$, such that each P_i is isomorphic to $\mathbb{R}^N$, $A \subset I$, for all $\alpha \in A, P_\alpha \cap M = M_\alpha$, and for all $i \in I - A, P_i \cap M = \emptyset$. Since F is a smooth ring, $F(P_i \cap M)$ is the homotopy limit in the diagram

$$\begin{tikzcd} F(P_i \cap M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond", near start] & F(M) \arrow[d] \\ F(P_i) \arrow[r] & F(\mathbb{R}^N). \end{tikzcd}$$

Thus it suffices to show that any element of $\pi_0 F(\mathbb{R}^N)$ is equal to the image of an element in some $\pi_0 F(P_i)$.

First suppose that $N = 1$, that $\cup_i P_i = \mathbb{R}$ is an open cover, and that $a \in \pi_0 F(\mathbb{R})$. We consider a as a map $C^\infty(\mathbb{R}) \to C^\infty(F)$. By definition, $C^\infty(F)$ is a local C^∞-ring, and it has a unique point $C^\infty(F) \to C^\infty(\bullet)$ by [22, 3.8]. By Lemma 3.4.3, there exists $i_0 \in I$ such that composition

$$C^\infty(\mathbb{R}) \xrightarrow{a} C^\infty(F) \to C^\infty(\bullet)$$

factors through $C^\infty(P_{i_0})$. Since all nonzero prime ideals in $C^\infty(\mathbb{R})$ are maximal, a is a local map. Hence, there exists a lift in the diagram

$$\begin{tikzcd} C^\infty(\mathbb{R}) \arrow[r, "a"] \arrow[d] & C^\infty(F) \arrow[d] \\ C^\infty(P_{i_0}) \arrow[r] \arrow[ur, dashed] & C^\infty(\bullet). \end{tikzcd}$$

This completes the case $N = 1$.

For the case of general N, proceed by projecting onto a coordinate axis $\mathbb{R}$. The images $\{P'_i\}_{i \in I}$ cover $\mathbb{R}$, and P'_i is a retract of P_i, so the composition

$$\pi_0 \coprod_{i \in I} F(P_i) \to \pi_0 \coprod_{i \in I} F(P'_i) \to \pi_0 F(\mathbb{R})$$

is surjective. Since $\pi_0 F$ is product preserving, the map $\pi_0 \coprod F(P_i) \to \pi_0 F(\mathbb{R}^N)$ is surjective, as desired.

□

3.5 Solving Equations

While smooth-ringed spaces can be very convenient objects for defining derived manifolds, they are nearly impossible to compute with. Instead, one would like to work in the setting of simplicial commutative $\mathbb{R}$-algebras. In this section we show that the functor which replaces a smooth ring with its underlying simplicial commutative $\mathbb{R}$-algebra commutes with solving equations in the smooth ring. That is, it commutes with certain homotopy colimits.

Definition 3.5.1. Recall (Proposition 2.1.14) that the functor $-(\mathbb{R})$, which takes a smooth ring to its underlying simplicial set, is a right Quillen functor, and its left adjoint is $- \otimes H_\mathbb{R} : \mathbf{sSets} \to \mathbf{SR}$. We call a smooth ring *free* if it is in the image of $- \otimes H_\mathbb{R}$. In other words, a free smooth ring is of the form $\coprod_{i \in I} H_\mathbb{R}$ for some simplicial set I.

A simplicial smooth ring $X_\bullet$ can be regarded as a bisimplicial functor $X_{ij} : \mathbf{Man} \to \mathbf{Sets}$.

Proposition 3.5.2. *Let X be a local smooth ring. There exists a functorial simplicial resolution $X'_\bullet \to X$, where X'_n is a free smooth ring for all n. If $f : X \to Y$ is a morphism of local smooth rings, then $f'_\bullet : X'_\bullet \to Y'_\bullet$ sends each component $H_\mathbb{R} \subset X'_{ij}$ to a component $H_\mathbb{R} \subset Y'_{ij}$ by the identity map.*

Proof. Let U denote the underlying simplicial set functor $-(\mathbb{R}) : \mathbf{SR} \to \mathbf{sSets}$, and let F denote its left adjoint. The cotriple FU gives rise to an augmented simplicial smooth ring,

$$\Phi = \cdots \substack{\longrightarrow\\[-1em] \longrightarrow\\[-1em] \longrightarrow} FUFU(X) \rightrightarrows FUX \longrightarrow X.$$

By [27, 8.6.10], the induced augmented simplicial set

$$U\Phi = \cdots \substack{\longrightarrow\\[-1em] \longrightarrow\\[-1em] \longrightarrow} UFUFU(X) \rightrightarrows UFU(X) \longrightarrow UX$$

is a homotopy equivalence. By Corollary 3.3.7, Φ is a homotopy equivalence as well.

The second assertion follows by construction.

$\square$

Lemma 3.5.3. *Suppose that $f : A \to B$ is a fibration of smooth rings, such that $\pi_0(A) \to \pi_0(B)$ is a surjection. Let $f'_\bullet : A'_\bullet \to B'_\bullet$ be the simplicial resolution provide by Proposition 3.5.2, considered as a bisimplicial functor $f'_{ij} : \mathbf{Man} \to \mathbf{Sets}$. For every i, j, the morphism $f'_{ij}(\mathbb{R}) : A_{ij}(\mathbb{R}) \to B_{ij}(\mathbb{R})$ is a surjection of sets.*

Proof. Let U denote the underlying simplicial set functor $-(\mathbb{R}) : \mathbf{SR} \to \mathbf{sSets}$, and let F denote its left adjoint. Recall that $F = -\otimes H_{\mathbb{R}}$, and that the simplicial resolution provided in Proposition 3.5.2 is that associated to the cotriple FU.

Since f is a fibration and U is a right adjoint, $U(f)$ is a fibration of simplicial sets which is surjective on π_0. It is easy to show that such a morphism must be surjective in every degree (i.e. $U(f)_i : U(A)_i \twoheadrightarrow U(B)_i$ is surjective for all $i \in \mathbb{N}$). Thus $FU(f)$ is the morphism

$$\coprod_{a \in U(A)} H_{\mathbb{R}} \to \coprod_{b \in U(B)} H_{\mathbb{R}}$$

of free smooth rings, indexed by the morphism $U(f)$ of simplicial sets. Though this is not a fibration, it is easy to see that applying U to this morphism again induces a levelwise surjection of simplicial sets. Continuing inductively, one sees that $U(FU)^n(f)$ is a levelwise surjection of simplicial sets, and it indexes the morphism $(FU)^{n+1}$ of free smooth rings. This completes the proof.

$\square$

Let $\mathcal{P}_\Sigma(\mathbf{Man}^{\mathrm{op}})$ denote the model category of (weakly) product preserving functors from manifolds to simplicial sets (see [19, Section 14,15]). Consider the Quillen pairs

$$\mathbf{sSets}^{\mathbf{Man}} \underset{U_1}{\overset{L_1}{\rightleftarrows}} \mathcal{P}_\Sigma(\mathbf{Man}^{\mathrm{op}}) \underset{U_2}{\overset{L_2}{\rightleftarrows}} \mathbf{SR},$$

in which L_1 and L_2 are localizations of simplicial model categories.

Recall that if $X_\bullet$ is a simplicial object in a simplicial model category, then the homotopy colimit of $X_\bullet$ is weakly equivalent to the geometric realization of $X_\bullet$. We use these two notions interchangeably.

Definition 3.5.4. Let **sPP** denote the full simplicial subcategory of $\mathbf{sSets}^{\mathbf{Man}}$ spanned by those functors F that are *strictly product-preserving* in the sense that for all manifolds M, N, the natural map

$$F(M \times N) \to F(M) \times F(N)$$

is an isomorphism (rather than just a weak equivalence) of simplicial sets.

Let $\mathbf{sC}\mathbb{R}$ denote the simplicial model category of (algebraic) $\mathbb{R}$-algebras. If $F : \mathbf{Man} \to \mathbf{sSets}$ is a strictly product preserving functor, then $F(\mathbb{R})$ is a simplicial commutative ring in a functorial way. We write this functor as $-(\mathbb{R}) : \mathbf{sPP} \to \mathbf{sC}\mathbb{R}$.

The proof of the following theorem is included in Section A.2 of the appendix.

Theorem A.2.11 1. The simplicial category **sPP** is a simplicial model category.

Let **sPP** denote the category of strictly product preserving functors from manifolds to simplicial sets; this is a simplicial model category by Theorem A.2.11. By [20, 4.2.4.1] and [19, Section 15], there is a Quillen equivalence of model categories $S : \mathcal{P}_\Sigma(\mathbf{Man}^{\mathrm{op}}) \to \mathbf{sPP}$, called *strictification*. Consider the composition of derived functors

$$\mathbf{SR} \xrightarrow{U_2} \mathcal{P}_\Sigma(\mathbf{Man}^{\mathrm{op}}) \xrightarrow{S} \mathbf{sPP} \xrightarrow{-(\mathbb{R})} \mathbf{sC}\mathbb{R}, \tag{3.1}$$

which we denote $|\cdot|$.

Definition 3.5.5. We refer to the functor $|\cdot| : \mathbf{SR} \to \mathbf{sC}\mathbb{R}$ defined above as the *underlying commutative* $\mathbb{R}$*-algebra* functor.

Let **Poly** denote the category whose objects are Euclidean spaces $\mathbb{R}^n$, $n \in \mathbb{N}$, and where $\mathrm{Hom}_{\mathbf{Poly}}(\mathbb{R}^n, \mathbb{R}^m)$ is the set of polynomial maps from $\mathbb{R}^n \to \mathbb{R}^m$ with real coefficients.

Proposition 3.5.6. *Let $\mathcal{C}$ denote the category of product preserving functors* $\mathbf{Poly} \to \mathbf{sSets}$*. There is a natural equivalence of categories $\mathcal{C} \to \mathbf{sC}\mathbb{R}$. Moreover, there is an adjunction* $\mathbf{sSets}^{\mathbf{Poly}} \rightleftarrows \mathbf{sC}\mathbb{R}$ *in which the right adjoint is fully faithful.*

Proof. If A is a simplicial commutative $\mathbb{R}$-algebra, define a functor $F_A : \mathbf{Poly} \to \mathbf{sSets}$ as follows. Let $F_A(\mathbb{R}^n) = A^n$, and for a polynomial map $p = (p_1, \ldots, p_m) : \mathbb{R}^n \to \mathbb{R}^m$, let $F_A(p) : A^n \to A^m$ be defined by

$$p(\bar{a}) = (p_1(\bar{a}), \ldots, p_n(\bar{a})).$$

If $G : \mathbf{Poly} \to \mathbf{sSets}$ is a product-preserving functor, define a simplicial commutative $\mathbb{R}$-algebra B_G as follows. The underlying simplicial set of B_G is $G(\mathbb{R})$. Define $+ : B_G^2 \to B_G$ by applying G to $+ : \mathbb{R}^2 \to \mathbb{R}$; similarly for $\times, -, 0,$

and 1. The commutativity of diagrams in B_G expressing associativity, distributivity, etc., are established by the functoriality of G and the commutativity of corresponding diagrams in **Poly**.

It is clear that F and B are inverse equivalences of categories.

Let $P_{\mathbb{R}^n} : \mathbf{Poly} \to \mathbf{sSets}$ denote the representable functor

$$P_{\mathbb{R}^n}(\mathbb{R}^m) = \mathbf{Poly}(\mathbb{R}^n, \mathbb{R}^m).$$

Any finite coproduct of these is small in $\mathbf{sSets}^{\mathbf{Poly}}$. Let Γ be the set of natural maps

$$\gamma : P_{\mathbb{R}^n} \amalg P_{\mathbb{R}^m} \to P_{\mathbb{R}^{m+n}}.$$

The category of simplicial commutative $\mathbb{R}$-algebras can be identified as the full subcategory of $\mathbf{sSets}^{\mathbf{Poly}}$ consisting of functors F such that $\mathrm{Map}(\gamma, F)$ is an isomorphism for all $\gamma \in \Gamma$.

The second assertion follows from Theorem A.2.7.

$\square$

Lemma 3.5.7. *Let X be a topological space and let $\mathcal{F}$ be a sheaf of smooth rings on X. Then the functor $|\cdot| : \mathbf{Pre}(X, \mathbf{SR}) \to \mathbf{Pre}(X, \mathbf{sC}\mathbb{R})$, defined in Equation 3.1, commutes with taking stalks. That is, for any $x : \{*\} \to X$, the diagram*

$$\begin{tikzcd}
\mathbf{Pre}(X, \mathbf{SR}) \arrow[r, "|\cdot|"] \arrow[d, "x^*"'] & \mathbf{Pre}(X, \mathbf{sC}\mathbb{R}) \arrow[d, "x^*"] \\
\mathbf{SR} \arrow[r, "|\cdot|"'] & \mathbf{sC}\mathbb{R}
\end{tikzcd}$$

commutes.

Proof. Recall that a stalk is just a filtered colimit. The functor $U_2 : \mathbf{SR} \to \mathcal{P}_\Sigma(\mathbf{Man}^{\mathrm{op}})$ commutes with taking filtered colimits because filtered colimits in $\mathbf{SR}$ and $\mathcal{P}_\Sigma(\mathbf{Man}^{\mathrm{op}})$ are computed objectwise. The strictification functor S is a left adjoint, so it commutes with all colimits. Thus it suffices to show that $-(\mathbb{R}) : \mathbf{sPP} \to \mathbf{sC}\mathbb{R}$ commutes with taking filtered colimits.

Let $\ell_1 : \mathbf{sSets}^{\mathbf{Man}} \to \mathbf{sPP}$ and $\ell_2 : \mathbf{sSets}^{\mathbf{Poly}} \to \mathbf{sC}\mathbb{R}$ be the left adjoints to the respective inclusions of full subcategories. A colimit in $\mathbf{sPP}$ (resp. in $\mathbf{sC}\mathbb{R}$) is computed by taking the corresponding colimit in $\mathbf{sSets}^{\mathbf{Man}}$ (resp. $\mathbf{sSets}^{\mathbf{Poly}}$) and then applying ℓ_1 (resp. ℓ_2). Since filtered colimits commute with finite products, the colimit in $\mathbf{sSets}^{\mathbf{Man}}$ (resp. $\mathbf{sSets}^{\mathbf{Poly}}$) of a diagram of product-preserving functors is already product-preserving. Thus, applying ℓ_1 (resp. ℓ_2) does not change the underlying functor. Therefore, taking filtered colimits commutes with application of $-(\mathbb{R})$.

$\square$

Proposition 3.5.8. *The functor $-(\mathbb{R}) : \mathbf{sPP} \to \mathbf{sC}\mathbb{R}$ preserves geometric realizations and filtered colimits.*

Proof. By Proposition 3.5.6, the category of simplicial commutative $\mathbb{R}$-algebras is just the category of product-preserving functors **Poly** $\to$ **sSets**. But **Poly** is a subcategory of **Man**, and the functor $-(\mathbb{R}) : \mathbf{sPP} \to \mathbf{sC}\mathbb{R}$ is given by restriction of functors.

By [19, A.1], the geometric realization of simplicial objects in both **sPP** and $\mathbf{sC}\mathbb{R}$ is equivalent to the diagonal of the corresponding bisimplicial objects. Since the functor $-(\mathbb{R})$ preserves the diagonal, it commutes with geometric realization.

Since filtered colimits preserve finite limits (and in particular preserve finite products), they are computed objectwise in **sPP**. In particular, $(\mathrm{colim}_i F_i)(\mathbb{R}) = \mathrm{colim}_i(F_i(\mathbb{R}))$.

∎

Let R be a commutative $\mathbb{R}$-algebra. Let $\mathbf{Ch}_R$ denote the simplicial model category of (unbounded) chain complexes of R-modules, and let $\mathbf{DGC}_R$ denote the simplicial model category of commutative differential graded algebras over R. See [12] for the definitions of these model structures. Let $\otimes_A^L$ denote the left derived functor of $\otimes$ in both $\mathbf{Ch}_A$ and $\mathbf{DGC}_A$.

Lemma 3.5.9. *Let A be a commutative $\mathbb{R}$-algebra, let B and C be commutative differential graded A-algebras, and let D be the homotopy pushout in the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, phantom, "\diamond"] & D. \end{tikzcd}$$

The homotopy groups of D are given by the formula

$$\pi_i(D) = \mathrm{Tor}_i^A(B, C).$$

Proof. Consider the Quillen pair $\mathrm{Sym}_A : \mathbf{Ch}_A \rightleftarrows \mathbf{DGC}_A : U$, where U forgets the algebra structure. The cofibrant replacement functor $X \mapsto \widetilde{X}$ on $\mathbf{DGC}_A$ replaces X with a graded algebra in which every component is projective. Thus, applying U to the acyclic fibration $\widetilde{X} \to X$ gives an acyclic fibration $U\widetilde{X} \to UX$ in which the source is cofibrant. We have

$$U(B \otimes_A^L C) \simeq U(\widetilde{B} \otimes_A \widetilde{C}) \simeq U\widetilde{B} \otimes_A U\widetilde{C} \simeq UB \otimes_A^L UC.$$

Since the homotopy groups of an algebra are isomorphic to those of the underlying chain complex, we have

$$\pi_i(B \otimes_A^L C) \cong \pi_i(UB \otimes_A^L UC) = \mathrm{Tor}_i^A(UB, UC).$$

A priori, $B \otimes_A^L C$ may not be weakly equivalent to D, unless A is cofibrant. However, by [12, 3.3.2], we have

$$D = B \otimes_{\widetilde{A}}^L C \simeq B \otimes_A^L C,$$

where $\widetilde{A} \to A$ is a cofibrant replacement. Therefore $\pi_i D = \pi_i(B \otimes_A^L C)$.

∎

The following lemma is an application of Hadamard's lemma. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth map and we impose the equivalence relation on the ring $C^\infty(\mathbb{R}^n)$ whereby we set $g \sim h$ if and only if $f(x) = 0$ implies $g(x) = h(x)$, for all $x \in \mathbb{R}^n$. There is also an ideal-theoretic equivalence relation on $C^\infty(\mathbb{R}^n)$ whereby $g \sim_I h$ if f divides $g - h$ (in the sense that there exists a smooth function $f' : \mathbb{R}^n \to \mathbb{R}$ with $ff' = g - h$).

Clearly, if $g \sim_I h$ then $g \sim h$, whereas the converse is false in general (take $f = x^2, g = x, h = 0$). Hadamard's lemma says that if $f = x_i$ is a coordinate function on $\mathbb{R}^n$, then $g \sim h$ implies $g \sim_I h$. This will be a crucial step in allowing us to transform calculations on smooth rings into calculations on simplicial commutative rings.

Lemma 3.5.10. *Let $x_1 : \mathbb{R}^n \to \mathbb{R}$ denote the first coordinate projection, let $g : \bullet \to \mathbb{R}$ be a point in $\mathbb{R}$, and consider the diagram*

$$\begin{tikzcd} H_{\mathbb{R}} \arrow[r, "x_1"] \arrow[d, "g"'] & H_{\mathbb{R}^n} \\ H_{\bullet} & \end{tikzcd}$$

of smooth rings. The homotopy colimit of the diagram is $H_{\mathbb{R}^{n-1}}$. Moreover, the underlying $\mathbb{R}$-algebra functor $|\cdot| : \mathbf{SR} \to \mathbf{sC}\mathbb{R}$ commutes with homotopy colimit, when applied to this diagram.

Proof. We may assume that $g = 0 : \bullet \to \mathbb{R}$. The diagram of smooth manifolds

$$\begin{tikzcd} \mathbb{R}^{n-1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \bullet \arrow[d, "0"] \\ \mathbb{R}^n \arrow[r, "x_1"'] & \mathbb{R} \end{tikzcd}$$

is a pullback. Since the bottom arrow is a submersion, it follows that

$$\begin{tikzcd} H_{\mathbb{R}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond", very near end] & H_{\mathbb{R}^n} \arrow[d] \\ H_{\bullet} \arrow[r] & H_{\mathbb{R}^{n-1}} \end{tikzcd}$$

is indeed a homotopy pushout of smooth rings.

Any representable smooth ring H_M is in fact strictly submersion-pullback preserving and in particular strictly product preserving. In other words, when one applies the underlying commutative $\mathbb{R}$-algebra functor to H_M, we get simply

$$|H_M| = S(U_2(H_M))(\mathbb{R}) \cong H_M(\mathbb{R}) = \mathrm{Hom}(M, \mathbb{R}) \in \mathbf{sC}\mathbb{R},$$

the (constant simplicial) ring of smooth functions on M.

Define a simplicial commutative $\mathbb{R}$-algebra D as the homotopy colimit in the diagram

$$\begin{tikzcd} |H_{\mathbb{R}}| \arrow[r] \arrow[d] & |H_{\mathbb{R}^n}| \arrow[d] \\ |H_{\bullet}| \arrow[r] \arrow[ur, phantom, "\diamond"] & D. \end{tikzcd}$$

We must show that $D \simeq |H_{\mathbb{R}^{n-1}}|$. Note that $|H_{\bullet}| \cong \mathbb{R}$. Since x_1 is a nonzerodivisor on $H_{\mathbb{R}^n}$, the homotopy groups of D are

$$\pi_0(D) = |H_{\mathbb{R}^n}| \otimes_{|H_{\mathbb{R}}|} \mathbb{R}; \text{ and } \pi_i(D) = 0, i > 0.$$

Thus, $\pi_0(D)$ can be identified with the equivalence classes of functions $f : \mathbb{R}^n \to \mathbb{R}$, where $f \sim g$ if $f - g$ is a multiple of x_1.

The smooth functions on $\mathbb{R}^{n-1}$ are the equivalence classes of functions $f : \mathbb{R}^n \to \mathbb{R}$, where $f \sim g$ if

$$f(0, x_2, x_3, \ldots x_n) = g(0, x_2, x_3, \ldots, x_n).$$

Thus to prove that the obvious map $D \to |H_{\mathbb{R}^{n-1}}|$ is a weak equivalence, we must only show that if a function $f(x_1, \ldots, x_n) : \mathbb{R}^n \to \mathbb{R}$ vanishes whenever $x_1 = 0$, then x_1 divides f. This is called Hadamard's lemma, and it follows from the definition of smooth functions. Indeed, we consider such an f, and define a function $g : \mathbb{R}^n \to \mathbb{R}$ by the formula

$$g(x_1, \ldots, x_n) = \lim_{a \to x_1} \frac{f(a, x_2, x_3, \ldots, x_n)}{a}.$$

It is clear that g is smooth, and $x_1 g = f$.

□

Theorem 3.5.11. *Let Ψ be a diagram*

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "g"'] & G \\ H & \end{tikzcd}$$

of local smooth rings, for which $\pi_0 g : \pi_0 F \to \pi_0 H$ is a surjection. Let U : $\mathbf{SR} \to \mathbf{sC}\mathbb{R}$ be the underlying $\mathbb{R}$-algebra functor, $U = |\cdot|$. Then application of U commutes with taking the homotopy colimit of Ψ; i.e.

$$\operatorname{hocolim}(U\Psi) \simeq U \operatorname{hocolim}(\Psi).$$

Proof. Let Ψ' be a simplicial resolution of Ψ as in Proposition 3.5.2; we have

$$\Psi' = \begin{tikzcd} F'_{\bullet} \arrow[r, "f'"] \arrow[d, "g'"'] & G'_{\bullet} \\ H'_{\bullet}, & \end{tikzcd}$$

where each F'_n, G'_n and H'_n are free smooth rings (i.e. the coproducts of copies of $H_{\mathbb{R}}$). The homotopy colimit of Ψ is the same as the homotopy colimit of Ψ', and the homotopy colimit of $U\Psi$ is the same as the homotopy colimit of $U\Psi'$, by Proposition 3.5.8. The homotopy colimits of both Ψ' and $U\Psi'$ may be taken levelwise. Moreover, we may replace g' by a fibration (and hence $U(g')$ with a fibration) without affecting the homotopy colimit. Since g' is surjective on π_0, it is in fact surjective in every degree (see Lemma 3.5.3). Thus it suffices to prove that U commutes with the homotopy colimit of the diagram

$$\begin{tikzcd} F' \arrow[r] \arrow[d, "g'"'] & G' \\ H', & \end{tikzcd}$$

where F', G', and H' are free smooth rings and that g' is surjective.

Now for any $i \neq 1$, the map $H_{\mathbb{R}^n} \xrightarrow{x_i \mapsto x_i - x_1} H_{\mathbb{R}^n}$ is an isomorphism. Thus, after recalling Proposition 3.5.2 and composing with a subtraction isomorphism as above, we may simplify g' a bit more. Namely, we may subdivide F' into two parts F'_{inj} and F'_0, such that $g' : F'_{\text{inj}} \to H'$ takes distinct components $H_{\mathbb{R}} \subset F'_{\text{inj}}$ to distinct components $H_{\mathbb{R}} \subset H'$, and such that $g' : F'_0 \to H'$ sends each component $H_{\mathbb{R}}$ to $H_{\bullet}$. For components in F'_0, Lemma 3.5.10 implies that the homotopy colimit commutes with U, and for components in F'_{inj} this fact is obvious.

□

Definition 3.5.12. We refer to the homotopy colimit of a diagram $F \xleftarrow{p} G \xrightarrow{0} H_{\bullet}$ of smooth rings as the *solution to the equation* $p = 0$ *in* F.

This is analogous to the terminology in commutative $\mathbb{R}$-algebras, in which the colimit (tensor product) of a diagram such as $\mathbb{R}[x, y] \xleftarrow{x+y} \mathbb{R}[f] \xrightarrow{0} \mathbb{R}$ is said to be the solution to the equation $x+y = 0$ in $\mathbb{R}[x, y]$. We might also write such a homotopy colimit as F/p. Theorem 3.5.11 is very useful for computational purposes, because simplicial commutative $\mathbb{R}$-algebras are so much easier to work with than are smooth rings. The theorem says that passing to the underlying simplicial commutative $\mathbb{R}$-algebra commutes with solving equations.

Chapter 4

Derived Manifolds

4.1 Definition and Properties

Definition 4.1.1. Let $\mathcal{X}$ be a local smooth-ringed space, and let $f : \mathcal{X} \to \mathbb{R}^p$ be a morphism of local smooth-ringed spaces. Let $\mathcal{Y}$ be the fiber product in the diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\ \mathcal{X} \arrow[r, "f"'] & \mathbb{R}^p \end{tikzcd}$$

of smooth-ringed spaces (see Proposition 2.3.21). We refer to $\mathcal{Y}$ as *the solution to the equation $f = 0$ in $\mathcal{X}$*, or as *the zeroset of f in $\mathcal{X}$*. We write $\mathcal{Y} = \mathcal{X}_{f=0}$.

Definition 4.1.2. A smooth-ringed space $\mathcal{U} = (U, \mathcal{O}_U)$ is called a *principal derived manifold* if it is the solution to an equation on $\mathbb{R}^m$; i.e. if $\mathcal{U}$ can be written as the homotopy fiber product in a diagram of smooth ringed spaces

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\ \mathbb{R}^n \arrow[r, "f"'] & \mathbb{R}^m \end{tikzcd}$$

for some smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$.

A Hausdorff smooth-ringed space $\mathcal{X} = (X, \mathcal{O}_X)$ is called a *derived manifold* if it can be covered by countably many principal derived manifolds.

A derived manifold $\mathcal{X}$ is called *non-disjoint* if, for every two points $p, q \in \mathcal{X}$, there exists a sequence $\mathcal{U}_1, \ldots, \mathcal{U}_n$ of principal derived submanifolds, such that $p \in \mathcal{U}_1$ and $q \in \mathcal{U}_n$ and for all $1 \leq i < n$, the intersection $\mathcal{U}_i \cap \mathcal{U}_{i+1}$ is nonempty.

A manifold is non-disjoint if and only if it is connected. However, a derived manifold $\mathcal{X}$ may be non-disjoint and disconnected (perhaps not even locally connected). However a non-disjoint derived manifolds have many of the good properties that connected manifolds have. For example, they have constant dimension (see Corollary 5.2.13), and every derived manifold can be written as the disjoint union (coproduct) of non-disjoint derived submanifolds.

Example 4.1.3. Every manifold is clearly a derived manifold (substitute $m = 0$ above). Our running example of the squared origin $\mathcal{G}$ from Example 2.3.26 is also a derived manifold.

Lemma 4.1.4. *Every derived manifold $\mathcal{X}$ is second countable, locally compact, and paracompact.*

Proof. We may assume that $\mathcal{X}$ is a principal derived manifold. It is then a closed subset of a locally compact Hausdorff space $\mathbb{R}^n$; hence it is locally compact and second countable, which implies that it is paracompact. $\square$

Remark 4.1.5. Let $\mathcal{L}$ denote the set of submersion-pullback diagrams in **Man**. Let $\mathbf{U} : \mathbf{Man} \to \mathbf{Top}$ be the forgetful functor. Then $\overline{\mathbf{Man}} = (\mathbf{Man}, \mathbf{U}, \mathcal{L}, \mathbb{R})$ is a category of local models, in the sense of Chapter 1. Smooth rings, smooth-ringed spaces, and local smooth-ringed spaces, as defined in Chapters 2 and 3, are homotopical versions of the notions defined in Chapter 1. Let $\mathcal{L}'$ consist of all Λ^2_2-shaped diagrams in **Man**. We have defined $\overline{\mathbf{Man}'} = \overline{\mathbf{Man}}[\lim \mathcal{L}']$ to be the extension of $\overline{\mathbf{Man}}$ by (homotopy) limits of $\mathcal{L}'$-shaped diagrams.

Proposition 4.1.6. *Let $\mathcal{X}$ be the homotopy fiber product of a diagram of smooth-ringed spaces*

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & P \arrow[d, "f"] \\ N \arrow[r, "g"'] & M, \end{tikzcd}$$

in which N, M, and P are smooth manifolds. Then $\mathcal{X}$ is a derived manifold.

Proof. Cover M by Euclidean spaces $\mathbb{R}^m$, and then cover their preimages in N and P by Euclidean spaces. It suffices to show that the pullback

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathbb{R}^p \arrow[d, "f"] \\ \mathbb{R}^n \arrow[r, "g"'] & \mathbb{R}^m \end{tikzcd}$$

is a principal derived manifold. Consider the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathbb{R}^m \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\ \mathbb{R}^n \times \mathbb{R}^p \arrow[r] & \mathbb{R}^m \times \mathbb{R}^m \arrow[r, "-"] & \mathbb{R}^m, \end{tikzcd}$$

where the bottom right map sends $(a, b) \mapsto a - b$. Both the left square and the right square are fiber product diagrams; hence the large rectangle is as well.

$\square$

Theorem 4.1.7. *If $\mathcal{X}$ is a local smooth-ringed space and $f : \mathcal{X} \to \mathbb{R}^n$ is a morphism of local smooth-ringed spaces, then the solution to the equation $f = 0$ is a local smooth-ringed space.*

Proof. Let $\mathcal{X}' = (X', \mathcal{O}_{X'}) := \mathcal{X}_{f=0}$ denote the solution to the equation $f = 0$ in $\mathcal{X}$; its structure sheaf is the homotopy colimit in the diagram

$$\begin{tikzcd} \mathcal{H}_{\mathbb{R}^n} \arrow[r, "0"] \arrow[d, "f"'] \arrow[dr, phantom, "\diamond"] & \mathcal{H}_{\bullet} \arrow[d] \\ \mathcal{O}_X \arrow[r] & \mathcal{O}_{X'} \end{tikzcd}$$

(see Notation 2.3.25). We want to show that $\mathcal{O}_{X'}$ is local; thus we may assume that X' is a point. Let $F = \mathcal{H}_{\mathbb{R}^n}(X'), G = \mathcal{H}_{\bullet}(X')$, and $K = \mathcal{O}_X(X')$ be the local smooth rings at that point, and let $L = \mathcal{O}_{X'(X')}$.

The map $F \to G$ is surjective on π_0, so by Theorem 3.5.11, $L(\mathbb{R}) \simeq G(\mathbb{R}) \amalg_{F(\mathbb{R})} K(\mathbb{R})$ as simplicial commutative $\mathbb{R}$-algebras. Moreover, $C^\infty(L)$ is a finitely presented C^∞-ring, and is the quotient of a local ring. Therefore it is local as a C^∞-ring, and by Theorem 3.4.5 $L(\mathbb{R})$ is also local as a smooth ring.

$\square$

Corollary 4.1.8. *If $\mathcal{X}$ is a derived manifold, then it is also a local smooth-ringed space.*

Proof. The property of being a local smooth-ringed space is local, so we may assume that $\mathcal{X}$ is a principal derived manifold. That is, $\mathcal{X}$ is $\mathbb{R}^n_{f=0}$ for some $f : \mathbb{R}^n \to \mathbb{R}^m$. Since manifolds are local smooth-ringed spaces, the result follows from Theorem 4.1.7

$\square$

Proposition 4.1.9. *The inclusion $i :$ **Man** $\to$ **dMan** is a fully faithful functor, and it preserves submersion pullbacks.*

Proof. That i is fully faithful was proven in Proposition 3.2.3. Suppose that $s : M \to N$ is a submersion and $a : P \to N$ is any smooth map; let Q be the pullback manifold

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[d] \\ P \arrow[r] & N. \end{tikzcd}$$

Let $\mathcal{X}$ be any derived manifold. By Theorem 3.3.3, we have

$$\mathrm{Map}(\mathcal{X}, i(Q)) \simeq \mathcal{O}_X(X, Q),$$

and similarly for M, N, and P in place of Q. Since $\mathcal{O}_X$ is a sheaf, $\mathcal{O}_X(X, -)$ is a fibrant smooth ring, which implies that $\mathcal{O}_X(X, Q)$ is the homotopy pullback in the diagram

$$\begin{tikzcd} \mathcal{O}_X(X,Q) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{O}_X(X,M) \arrow[d] \\ \mathcal{O}_X(X,P) \arrow[r] & \mathcal{O}_X(X,N). \end{tikzcd}$$

In other words, $i(Q)$ represents the pullback of the diagram $i(M) \to i(N) \leftarrow i(P)$, as desired.

□

We will refer to a derived manifold $\mathcal{X}$ as a manifold if it is in the essential image of the inclusion $i : \mathbf{Man} \to \mathbf{dMan}$.

Proposition 4.1.10. *The category of derived manifolds is closed under taking products.*

Proof. Let $\mathcal{X} = (X, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, \mathcal{O}_Y)$ be derived manifolds. Their product exists in the category of smooth-ringed spaces; we must show that it is a derived manifold. Taking small enough open sets, we may assume that $\mathcal{X} = \mathbb{R}^m_{f=0}$ and $\mathcal{Y} = \mathbb{R}^p_{g=0}$, where $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^p \to \mathbb{R}^q$ are smooth functions. Let $h : \mathbb{R}^{m+p} \to \mathbb{R}^{n+q}$ be the Cartesian product of f and g. Let $\mathcal{Z} = \mathbb{R}^{m+p}_{h=0}$. Clearly $\mathcal{Z}$ is a derived manifold, and it is purely formal to show that $\mathcal{Z}$ is the product of $\mathcal{X}$ and $\mathcal{Y}$.

□

Definition 4.1.11. Let $\mathcal{U}$ and $\mathcal{Y}$ be local smooth-ringed spaces. A map $i : \mathcal{U} \to \mathcal{Y}$ is called a *principal closed immersion* if there exists a map $g : \mathcal{Y} \to \mathbb{R}^n$ such that the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\ \mathcal{Y} \arrow[r, "g"'] & \mathbb{R}^n \end{tikzcd}$$

is Cartesian.

A map $i : \mathcal{X} \to \mathcal{Y}$ of local smooth-ringed spaces is called a *closed immersion* if $\mathcal{X}$ can be covered by open subspaces $\mathcal{U}$ for which the induced map $\mathcal{U} \to \mathcal{Y}$ is a principal closed immersion.

A map $i : \mathcal{X} \to \mathcal{Y}$ of local smooth-ringed spaces is called an *open immersion* if $i(X)$ is an open subset of Y and $\mathcal{O}_X \simeq i^*\mathcal{O}_Y$.

If $\mathcal{X} = (X, \mathcal{O}_X)$ and $\mathcal{Y} = (Y, \mathcal{O}_Y)$ are derived manifolds, we refer to $\mathcal{X}$ as a *derived submanifold of* $\mathcal{Y}$ if there exists a subset $i : V \subset Y$ and an isomorphism $\mathcal{X} \cong (V, i^*\mathcal{O}_Y)$.

Example 4.1.12. Note that our example of the squared origin (2.3.26), the map $\mathcal{G} \to \bullet$ is a (principle) closed immersion.

Proposition 4.1.13. *Suppose that $i : \mathcal{X} \to \mathcal{Y}$ is a closed immersion of local smooth-ringed spaces, and let M be a manifold. Any morphism $g : \mathcal{X} \to M$ can be locally extended to $g' : \mathcal{Y} \to M$.*

Proof. We must show that $\mathrm{Map}(\mathcal{Y}, M) \to \mathrm{Map}(\mathcal{X}, M)$ is locally a surjection on connected components. By the structure Theorem 3.3.3, it suffices to show that the morphism of sheaves

$$\pi_0(i^\flat(-, M)) : \pi_0(i^*)_Y(-, M)) \to \pi_0(\mathcal{O}_X(-, M))$$

is surjective. Since we can check this locally on $\mathcal{X}$, we may assume that $M = \mathbb{R}^p$, that $X = \{*\}$, and that $\mathcal{O}_X$ is a local smooth ring.

By definition, $\mathcal{O}_X$ is the pushout in a diagram of sheaves on X of the form

$$\begin{tikzcd} \mathcal{H}_{\mathbb{R}^n} \arrow[r, "0"] \arrow[d, "f"'] \arrow[dr, phantom, "\diamond"] & \mathcal{H}_\bullet \arrow[d] \\ \mathcal{O}_Y \arrow[r, "i^\flat"'] & \mathcal{O}_X \end{tikzcd}$$

(see Notation 2.3.25). By Theorem 3.5.11, the diagram remains a homotopy colimit after applying the functor $-(\mathbb{R})$. It again remains a colimit after applying π_0. Since the top morphism is surjective, so is $\pi_0(i^\flat(-, \mathbb{R})$. Since smooth rings are product preserving (up to weak equivalence), $\pi_0(i^\flat(-, \mathbb{R}^p))$ is also surjective. $\square$

Theorem 4.1.14. *Let $\mathcal{X}$ and $\mathcal{X}'$ be derived manifolds.*

1. *If $g : \mathcal{X} \to \mathbb{R}^p$ is a morphism of derived manifolds, then the solution to the equation $g = 0$ in $\mathcal{X}$ is again a derived manifold.*

2. *If M is a manifold and $f : \mathcal{X} \to M$ and $g : \mathcal{X}' \to M$ are morphism of derived manifolds, then the fiber product*

$$\begin{tikzcd} \mathcal{X} \times_M \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{X} \arrow[d, "f"] \\ \mathcal{X}' \arrow[r, "g"'] & M, \end{tikzcd}$$

   exists in the category of derived manifolds.

3. *Suppose that $\mathcal{U} = (U, \mathcal{O}_X|_U) \subset \mathcal{X}$ and $\mathcal{U}' = (U', \mathcal{O}_X|_{U'}) \subset \mathcal{X}'$ are open derived submanifolds, and suppose that there is an isomorphism of topological spaces $\phi : U \to U'$ and a weak equivalence of sheaves $\phi^\flat : \phi^*\mathcal{O}'_U \to \mathcal{O}_U$. Then the union of $\mathcal{X}$ and $\mathcal{X}'$ along $\mathcal{U}$ exists in the category of derived manifolds, if the underlying space of the union is Hausdorff.*

Proof. We begin with part 1. The condition of a smooth-ringed space being a derived manifold is local, so we may assume that $\mathcal{X} = \mathbb{R}^n_{f=0}$ is a principal derived manifold (where $f : \mathbb{R}^n \to \mathbb{R}^m$). There is a natural closed immersion $i : \mathcal{X} \to \mathbb{R}^n$. By Proposition 4.1.13, there exists a map $g' : \mathbb{R}^n \to \mathbb{R}^p$ such that $g'i$ is homotopic to g. Let $\mathcal{Y}$ denote the zeroset of g in and let $\mathcal{Y}'$ denote the zeroset of $g'i$ in $\mathcal{X}$. First note that $\mathcal{Y}$ is equivalent to $\mathcal{Y}'$. Indeed, since g and $g'i$ are in the same path component of $\mathrm{Map}(\mathcal{X}, \mathbb{R}^p)$, they represent the same map of underlying topological spaces $X \to \mathbb{R}^p$, so $Y \cong Y'$. On structure sheaves, $\mathcal{O}_Y$ and $\mathcal{O}_{Y'}$ are each the homotopy colimit of a diagram

$$\begin{tikzcd}
\mathcal{H}_{\mathbb{R}^p} \arrow[r] \arrow[d] & \mathcal{H}_\bullet \\
\mathcal{O}_X &
\end{tikzcd}$$

in which the vertical maps are the same up to homotopy. Using basic model category theory, these homotopy colimits are weakly equivalent.

It suffices to show that $\mathcal{Y}'$ is a derived manifold; we will show that it is the zeroset of the map $(f, g') : \mathbb{R}^n \to \mathbb{R}^{m+p}$. We do so by forming the all-Cartesian diagram

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] & \\
\mathcal{X} \arrow[r, "g'i"] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathbb{R}^p \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\
\mathbb{R}^n \arrow[r, "(f,g')"] & \mathbb{R}^m \times \mathbb{R}^p \arrow[r, "\pi_1"] & \mathbb{R}^m.
\end{tikzcd}$$

The bottom right square is a fiber product, and by definition the bottom rectangle and the top square are fiber products. Therefore, the left rectangle is a fiber product, proving part 1.

For part 2, since the issue is again a local one, we may assume that $M = \mathbb{R}^p$. The diagram

$$\begin{tikzcd}
\mathcal{X} \times_{\mathbb{R}^p} \mathcal{Y} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\diamond"] & & \bullet \arrow[d, "0"] \\
\mathcal{X} \times \mathcal{Y} \arrow[r] & \mathbb{R}^p \times \mathbb{R}^p \arrow[r, "-"] & \mathbb{R}^p
\end{tikzcd}$$

is Cartesian, where $- : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$ is coordinate-wise subtraction. Since $\mathcal{X} \times \mathcal{Y}$ is a derived manifold by Proposition 4.1.10 and since derived manifolds are closed under solving equations by part 1, the fiber product $\mathcal{X} \times_{\mathbb{R}^p} \mathcal{Y}$ is a derived manifold.

In part 3, we need to construct the union of $\mathcal{X}$ and $\mathcal{X}'$ along a common open subspace $\mathcal{U}$. To do so, we let the underlying space of the union be the union of the underlying spaces $X \cup_U X'$. Consider the mapping cylinder $C = \mathrm{Cyl}(\phi^\flat)$. The natural inclusions of $\mathcal{O}_U$ and $\mathcal{O}'_U$ into C are weak equivalences. We can now define the sheaf $\mathcal{F} = \mathcal{O}_{X \cup X'}$ on a basis of $X \cup X'$ as follows. On open sets $W \subset U$, let $\mathcal{F}(W) = C(W)$. On open sets W such that $W \subset X$ and $W \not\subset U$, let $\mathcal{F}(W) = \mathcal{O}_X(W)$. Finally, on open sets W such that $W \subset X'$ and $W \not\subset U$, let $\mathcal{F}(W) = \mathcal{O}_{X'}(W)$. We can then complete this to a sheaf $\mathcal{F}$ on $X \cup X'$, and it is clear that the restrictions of $\mathcal{F}$ to X and X' are weakly equivalent to $\mathcal{O}_X$ and $\mathcal{O}_{X'}$ respectively. The Hausdorff condition is simply part of the definition of derived manifold.

□

Lemma 4.1.15. *Suppose that M is a manifold, that $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Y}'$ are derived manifolds, and that the diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\diamond"] & \mathcal{Y}' \arrow[d] \\ \mathcal{Y} \arrow[r] & M \end{tikzcd}$$

is homotopy Cartesian. If $\mathcal{U}$ is a derived submanifold of $\mathcal{Y}$, then the preimage of $\mathcal{U}$ in $\mathcal{X}$ is a derived submanifold of $\mathcal{X}$. In particular, if $X \subset U$ as topological spaces, then the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\diamond"] & \mathcal{Y}' \arrow[d] \\ \mathcal{U} \arrow[r] & M \end{tikzcd}$$

is homotopy Cartesian.

Proof. This follows from the fact that fiber products can be constructed locally.

□

Definition 4.1.16. We say that morphisms $a_0 : M_0 \to M$ and $a_1 : M_1 \to M$ of manifolds are *transverse* if the following condition is satisfied:

- For any chart $\mathbb{R}^n \subset M$, let $b_0 : N_0 \to \mathbb{R}^n$ and $b_1 : N_1 \to \mathbb{R}^n$ be the pullbacks of a_0 and a_1.
- Then the composite

$$(b_0 - b_1) : N_0 \times N_1 \xrightarrow{(b_0,b_1)} \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{-} \mathbb{R}^n$$

  is a submersion.

Remark 4.1.17. This is equivalent to the usual definition but is a more useful formulation for us.

Corollary 4.1.18. *Suppose that $a : M_0 \to M$ and $b : M_1 \to M$ are morphisms of manifolds. Suppose that they are transverse, so that a fiber product N exists* *in the category of manifolds. If $\mathcal{X}$ is the fiber product*

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & M_1 \arrow[d, "f"] \\
M_2 \arrow[r, "g"'] & M
\end{tikzcd}$$

taken in the category of derived manifolds, then the natural map $N \to \mathcal{X}$ is an isomorphism of derived manifolds.

Proof. It suffices to check this locally, so we may assume that $M = \mathbb{R}^n$. Let $D : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be the diagonal map. We may write $\mathcal{X}$ as the fiber product of the left square in the diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r] \arrow[d] & \mathbb{R}^n \arrow[r] \arrow[d, "D"] & \bullet \arrow[d, "0"] \\
M_1 \times M_2 \arrow[r, "(f,g)"] & \mathbb{R}^n \times \mathbb{R}^n \arrow[r, "-"] & \mathbb{R}^n.
\end{tikzcd}$$

The left and right squares are fiber products, so the large rectangle is as well. Since a and b are transverse, the composition of the bottom arrows is a submersion, and N is the pullback of $0 : \bullet \to \mathbb{R}^n$ along this composition (by Corollary A.3.4) in the category of smooth manifolds. Thus $\mathcal{X} \cong N$ by Proposition 4.1.9. $\square$

4.2 Perspective

One may think of the category **dMan** of derived manifolds as an enlargement of the category of manifolds such that all fiber products of manifolds exist and are "correct" in **dMan**. (Evidence that fiber products are correct in **dMan** is found in Corollaries 6.3.4 and 6.3.5, where we show that their fundamental classes behave in accordance with various cohomological constructions.) Then the above Corollary 4.1.18 can be interpreted as saying that if submanifolds are transverse then their intersection (in the category of manifolds) is already correct. More generally, Proposition 4.1.9 says that the fiber product of maps s and a in the category of manifolds are correct if s is a submersion.

This was of course woven in to the structure of derived manifolds. In Chapter 2 we defined smooth rings in such a way that they were submersion-pullback preserving. The structure Theorem 3.3.3 showed that the structure sheaf of smooth rings on a local smooth-ringed space X gives maps from X to manifolds. Together, these ideas forced Proposition 4.1.9 to hold.

Many fiber products do not exist in the category of manifolds; for example one takes a generic height function $h : T^2 \to \mathbb{R}$ on a torus and can find two points p, q in $\mathbb{R}$ for which the fiber product

$$\begin{tikzcd} ? \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \bullet \arrow[d] \\ T^2 \arrow[r, "h"'] & \mathbb{R} \end{tikzcd}$$

is singular and hence does not exist in **Man**. Or much more generally, any closed subset of a manifold can be realized as a fiber product of smooth manifolds.

On the other hand, some fiber products exist in the category of manifolds but are "incorrect" from the standpoint of intersection theory. For example, if curves C and C' in $\mathbb{R}^2$ meet tangentially at a single point p, then $\{p\}$ will be the fiber product in the category of manifolds, but the fundamental cohomology class of $\{p\}$ will not be the cup product of the fundamental cohomology classes of C and C'. In this case the fiber product has the correct dimension, but somehow loses track of nilpotents. For many cases even this is too much to expect. For example the intersection of the origin with itself in $\mathbb{R}$ has codimension 1, when it should (intersection-theoretically) have codimension 2.

Again, we take the standpoint that fiber products taken in the category of derived manifolds are correct. We have already shown that transverse pullbacks exist and are correct in the category of manifolds. We now show the converse, that if $M \to N \leftarrow M'$ is a *non-transverse* diagram, then the fiber product taken in the category of manifolds either does not exist or is incorrect (i.e. differs from the fiber product taken in the category of derived manifolds). The proof uses the theory of cotangent complexes, which will be developed in the next chapter (and Theorem 4.2.1 will not be applied until Chapter 6).

Theorem 4.2.1. *Suppose that $a : M_0 \to N$ and $b : M_1 \to N$ are maps of manifolds, and suppose that a manifold P exists such that the diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & M_1 \arrow[d, "b"] \\ M_0 \arrow[r, "a"'] & N \end{tikzcd}$$

is a pullback in the category of manifolds. Suppose that

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond", very near start] & M_1 \arrow[d] \\ M_0 \arrow[r] & N \end{tikzcd}$$

is a pullback in the category of derived manifolds. Then a and b are transverse if and only if the natural map

$$\phi : P \to \mathcal{X}$$

is an isomorphism of derived manifolds.

Proof. If a and b are transverse then ϕ is an isomorphism by Corollary 4.1.18. For the converse, suppose that ϕ is an isomorphism. After covering M_0, M_1 and N with charts and proceeding as in the proof of Proposition 4.1.18, we may reduce to the case in which $M_0 = \mathbb{R}^n, M_1 = \bullet$, and $N = \mathbb{R}^m$. We must show that $a : \mathbb{R}^n \to \mathbb{R}^m$ is a submersion.

Let $p : \bullet \to P$ be a point in P and let $q = \phi(p) : \bullet \to \mathcal{X}$ be its image in $\mathcal{X}$. Then $L_p \cong L_q$, which means that L_q has homology concentrated in degree 1, by Lemmas 5.2.10 and 5.2.11. The situation can now be summed up by the diagram

$$\begin{tikzcd} \bullet \arrow[r, "q"] \arrow[d, equal] & \mathcal{X} \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d] \\ \bullet \arrow[r, "r"] & \mathbb{R}^n \arrow[r, "a"] & \mathbb{R}^m, \end{tikzcd}$$

in which the right square is Cartesian. Since $L_q \cong r^* L_a[1]$, there is a distinguished triangle

$$L_q[-1] \to L_r \to L_{a \circ r} \to L_q.$$

Since L_r and $L_{a\circ r}$ have homology concentrated in degree 1, the map $H_1(L_r) \to H_1(L_{a\circ r})$ must be injective. But $H_1(L_r)$ and $H_1(L_{a\circ r})$ are the cotangent spaces $T_r\mathbb{R}^n$ and $T_{a(r)}\mathbb{R}^m$ respectively, and $H_1(L_r) \to H_1(L_{a\circ r})$ is the map on cotangent spaces induced by a. Since it is injective, a is a submersion by definition. $\square$

Chapter 5

The Cotangent Complex

The cotangent complex on a derived manifold is a homotopical version of a cotangent bundle on a manifold. It is a sheaf-theoretic linearization of the derived manifold. The cotangent complex will play a critical role in the theory of derived manifolds. First, it provides a way of finding the dimension of a derived manifold. Second, it will be crucial in proving one of our main results, that the cobordism ring of derived manifolds is isomorphic to the cobordism ring of manifolds.

5.1 Definition of the Cotangent Complex

We first recall Schwede's definition of spectra from [25]. Let $\mathcal{M}$ be a pointed simplicial model category which admits the small object argument. Define $\mathcal{M}^\infty$ to be the category whose objects are sequences of objects $X_i, i \in \mathbb{N}$ and maps $\Sigma X_i \to X_{i+1}$, where $\Sigma : \mathcal{M} \to \mathcal{M}$ is the functor which takes an object X to the homotopy pushout in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\diamond"] & \Sigma X. \end{tikzcd}$$

An Ω-spectrum is a spectrum in which each X_i is fibrant, and each of the adjoint maps $X_i \to \Omega(X_{i+1})$ is a weak equivalence in $\mathcal{M}$. There is a functor $Q : \mathcal{M}^\infty \to \mathcal{M}^\infty$ that replaces a spectrum with an Ω spectrum.

The category $\mathcal{M}^\infty$ of spectra in $\mathcal{M}$ can be given a simplicial model category structure as follows. Define a map $f : X \to Y$ to be a cofibration if the induced maps

$$X_0 \to Y_0 \text{ and } X_n \amalg_{\Sigma X_{n-1}} \Sigma Y_{n-1} \to Y_n$$

are cofibrations in $\mathcal{M}$. Define f to be a weak equivalence if $Qf : QX \to QY$ is a levelwise weak equivalence. Define fibrations by the right lifting property.

Suppose that $F : \mathcal{L} \to \mathcal{M}$ is a simplicial (pointed) functor between pointed simplicial model categories. Then there is an induced functor $F^\infty : \mathcal{L}^\infty \to \mathcal{M}^\infty$ given by levelwise application of F. That is, for $L \in \mathcal{L}^\infty$, define $F^\infty(L)_i = F(L_i)$. There is a left Quillen functor

$$\Sigma^\infty : \mathcal{M} \to \mathcal{M}^\infty$$

which takes an object $X \in \mathcal{M}$ to the sequence $X_i = \Sigma^i X$ with identity maps $\Sigma X_i \to X_{i+1}$. There is a natural transformation $\Sigma^\infty \circ F \to F^\infty \circ \Sigma^\infty$ which may not be an isomorphism. However, it is an isomorphism if $\Sigma \circ F \xrightarrow{\cong} F \circ \Sigma$ is an isomorphism.

Suppose that $\mathcal{N}$ is a simplicial model category which is not necessarily pointed. Let $\emptyset$ and $*$ represent the initial and terminal object in $\mathcal{N}$, respectively. There is an initial pointed simplicial model category $-_+ : \mathcal{N} \to \mathcal{N}_+$ under $\mathcal{N}$, where $-_+$ takes an object $X \in \mathcal{N}$ to the homotopy pushout

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\diamond"] & X_+. \end{tikzcd}$$

The zero object in $\mathcal{N}_+$ is $\emptyset_+$. We denote by $\Sigma_+^\infty : \mathcal{N} \to \mathcal{N}_+^\infty$ the composition $\mathcal{N} \to \mathcal{N}_+ \xrightarrow{\Sigma^\infty} \mathcal{N}_+^\infty$. There is a distinguished object $*_+ \in \mathcal{N}_+$, the image of the terminal object in $\mathcal{N}$, and similarly there is a distinguished object $*_+^\infty \in \mathcal{N}_+^\infty$.

For any map $f : X \to Y$ in $\mathcal{N}$, let $\mathcal{N}_f = \mathcal{N}_{X//Y}$ denote the category of objects under X and over Y in $\mathcal{N}$. For ease of notation, we denote the category $((\mathcal{N}_f)_+)^\infty$ of spectra in $(\mathcal{N}_f)_+$ simply by $\mathcal{N}_f^\infty$. Clearly, the distinguished object in $\mathcal{N}_f$ is Y_+, which is the homotopy colimit in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Y \arrow[r] \arrow[ur, phantom, "\diamond"] & Y_+, \end{tikzcd}$$

and the distinguished object in $\mathcal{N}_f^\infty$ is $\Sigma^\infty Y_+$.

Definition 5.1.1. Let $\mathcal{N}$ be a simplicial model category and $f : X \to Y$ a map in $\mathcal{N}$. The distinguished object Y_+^∞ in $\mathcal{N}_f^\infty$ is called *the cotangent complex associated to* f and is denoted by $L_f^{\mathcal{N}}$, L_f, or $L_{X/Y}$ depending on whether $\mathcal{N}$ and/or f are clear from context.

Suppose that $\mathcal{N}$ is the category of simplicial commutative $\mathbb{R}$-algebras and $f : A \to B$ is a morphism in $\mathcal{N}$. Then Schwede proves in [25, pp. 102-104] that the spectrum L_f as defined above coincides with the usual notion of cotangent complex (often denoted $L_{B/A}$), as defined by Quillen [23] and Illusie [15]. In particular, L_f is a B-module in the usual sense.

One may think of B-modules as the abelian group objects in $\mathbf{sC}\mathbb{R}_{A//B} = \mathbf{sC}\mathbb{R}_f$. The forgetful functor B-mod $\to \mathbf{sC}\mathbb{R}_{A//B}$ is denoted

$$M \mapsto B \oplus M,$$

where $B \oplus M$ is considered as ring over B with trivial multiplication on the M summand. Its left adjoint is denoted $Ab_{A//B} : \mathbf{sC}\mathbb{R}_{A//B} \to B$-mod, and the left derived functor of $\mathrm{Ab}_{A//B}$, as applied to B, is L_f. This is all explained more thoroughly in [25].

5.2 Cotangent complexes in our context

Definition 5.2.1. If $f : F \to G$ is a map of smooth rings, we let $\mathbb{L}_f := L_f^{\mathbf{SR}}$ and refer to it as the **SR**-*theoretic cotangent complex associated to* f. We let $L_f := L_{Uf}^{\mathbf{sC}\mathbb{R}}$ and refer to it as the *ring-theoretic cotangent complex associated to* f.

Taking the ring-theoretic cotangent complex may not always be a good notion (in a sense indicated by Proposition 5.2.3), but it is if f is surjective on π_0 or if there exists $g : G \to F$ with $g \circ f = \mathrm{id}_F$ (see Proposition 5.2.3 and Lemma 5.2.10 below). However, since these cases are enough to handle closed immersions of derived manifolds, and since ring-theoretic cotangent complexes are much easier to compute with, they will end up playing the more significant role in what follows.

Proposition 5.2.2. *Let $R \xrightarrow{f} S \xrightarrow{g} T$ be a pair of maps in* $\mathbf{sC}\mathbb{R}$*. Then the diagram*

$$\begin{tikzcd} L_f \otimes g \arrow[r] \arrow[d] & L_{g\circ f} \arrow[d] \\ 0 \arrow[r] & L_g \end{tikzcd}$$

is a homotopy pullback square and a homotopy pushout square in $\mathbf{sC}\mathbb{R}^{\infty}_{/T}$.

Proof. This follows from the fact that $\mathbf{sC}\mathbb{R}^{\infty}_{/T}$ is equivalent to the category of T-modules and the above proposition is proven by Illusie, [15, 2.1.2]. However, we prove it explicitly below to gain familiarity with the objects involved.

Let N be any object in $\mathbf{sC}\mathbb{R}^{\infty}_{/T}$. We will prove that the diagram

$$\begin{tikzcd} \mathrm{Map}_{T\text{-mod}}(L_f \otimes g, N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathrm{Map}_{T\text{-mod}}(L_{g\circ f}, N) \arrow[d] \\ \Delta^0 \arrow[r] & \mathrm{Map}_{T\text{-mod}}(L_g, N) \end{tikzcd} \tag{5.1}$$

is a homotopy pullback, which also implies that

$$\begin{tikzcd}
L_f \otimes g \arrow[r] \arrow[d] & L_{g\circ f} \arrow[d] \\
0 \arrow[r] & L_g
\end{tikzcd}$$

is a homotopy pushout. This will prove the theorem because every homotopy pushout square is a homotopy pullback square in a stable model category.

Let A, B, C, and D respectively denote the following four diagrams, each of which is a fiber product of simplicial sets:

$$\begin{tikzcd}
\mathrm{Map}_{S//T}(T, T\oplus N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}_{\mathbb{R}//T}(T, T\oplus N) \arrow[d, "g"] \\
\Delta^0 \arrow[r, "g"'] & \mathrm{Map}_{\mathbb{R}//T}(S, T\oplus N),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}_{R//T}(T, T\oplus N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}_{\mathbb{R}//T}(T, T\oplus N) \arrow[d, "g\circ f"] \\
\Delta^0 \arrow[r, "g\circ f"'] & \mathrm{Map}_{\mathbb{R}//T}(R, T\oplus N),
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}_{S//T}(S, T\oplus N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}_{\mathbb{R}//T}(S, T\oplus N) \arrow[d, "\mathrm{id}"] \\
\Delta^0 \arrow[r, "g"'] & \mathrm{Map}_{\mathbb{R}//T}(S, T\oplus N), \mathit{and}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Map}_{R//T}(S, T\oplus N) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Map}_{\mathbb{R}//T}(S, T\oplus N) \arrow[d, "f"] \\
\Delta^0 \arrow[r, "g\circ f"'] & \mathrm{Map}_{\mathbb{R}//T}(R, T\oplus N).
\end{tikzcd}$$

The four northwest corners (from A, B, C, and D) come together form the diagram

$$\begin{tikzcd}
\mathrm{Map}_{S//T}(T, T\oplus N) \arrow[r] \arrow[d] & \mathrm{Map}_{R//T}(T, T\oplus N) \arrow[d] \ , \\
\mathrm{Map}_{S//T}(S, T\oplus N) \arrow[r] & \mathrm{Map}_{R//T}(S, T\oplus N)
\end{tikzcd}$$

and the other three corners follow suit. The total diagram commutes. Moreover, the southwest, northeast, and southeast diagrams are pullback squares. Thus it follows that the northwest diagram is a pullback square as well. Taking adjoints, one finds that Diagram 5.1 is indeed a pullback for any N.

$\square$

Proposition 5.2.3. *Suppose that $f : F \to G$ is a morphism of smooth rings that is surjective on π_0. Let $U = |\cdot| : \mathbf{SR} \to \mathbf{sC}\mathbb{R}$ denote the underlying simplicial commutative $\mathbb{R}$-algebra functor. Then the diagram*

$$\begin{tikzcd} \mathbf{SR}_f \arrow[r, "\Sigma^\infty_+"] \arrow[d, "U"'] & \mathbf{SR}^\infty_f \arrow[d, "U^\infty"] \\ \mathbf{sC}\mathbb{R}_f \arrow[r, "\Sigma^\infty_+"'] & \mathbf{sC}\mathbb{R}^\infty_{Uf} \end{tikzcd}$$

commutes up to weak equivalence. Moreover, there is a natural weak equivalence

$$L_f \simeq U^\infty \mathbb{L}_f.$$

Proof. The second assertion follows from the first because then

$$L_f = \Sigma^\infty_+ U(G \otimes_F G) \simeq U^\infty \Sigma^\infty_+ (G \otimes_F G) = U^\infty \mathbb{L}_f.$$

The functor $-_+ : \mathbf{SR}_f \to \mathbf{SR}_f$ takes an object $F \xrightarrow{a} X \xrightarrow{b} G$ to $F \to X \otimes_F G \to G$, where $X \otimes_F G$ is the homotopy pushout in the diagram

$$\begin{tikzcd} F \arrow[r, "f"] \arrow[d, "a"'] & G \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "\diamond"] & X \otimes_F G. \end{tikzcd}$$

Since f is surjective on π_0, this homotopy pushout commutes with U by Theorem 3.5.11, up to weak equivalence. Since the induced map

$$G \to X \otimes_F G \to G$$

is a retract, the map $X \otimes_F G \to G$ is surjective on π_0. Hence suspension also commutes with U, and the result follows by definition of the suspension spectrum.

□

Definition 5.2.4. Suppose that X is a topological space and $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves on X. We define *the* $\mathbf{SR}$*-theoretic cotangent complex associated to* f to be the sheaf associated to the presheaf

$$V \mapsto \mathbb{L}_{f|V},\ V \in \mathrm{Op}(X).$$

We define *the ring-theoretic cotangent complex associated to* f to be the sheaf associated to the presheaf

$$V \mapsto L_{f|V},\ V \in \mathrm{Op}(X).$$

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of derived manifolds and $f^\flat : f^*\mathcal{O}_Y \to \mathcal{O}_X$ is the corresponding morphism of sheaves on X, we define $\mathbb{L}_f := \mathbb{L}_{f^\flat}$ and $L_f := L_{f^\flat}$.

Lemma 5.2.5. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a topological space X, and let $p : \{*\} \to X$ be a point. There are weak equivalences on stalks*

$$p^*(\mathbb{L}_f) \simeq \mathbb{L}_{p^*(f)} \text{ and } p^*(L_f) \simeq L_{p^*(f)}.$$

Proof. In both cases, p^* is a left Quillen functor and the cotangent complex functor is obtained by a series of homotopy colimits. These two functors commute.

□

Lemma 5.2.6. *Let X be a topological space, let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves of smooth rings on X, and let $\widehat{f} : \widehat{\mathcal{F}} \to \widehat{\mathcal{G}}$ be the corresponding morphism of their sheafification on X. Then $\mathbb{L}_f \simeq \mathbb{L}_{\widehat{f}}$ and $L_f \simeq L_{\widehat{f}}$.*

Proof. This becomes clear after taking stalks.

□

Corollary 5.2.7. *Suppose that $f : \mathcal{X} \to \mathcal{Y}$ is a closed immersion of derived manifolds, and let $U = |\cdot|$ denote the underlying $\mathbb{R}$-algebra functor. Then $U^\infty \mathbb{L}_f \simeq L_f$.*

Proof. Let $p : \bullet \to \mathcal{X}$ be an arbitrary point. It suffices to show that the map

$$p^* U^\infty \mathbb{L}_{f^\flat} \to p^* L_{(Uf)^\flat}$$

is a weak equivalence. By Lemma 3.5.7, there is a natural isomorphism of functors $p^*U \xrightarrow{\cong} Up^*$. Also, note that $p^*(f^\flat)$ is surjective on π_0 by definition of f being a closed immersion. Therefore, by Lemma 5.2.5 and Proposition 5.2.3, we have

$$\begin{aligned} p^* U^\infty \mathbb{L}_{f^\flat} \simeq U^\infty p^* \mathbb{L}_{f^\flat} &\simeq U^\infty \mathbb{L}_{p^*(f^\flat)} \\ &\simeq L_{U(p^*(f^\flat))} \\ &\simeq L_{p^*(Uf^\flat)} \\ &\simeq p^* L_{(Uf^\flat)} \end{aligned}$$

as desired.

□

Definition 5.2.8. Let $f : A \to B$ be a morphism of $\mathbb{R}$-algebras. Define the *$\mathbb{R}$-module of Kahler differentials of f* as

$$\Omega^1_f := \pi_0 L_f,$$

where L_f is the cotangent complex associated to f, as defined in Definition 5.2.1.

Define the *relative dimension of f* to be the Euler characteristic

$$\chi(L_f) = \Sigma_{i=0}^{\infty} (-1)^i \dim(\pi_i L_f)$$

of the cotangent complex associated to f. Here, $\pi_i L_f$ is an $\mathbb{R}$-module and the dimension is taken as such.

Suppose X is a topological space and $f : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of $\mathbb{R}$-algebras on X. Define Ω^1_f as the sheaf associated to the presheaf

$$V \mapsto \Omega^1_{f|V},\ V \in \mathrm{Op}(X).$$

Let $\mathcal{X}$ be a derived manifold and $t : \mathcal{X} \to \bullet$ be the terminal map. If $p : \bullet \to \mathcal{X}$ is a point, we define the *dimension of $\mathcal{X}$ at p*, denoted by $\dim_p(\mathcal{X})$, to be $\chi(p^* L_t)$, the Euler characteristic of the pullback along p of the ring-theoretic cotangent complex associated to t.

Remark 5.2.9. Note that if $t : \mathcal{X} \to \bullet$ is the terminal map out of a derived manifold $\mathcal{X}$, then the ring-theoretic cotangent complex L_t is not necessarily the spectrum underlying $\mathbb{L}_t$ (see Definition 5.2.1). However, if $p : \bullet \to X$ is a point, then the pullback $p^* L_t$, which is referenced in Definition 5.2.8 is well-behaved: See Lemma 5.2.10 below. (Alternatively, we could have defined $\dim_p(\mathcal{X})$ to be the additive inverse of $\chi(L_p)$ and avoided this issue.)

Lemma 5.2.10. *Let A and B be smooth rings, and suppose that the composition $A \xrightarrow{i} B \xrightarrow{p} A$ is equal to the identity. Then there is a natural isomorphism*

$$L_p \xrightarrow{\cong} L_i[1] \otimes p.$$

Furthermore, one has weak equivalences $L_p \simeq U^\infty \mathbb{L}_p$ and $L_i \simeq U^\infty \mathbb{L}_i$.

Proof. By Proposition 5.2.2, the diagram

$$\begin{tikzcd} L_i \otimes p \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & L_{\mathrm{id}} \arrow[d] \\ 0 \arrow[r] & L_p \end{tikzcd}$$

is a pullback. The module L_{id} is isomorphic to 0, and the first assertion follows from the long exact sequence of homotopy groups. The second assertion follows from Proposition 5.2.3, since p is automatically surjective on π_0.

□

Lemma 5.2.11. *Let $t : \mathbb{R}^n \to \bullet$ be the terminal map. Then L_t is locally free, 0-truncated, and of rank n on $\mathbb{R}^n$. Thus, for any point $p : \bullet \to \mathbb{R}^n$, we have*

$$\dim_p \mathbb{R}^n = n.$$

Proof. For any point $p : \bullet \to \mathbb{R}^n$, $p^* L_t[1] \cong L_p$. Each L_p is free (by definition), so L_t is locally free. It suffices to show that L_p is perfect, has homology concentrated in degree 1, and that $\dim(H_1(L_p)) = n$. We may assume that p is the origin. Let C be the local ring

$$C = C^\infty_p(\mathbb{R}^n),$$

and let I be the ideal $(x_1, \ldots, x_n)$, where x_i is the ith coordinate function on any neighborhood of p. By Hadamard's Lemma (3.5.10) $C/I \cong \mathbb{R}$, and we are interested in the cotangent complex of $|p^\flat| : C \to C/I$.

Since $(x_1, \ldots, x_n)$ is a regular sequence, the Koszul complex for C/I as a C-module is a resolution of C/I. By [15, III.3.2.4], there is a quasi-isomorphism

$$L_p \to I/I^2[1].$$

The vector space I/I^2 has dimension n, completing the proof.

$\square$

Proposition 5.2.12. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map, and let $\mathcal{U} = \mathbb{R}^n_{f=0}$ be the corresponding principal derived manifold. Then for any point $p : \bullet \to \mathcal{U}$,*

$$\dim_p \mathcal{U} = n - m.$$

Proof. We have a pullback diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, "t"] \arrow[d, "i"'] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "z"] \\ \mathbb{R}^n \arrow[r, "f"'] & \mathbb{R}^m. \end{tikzcd} \tag{5.2}$$

By abuse of notation, let p denote not just the map $p : \bullet \to \mathcal{U}$ but all of it's compositions with the morphisms in Diagram 5.2. The diagram

$$\begin{tikzcd} p^*\mathcal{H}_{\mathbb{R}^m}(\mathbb{R}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & p^*\mathcal{H}_{\bullet}(\mathbb{R}) \arrow[d] \\ p^*\mathcal{H}_{\mathbb{R}^n}(\mathbb{R}) \arrow[r] & p^*\mathcal{O}_U(\mathbb{R}) \end{tikzcd}$$

is a pushforward in $\mathbf{sC}\mathbb{R}$ by Theorem 3.5.11. By [15, II.2.2.1] the cotangent complex for the top map is an extension of scalars applied to the cotangent complex of the bottom map, p^*L_f (and thus has the same Euler characteristic). By Lemmas 5.2.10 and 5.2.11, one can easily show that the Euler characteristic of p^*L_f is $n - m$.

$\square$

Corollary 5.2.13. *If a derived manifold $\mathcal{X}$ is non-disjoint (see Definition 4.1.2), then it has constant dimension.*

Proof. By Proposition 5.2.12, every two points in a principal derived manifold have the same dimension; since isomorphisms of derived manifolds preserve dimension, the result follows from a standard topological argument.

$\square$

Proposition 5.2.14. *Suppose that $\mathcal{B}, \mathcal{C}$, and $\mathcal{D}$ are derived manifolds, equidimensional of dimension b, c, and d, respectively, and that the fiber product*

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{B} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

exists in the category of derived manifolds. Then $\mathcal{A}$ is equidimensional of dimension a and

$$a + d = b + c$$

Proof. We may instead write

$$\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d, "x"'] \arrow[dr, phantom, "\diamond"] & \mathcal{D} \arrow[d, "y"] \\ \mathcal{B} \times \mathcal{C} \arrow[r] & \mathcal{D} \times \mathcal{D} \end{tikzcd}$$

where $\mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is the diagonal map and in particular a closed immersion. The map $\mathcal{A} \to \mathcal{B} \times \mathcal{C}$ is also a closed immersion so the ring-theoretic cotangent complex L_x is a pullback of, and hence the same dimension as, L_y. Thus $a - (b + c) = d - (d + d)$ as desired.

$\square$

Example 5.2.15. Let $\mathcal{G}$ be the squared origin of Example 2.3.26. Recall that $\mathcal{G}$ is the fiber product in the diagram

$$\begin{tikzcd} \mathcal{G} \arrow[r, "a"] \arrow[d, "a"'] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "p"'] & \mathbb{R}, \end{tikzcd}$$

where $p : \bullet \to \mathbb{R}$ is the point corresponding to the origin in $\mathbb{R}$. By [15, 2.2.1], $L_a = a^* L_p$, which is an $\mathbb{R}$-vector space of dimension 1 concentrated in degree 1. We see that $\mathcal{G}$ has dimension -1.

5.3 Other calculations

This section includes some technical calculations which will be useful later.

Let $f : S \to T$ be a morphism of simplicial commutative rings. Recall that

$$\mathrm{Ab}_{S//T} : \mathbf{sC}\mathbb{R}_{S//T} \to T\text{-mod}$$

is the left adjoint of the functor

$$M \mapsto T \oplus M : T\text{-mod} \to \mathbf{sC}\mathbb{R}_{S//T}.$$

Let $ker : \mathbf{sC}\mathbb{R}_{S//T} \to S$-mod be the functor which takes $S \xrightarrow{i} A \xrightarrow{p} T$ to $\ker(p)$. Let $U : \mathbf{sC}\mathbb{R}_{S/} \to S$-mod denote the forgetful functor, which is right adjoint to Sym_S. We will denote $\mathrm{Map}_{\mathbf{sC}\mathbb{R}_{S//T}}$ simply by $\mathrm{Map}_{S//T}$.

Lemma 5.3.1. *Let $f : S \to T$ be a morphism of simplicial commutative rings. The functor* $\ker : \mathbf{sC}\mathbb{R}_{S//T} \to S$-mod *is right adjoint to a functor*

$$\mathrm{Sym}_{S//T} : S\text{-mod} \to \mathbf{sC}\mathbb{R}_{S//T}.$$

Proof. Choose an S-module M and an object $S \xrightarrow{i} A \xrightarrow{p} T \in \mathbf{sC}\mathbb{R}_{S//T}$. We begin with the natural isomorphism

$$\mathrm{Map}_{S\text{-mod}}(M, \ker p) \cong \mathrm{Map}_{S\text{-mod}/T}(M \xrightarrow{0} UT, UA \xrightarrow{Up} UT).$$

It is not hard to show using the adjointness of Sym_S and U that there is another natural isomorphism

$$\mathrm{Map}_{S\text{-mod}/T}(M \xrightarrow{0} U(T), U(A) \xrightarrow{U(p)} U(T)) \cong \mathrm{Map}_{S//T}(\mathrm{Sym}(M) \xrightarrow{\mathrm{Sym}(0)} T, A \xrightarrow{p} T).$$

The composition of these isomorphisms yields the result.

□

Lemma 5.3.2. *Let $f : S \to T$ be a morphism of simplicial commutative rings. The composition of functors*

$$\mathrm{Ab}_{S//T} \circ \mathrm{Sym}_{S//T} : S\text{-mod} \to T\text{-mod}$$

is isomorphic to the extension of scalars functor $- \otimes_S T$.

Proof. Let $U : T$-mod $\to S$-mod be restriction of scalars. For any S-module N, we have

$$\begin{aligned}\mathrm{Map}_{T\text{-mod}}(\mathrm{Ab}_{S//T}(\mathrm{Sym}_{S//T}(M)), N) &\cong \mathrm{Map}_{S//T}(\mathrm{Sym}_{S//T}(M), T \oplus N)\\ &\cong \mathrm{Map}_{S\text{-mod}}(M, U(N)).\end{aligned}$$

□

Lemma 5.3.3. *Let $f : S \to T$ be a morphism of simplicial commutative rings. Let N be a free S-module, let $R = \mathrm{Sym}_S(N)$ be the symmetric algebra of N, and let $p : S \to R$ be the canonical map. Let $z : R \to S$ be the adjoint of the zero section $N \xrightarrow{0} S$. Finally consider the simplicial set $K = \mathrm{Map}_{S//T}(R, S)$, which consists of sections $s : R \to S$ of p such that the diagram*

$$\begin{tikzcd} R \arrow[r, "s"] \arrow[d, "z"'] & S \arrow[d, "f"] \\ S \arrow[r, "f"'] & T \end{tikzcd}$$

commutes. Then K naturally has the structure of an S-module, and the map

$$\lambda : \mathrm{Map}_{S//T}(R, S) \to \mathrm{Map}_{T\text{-mod}}(L_z \otimes f, L_f)$$

is an S-module homomorphism.

Proof. By Lemma 5.3.1, we have a natural isomorphism

$$\mathrm{Map}_{S//T}(R, S) \cong \mathrm{Map}_{S\text{-mod}}(N, \ker f),$$

giving K the structure of an S-module.

By Lemma 5.3.2, the extension of scalars $L_p \otimes z$ is naturally isomorphic to N. The distinguished triangle induced by the retraction $S \xrightarrow{p} R \xrightarrow{z} S$ provides a natural isomorphism $L_z[1] \to L_p \otimes z$. Thus $L_z[1] \otimes f$ is naturally isomorphic to $M \otimes f$. After making these replacements on the source and target, we can replace λ with the naturally isomorphic map

$$\mathrm{Map}_{S\text{-mod}}(N, \ker f) \xrightarrow{\lambda} \mathrm{Map}_{T\text{-mod}}(L_z \otimes f, L_f) \cong \mathrm{Map}_{T\text{-mod}}(N \otimes f, L_f[-1]),$$

and it suffices to show that this is an S-module homomorphism. The S-module structure on the source (resp. target) in effect comes from the diagonal map $N \to N \oplus N$ (resp. $N \otimes f \to (N \otimes f) \oplus (N \otimes f)$), and it is easy to show that λ preserves this structure.

□

Chapter 6

The Fundamental Class

In this chapter we prove that every derived manifold has a fundamental homology class. To do so, we first must prove an imbedding theorem which says that every derived manifold can be imbedded into Euclidean space. This is proved in the standard way; in fact it is easier for us because we do not require that the derivative of the inclusion be non-singular in any sense.

We then move on to define derived manifolds with boundary (or DMBs) and the corresponding notion of derived cobordism. The so-called Collar neighborhood theorem, which states that a neighborhood around the boundary ∂M of a manifold M is isomorphic to $\partial M \times [0,1)$, is not true for derived manifolds. Thus the usual argument that cobordism is an equivalence relation (transitive) fails. However, we get around that by proving that if $A \sim B$ and $B \sim C$ are cobordisms of derived manifolds then one can construct a cobordism $A \sim C$.

Our main result is that the inclusion $i : \mathbf{Man} \to \mathbf{dMan}$ induces an *isomorphism* on the cobordism groups over any absolute neighborhood retract. That is, every derived manifold is derived cobordant to a smooth manifold (surjectivity), and any two smooth manifolds which are derived cobordant are in fact cobordant (injectivity). This is the result that allows us to construct a fundamental class on a derived manifold.

6.1 Imbedding theorem

Definition 6.1.1. A map $f : \mathcal{X} \to \mathcal{Y}$ of derived manifolds is called an *imbedding* if it is a closed immersion and is injective on points.

Lemma 6.1.2. *The zero set of n different coordinate functions $x_1, \ldots, x_n$ on the manifold $\mathbb{R}^{N+n}$ (taken in the category of derived manifolds) is isomorphic to $\mathbb{R}^N$.*

Proof. By induction, we may assume $n = 1$. The big rectangle and the bottom square in the diagram

$$
\begin{tikzcd}
\mathbb{R}^N \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\mathbb{R}^{N+1} \arrow[r, "x_1"] \arrow[d] & \mathbb{R} \arrow[d] \\
\mathbb{R}^N \arrow[r] & \bullet
\end{tikzcd}
$$

are Cartesian, by Proposition 4.1.9. Therefore, the top square is Cartesian as well.

□

Definition 6.1.3. A local smooth-ringed space $(X, \mathcal{O}_X)$ is called *compact* if the underlying topological space X is compact.

The following theorem and proof are reformulations of [5, II.10.7].

Theorem 6.1.4. *If $\mathcal{X}$ is a compact derived manifold, then it can be imbedded in to Euclidean space, $\mathbb{R}^N$ for some N.*

Proof. Every derived manifold can be written as the disjoint union of non-disjoint derived submanifolds (see Definition 4.1.2). Thus we may assume that $\mathcal{X}$ is non-disjoint and has constant dimension n, by Corollary 5.2.13. By definition, we can cover $(X, \mathcal{O}_X)$ by finitely many principal derived manifolds $\mathcal{U}_i, i = 1, \ldots, k$. For each i there is a Cartesian diagram

$$
\begin{tikzcd}
\mathcal{U}_i \arrow[r] \arrow[d, "x^i"'] & \bullet \arrow[d] \\
\mathbb{R}^{n_i} \arrow[r, "f^i"'] & \mathbb{R}^{\ell_i}.
\end{tikzcd}
$$

Here, f^i is the sequence of functions $f^i_1, \ldots, f^i_{\ell_i} : \mathbb{R}^{n_i} \to \mathbb{R}$, and $n = n_i - \ell_i$ by Proposition 5.2.12. The coordinate functions on $\mathbb{R}^{n_i}$ extend along x^i to give functions $x^i_1, \ldots, x^i_n : \mathcal{U}_i \to \mathbb{R}$.

Let $\zeta^i : \mathbb{R}^{n_i} \to \mathbb{R}$ be a smooth function that is 1 on some open disc, 0 outside of some bigger disc, and nonnegative everywhere; let $z^i = \zeta^i \circ x^i : U_i \to \mathbb{R}$. So z_i is identically 1 on an open subset $V_i \subset U_i$ and identically 0 outside some closed neighborhood of V_i in U_i. Define functions $y^i_j : U_i \to \mathbb{R}$ by multiplication

$$y^i_j = z^i x^i_j.$$

We have

$$y^i_j|_{V_i} = x^i_j|_{V_i}, \quad j = 1, \ldots, n_i,$$

and each y^i_j is equal to 0 outside of a closed neighborhood of V_i in U_i. We can thus extend the z^i and the y^i_j to all of X by making them zero outside of U_i.

Let $N = k + \sum_{i=1}^{k} n_i$. The sequence $(z, y^1, \ldots, y^k)$ gives a map

$$W : \mathcal{X} \to \mathbb{R}^N.$$

Let $(b^0, \ldots, b^k)$ be the coordinates on $\mathbb{R}^N$. So $W^\flat$ sends $b^0 \mapsto z$ and $b^i \mapsto y^i (1 \leq i \leq k)$. Finally, let

$$c^i = (b^0, \ldots, \widehat{b^i}, \ldots, b^k, f^i) : \mathbb{R}^N \to \mathbb{R}^{N-n_i} \times \mathbb{R}^{\ell_i}.$$

We first show that W is injective on points. If $W(p) = W(q)$ then $z_j(p) = z_j(q)$ for each j. For some i, $p \in V_i$, so $z_i(p) = 1$, which implies that $q \in V_i$. Finally, since x^i is a monomorphism, $x^i(p) = y^i(p) = y^i(q) = x^i(q)$ implies that $p = q$.

It remains to show that W is a closed immersion (i.e. that it is locally the zeroset of smooth functions on $\mathbb{R}^N$). We will show that for each i, $\mathcal{U}_i$ is the zeroset of c^i. We first construct the following diagram

$$\begin{tikzcd}
\mathcal{U}_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] & \\
\mathbb{R}^{n_i} \arrow[r, "f^i"] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathbb{R}^{\ell_i} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\
\mathbb{R}^N \arrow[r, "c^i"] & \mathbb{R}^{(N-n_i)+\ell_i} \arrow[r, "\pi_1"] & \mathbb{R}^{N-n_i}.
\end{tikzcd}$$

The bottom right square and bottom rectangle are pullbacks by Lemma 6.1.2, so the bottom left square is as well. The pullback of c^i along the vertical map $\mathbb{R}^{\ell_i} \to \mathbb{R}^{N-n_i} \times \mathbb{R}^{\ell_i}$ is f^i, so the top square is Cartesian. Therefore the left rectangle is Cartesian, as desired.

□

Theorem 6.1.5. *If $\mathcal{X}$ is a derived manifold, then it can be imbedded as a closed subset in $\mathbb{R}^N$ for some N.*

Proof. One easily adapts the proof of [5, II.10.8] as in Theorem 6.1.4.

□

Corollary 6.1.6. *If $\mathcal{X}$ is a derived manifold and $f : \mathcal{X} \to \mathbb{R}^m$ is a morphism of derived manifolds, then there exists a closed imbedding $i : \mathcal{X} \to \mathbb{R}^N$ and an extension $f' : \mathbb{R}^N \to \mathbb{R}^m$ such that $f' \circ i = f$.*

Proof. By Theorem 6.1.5, we can find a closed imbedding $j : \mathcal{X} \to \mathbb{R}^n$. Let $N = n + m$, let $i = (j, f) : \mathcal{X} \to \mathbb{R}^N$, and let $f' : \mathbb{R}^N \to \mathbb{R}^m$ be the second projection. Then i is an imbedding and $f' \circ i = f$.

□

Notation 6.1.7. Fix a derived manifold $\mathcal{X}$ of dimension n and an imbedding $W : \mathcal{X} \to \mathbb{R}^N$ for the remainder of this section. Let L_W denote the (ring theoretic) cotangent complex sheaf associated the imbedding, and let $\mathcal{N}_W^\vee = H_1(L_W)$ denote its first homology group, which we call the *conormal sheaf* on $\mathcal{X}$.

Lemma 6.1.8. *The conormal sheaf $\mathcal{N}_W^\vee$ on $\mathcal{X}$ is locally free of dimension $N-n$.*

Proof. Locally, $\mathcal{X}$ is the fiber product in the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "W"'] \arrow[dr, phantom, "\ulcorner", very near start] & \bullet \arrow[d, "z"] \\ \mathbb{R}^N \arrow[r, "f"'] & \mathbb{R}^n \end{tikzcd}$$

and by [15, II.2.2.1], L_W is the pullback of L_z along f. The map z is a section of $t : \mathbb{R}^m \to \bullet$, so Lemmas 5.2.10 and 5.2.11 imply that L_W is a pullback of a translation of L_t, and is thus locally free of rank n.

□

Lemma 6.1.9. *Suppose given a diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "g"] \arrow[d, "f"'] & \mathcal{X}' \arrow[dl, "f'"] \\ \mathcal{Y} & \end{tikzcd}$$

of local smooth-ringed spaces such that g, f, and f' are closed immersions, the underlying map of topological spaces $g : X \to X'$ is onto, and the induced map $g^*L_{f'} \to L_f$ is an isomorphism. Then g is an isomorphism.

Proof. It suffices to prove this on stalks; thus we may assume that X and X' are points and that $|\mathcal{O}_X|$ and $|\mathcal{O}_{X'}|$ are local simplicial commutative rings. Let I be the (stalk of the) ideal sheaf of g. By the distinguished triangle associated to the composition $f' \circ g = f$, we find that $L_g = 0$, so $I/I^2 = 0$. Thus, by Nakayama's lemma, I=0, so g induces an isomorphism of sheaves of underlying simplicial commutative rings. The result now follows from Corollary 3.3.7.

□

If X is a topological space and F is a locally free sheaf on X, let $V(F)$ denote the associated vector bundle on X.

Underlying the imbedding $W : \mathcal{X} \to \mathbb{R}^N$ is a closed imbedding $X \to \mathbb{R}^N$ of topological spaces. Since $\mathbb{R}^N$ is paracompact, we can extend the normal bundle on X to an open neighborhood of X in $\mathbb{R}^N$; that is we can find an open submanifold $M \subset \mathbb{R}^N$ with $X \subset M$ and a vector bundle $p : E \to M$ such that $W^*E \cong V(\mathcal{N}_W)$.

Notation 6.1.10. The manifold M and vector bundle $p : E \to M$ obtained above are non-canonical, so we choose a vector bundle E as above and fix it for the remainder of this section. Let $f : \mathcal{X} \to M$ denote the inclusion, and let $z : M \to E$ denote the zero section of p. Note that, because $M \subset \mathbb{R}^N$ is an open neighborhood, we have $\mathcal{N}_f = \mathcal{N}_W$.

Proposition 6.1.11. *Suppose that $s : M \to E$ is a section of p such that the diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "f"'] & M \arrow[d, "s"] \\ M \arrow[r, "z"'] & E \end{tikzcd} \tag{6.1}$$

commutes. It induces a map

$$\lambda_s : f^*(E) \to \mathcal{N}_f^\vee,$$

which is an isomorphism if and only if Diagram 6.1 is a pullback.

Proof. On any open subset U of X, we are in the situation of Lemma 5.3.3, with $R = \mathcal{O}_E(U), S = \mathcal{O}_M(U)$, and $T = \mathcal{O}_X(U)$. The section s gives rise to a map $\lambda_s : L_z \otimes f, L_f$. Both the source and target have homology concentrated in degree 1; the first can be identified with f^*E and the second can be identified with $\mathcal{N}_f^\vee$.

The map λ_s comes to us via Lemma 5.3.3. If the diagram above is a pullback, then by [15, II.2.2.1], we know that λ_s is an isomorphism, so we have only to prove the converse.

Suppose that λ_s is an isomorphism, let $\mathcal{X}'$ be the fiber product in the diagram

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "f'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[d, "s"] \\ M \arrow[r, "z"'] & E, \end{tikzcd}$$

and let $g : \mathcal{X} \to \mathcal{X}'$ be the induced map. Since the composition $X \xrightarrow{g} X' \xrightarrow{f'} M$, namely f, is a closed immersion, so is g. By assumption, g induces an isomorphism on cotangent complexes $g^*L_{f'} \xrightarrow{\cong} L_f$. The result follows from Lemma 6.1.9.

□

The following is a very important improvement on the imbedding Theorem 6.1.5. It says that every compact derived manifold $\mathcal{X}$ can be cut out of a manifold M by taking the zeroset of a section $s : M \to E$ of a vector bundle $p : E \to M$. We can make this explicit if case $\mathcal{X}$ is a compact manifold. Then, M will be a tubular neighborhood of $\mathcal{X}$ in some big enough $\mathbb{R}^N$, the vector bundle E will be the normal bundle pulled back along $M \to \mathcal{X}$, and the section $s : M \to E$ is the tautological section which cuts out $\mathcal{X}$.

Theorem 6.1.12. *Let $\mathcal{X}$ be a compact derived manifold. Then there exists a manifold M, an imbedding $f : \mathcal{X} \to M$, a vector bundle $p : E \to M$, and a* *section $s : M \to E$ such that $\mathcal{X}$ is the fiber product in the diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\ulcorner"] & M \arrow[d, "s"] \\ M \arrow[r, "z"'] & E, \end{tikzcd} \tag{6.2}$$

where z is the zero section.

Proof. We choose M, E, $f : \mathcal{X} \to M$, and $p : E \to M$ as in Notation 6.1.10. Given a section $s : M \to E$, let $\lambda_s : f^*E \to \mathcal{N}_f^\vee$ be as in Proposition 6.1.11. Since f is an imbedding, we can locally choose s so that Diagram 6.2 is a fiber product, so by Proposition 6.1.11, we can locally choose s so that λ_s is an isomorphism. By Lemma 5.3.3, the map λ_s is linear in the choice of s. Therefore, using a partition of unity, we may patch these maps together so as to give a global section $s : M \to E$ such that λ_s is an isomorphism. The result follows from one more application of Proposition 6.1.11.

□

6.2 Derived Cobordism

In order to define derived cobordism, we must define derived manifolds with boundary (or DMBs). It is easy to show that if M is a manifold with boundary then there exists a manifold (without boundary) N and a function $f : N \to \mathbb{R}$ such that M is isomorphic to the preimage $f^{-1}([0, \infty))$ and ∂M is isomorphic to the preimage $f^{-1}(\{0\})$. It is this global fact about manifolds with boundary that we mimic to define DMBs.

Let x_N denote a coordinate of $\mathbb{R}^N$. Let $\mathbb{R}^N_{\geq 0}$ denote the half space for which $x_N \geq 0$ and let $\mathbb{R}^N_{=0}$ denote the hyperplane on which $x_N = 0$. We let $h_{=0} : \mathbb{R}^N_{=0} \to \mathbb{R}^N$ and $h_{\geq 0} : \mathbb{R}^N_{\geq 0} \to \mathbb{R}^N$ denote the inclusions. We consider $\mathbb{R}^N_{\geq 0}$ as a local smooth-ringed space with structure sheaf $\mathcal{O}_{\mathbb{R}^N_{\geq 0}} = h^*_{\geq 0}\mathcal{H}_{\mathbb{R}^N}$.

Definition 6.2.1. A sequence

$$\overline{\mathcal{P}} = (\mathcal{P}, \mathcal{Z}, j) = ((P, \mathcal{O}_P), (Z, \mathcal{O}_Z), j)$$

is called a *derived manifold with boundary (or DMB)* if

- $\mathcal{P}$ is a local smooth-ringed space, $\mathcal{Z}$ is a derived manifold, and $j : \mathcal{Z} \to \mathcal{P}$ is a closed immersion, such that
- there exists a derived manifold $\mathcal{X} = (X, \mathcal{O}_X)$ and a function $f : \mathcal{X} \to \mathbb{R}$, with
- $P = f^{-1}([0, \infty))$, $i : P \to X$ the inclusion, $\mathcal{O}_P = i^*\mathcal{O}_X$, and
- $\mathcal{Z} = \mathcal{X}/f \cong \mathcal{P}/f$, the zeroset of f.

We refer to $\mathcal{Z}$ as the *boundary* of $\overline{\mathcal{P}}$, and we may write $\mathcal{Z} = \partial\overline{\mathcal{P}}$. Finally we say that the pair $(\mathcal{X}, f)$ is a *presentation* of $\overline{\mathcal{P}}$. We sometimes write $\overline{\mathcal{P}} = \mathcal{X}_{f\geq 0}$.

A morphism of DMBs $f : \overline{\mathcal{P}} \to \overline{\mathcal{P}'}$ is a pair (f_Z, f_P) such that $f_Z : \mathcal{Z} \to \mathcal{Z}'$, $f_P : \mathcal{P} \to \mathcal{P}'$ and $f_P \circ j = j' \circ f_Z$. The category of DMBs is denoted **DMB**.

A DMB $(\mathcal{P}, \mathcal{Z}, j)$ is called *compact* if $\mathcal{P}$ (and hence $\mathcal{Z}$) is compact.

Lemma 6.2.2. *If $(\mathcal{X}, f)$ is a presentation for a DMB $(\mathcal{P}, \mathcal{Z}, j)$, then the following diagram of smooth-ringed spaces is all-Cartesian:*

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d, "j"'] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\
\mathcal{P} \arrow[r, "b"] \arrow[d, "i"'] \arrow[dr, phantom, "\diamond"] & \mathbb{R}_{\geq 0} \arrow[d, "h_{\geq 0}"] \\
\mathcal{X} \arrow[r, "f"] & \mathbb{R}.
\end{tikzcd}$$

Proof. The big rectangle is Cartesian by definition, so it suffices to show that the bottom square is Cartesian. On underlying topological spaces, this square is Cartesian. Thus, if

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "a"] \arrow[d, "g"'] & \mathbb{R}_{\geq 0} \arrow[d, "h_{\geq 0}"] \\
\mathcal{X} \arrow[r, "f"'] & \mathbb{R}
\end{tikzcd}$$

is a commutative diagram, then there is a unique map on underlying topological spaces $h : Y \to P$, such that $i \circ h = g$ and $b \circ h = a$. There is also an induced map

$$h^\flat : h^*\mathcal{O}_P = h^*i^*\mathcal{O}_X = g^*\mathcal{O}_X \to \mathcal{O}_Y,$$

which is unique such that $h^\flat \circ i^\flat = g^\flat$. It is easy to show that $h^\flat \circ b^\flat = a^\flat$ as well. Thus there is a unique map $\mathcal{Y} \to \mathcal{P}$ making the appropriate diagrams commute, so $\mathcal{P}$ is indeed the fiber product in the bottom square.

□

Remark 6.2.3. The local smooth-ringed space $\mathcal{P}$ in a DMB $\overline{\mathcal{P}} = (\mathcal{P}, \mathcal{Z}, j)$ is typically *not* a derived manifold. If $(\mathcal{X}, f)$ is a presentation for $\overline{\mathcal{P}}$ and $c : \mathbb{R} \to \mathbb{R}$ is a characteristic function for the closed subset $\mathbb{R}_{\geq 0} \subset \mathbb{R}$, then the fiber product $\mathcal{Y}$ in the diagram

$$\begin{tikzcd}
\mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\
\mathcal{X} \arrow[r, "c\circ f"'] & \mathbb{R}
\end{tikzcd}$$

is a derived manifold. Moreover, the underlying space of $\mathcal{Y}$ is isomorphic to that of $\mathcal{P}$. However, the natural map $\mathcal{P} \to \mathcal{Y}$ is typically not an isomorphism of local smooth-ringed spaces because $\mathbb{R}_{\geq 0} \to \mathbb{R}/c$ is not an isomorphism of local smooth-ringed spaces.

Remark 6.2.4. The definition of DMB generalizes the usual definition of manifold with boundary. Indeed, suppose that M is a manifold and $f : M \to \mathbb{R}$ is a smooth morphism for which $0 \in \mathbb{R}$ is a regular value. Then (M, f) is a presentation for a manifold with boundary $M_{f \geq 0}$. Similarly, any manifold with boundary P has such a presentation because we can imbed P into a half-space $\mathbb{R}^N_{\geq 0}$ such that the boundary of P is on the hyperplane $\mathbb{R}^N_{=0}$.

Remark 6.2.5. We defined DMBs by a global property, but one could instead try to define DMB by a local property. For example, one could define a "local DMB" as a local smooth-ringed space that was locally of the form $\mathcal{X}_{f \geq 0}$ (instead of globally of that form). A local DMB is not necessarily a DMB, unlike in manifolds where a "local manifold with boundary" is just a manifold with boundary.

Despite the fact that our definition of DMB is less flexible than the local version, it is the correct version for doing derived cobordism theory.

Lemma 6.2.6. *Suppose that $\overline{\mathcal{P}}$ is a DMB, M is a manifold, and $g : \mathcal{P} \to M$ is a morphism of local smooth-ringed spaces. Then there exists a presentation $(\mathcal{X}, f)$ for $\overline{\mathcal{P}}$ and a map $g' : \mathcal{X} \to M$ such that $g'|_{\mathcal{P}} = g$.*

Proof. Let $(\mathcal{X}', f')$ be a presentation of $\overline{\mathcal{P}}$, let $i : \mathcal{P} \to \mathcal{X}'$ be the inclusion. By definition, $\mathcal{O}_P(-, M) = i^*\mathcal{O}_{X'}(-, M)$, so the result follows by the structure Theorem 3.3.3. □

Definition 6.2.7. Let $(\mathcal{P}, \mathcal{Z}, j)$ be a DMB and let $g : \mathcal{P} \to \mathbb{R}^n$ be a morphism of structured spaces. The *zeroset of g* is the triple $(\mathcal{P}', \mathcal{Z}', j')$ in which $\mathcal{P}', \mathcal{Z}'$, and $j : \mathcal{Z}' \to \mathcal{P}'$ are the fiber products in the diagram of smooth-ringed spaces

$$\begin{tikzcd}
\mathcal{Z}' \arrow[r, "j'"] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{P}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\
\mathcal{Z} \arrow[r, "j"] & \mathcal{P} \arrow[r, "g"] & \mathbb{R}^n.
\end{tikzcd}$$

The zeroset of a function on a DMB is not *a priori* a DMB, but Lemma 6.2.8 shows that it is.

Lemma 6.2.8. *If $\overline{\mathcal{P}} = (\mathcal{P}, \mathcal{Z}, j)$ is a DMB, and $g : \mathcal{P} \to \mathbb{R}^n$ is a map of local smooth-ringed spaces, then the zeroset of g in $\overline{\mathcal{P}}$ is a DMB. If $\mathcal{P}$ is compact then the zeroset of g is compact.*

Proof. By Lemma 6.2.6, we can find a presentation $(\mathcal{X}, f)$ of $\overline{\mathcal{P}}$, with $i : \mathcal{P} \to \mathcal{X}$ the inclusion, and a map $g' : \mathcal{X} \to \mathbb{R}^n$ such that $g' \circ i = g$. Form the all-Cartesian diagram

$$\begin{tikzcd}
\mathcal{W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\
\mathcal{Z} \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{P} \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{X} \arrow[r, "g'"] \arrow[d, "f"] & \mathbb{R}^n \\
\bullet \arrow[r, "0"] & \mathbb{R}_{\geq 0} \arrow[r, "h_{\geq 0}"] & \mathbb{R}. &
\end{tikzcd}$$

All the named morphisms are given and all the unnamed morphisms are formed as in Remark A.3.2. Let $k : \mathcal{W} \to \mathcal{Q}$ be the top left map in the diagram. Looking at the top rectangles, $\overline{\mathcal{Q}} = (\mathcal{Q}, \mathcal{W}, k)$ is the zeroset of g in $\overline{\mathcal{P}}$. Let $f' : \mathcal{Y} \to \mathbb{R}$ be the vertical composition in the diagram. It is clear that $(\mathcal{Y}, f')$ is a representative for $\overline{\mathcal{Q}}$.

Since $\bullet \to \mathbb{R}^n$ is proper, so is $\mathcal{Q} \to \mathcal{P}$; thus $\mathcal{Q}$ is compact if $\mathcal{P}$ is.

□

Lemma 6.2.9. *Suppose that $\mathcal{Y}$ is a derived manifold, $(\mathcal{P}, \mathcal{Z}, j)$ is a DMB, M is a manifold, and $g : \mathcal{P} \to M$ and $h : \mathcal{Y} \to M$ are morphisms of local smooth-ringed spaces. Then the fiber product $(\mathcal{Q}, \mathcal{W}, k)$ in the all-Cartesian diagram*

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{Y} \arrow[d, "h"] \\
\mathcal{Z} \arrow[r] & \mathcal{P} \arrow[r, "g"] & M
\end{tikzcd}$$

is a DMB.

Proof. Let $(\mathcal{X}, f)$ be a presentation of $\mathcal{P}$, and $i : \mathcal{P} \to \mathcal{X}$ the induced inclusion. By Proposition 6.2.6, we can extend g to a map $g' : \mathcal{X} \to M$ with $g' \circ i = g$. Form the all-Cartesian diagram

$$\begin{tikzcd}
\mathcal{W} \arrow[r, "k"] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{Q} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{Q}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{Y} \arrow[d, "h"] \\
\mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{P} \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{X} \arrow[r, "g'"] \arrow[d, "f"] & M \\
\bullet \arrow[r, "0"] & \mathbb{R}_{\geq 0} \arrow[r, "h_{\geq 0}"] & \mathbb{R}. &
\end{tikzcd}$$

Since $\mathcal{Q}'$ is a derived manifold (Theorem 4.1.14) and the two vertical rectangles are Cartesian, $(\mathcal{Q}, \mathcal{W}, k)$ is a DMB.

□

Let $[0, 1]$ denote the unit interval, which is a manifold with boundary and hence a DMB.

Corollary 6.2.10. *Let $\mathcal{X}$ and $\mathcal{Y}$ be derived manifolds, let M be a manifold, and let $h : \mathcal{Y} \to M$ be a morphism of local smooth-ringed spaces. Suppose that $H : \mathcal{X} \times [0,1] \to M$ is also a morphism of local smooth-ringed spaces. Then the fiber product $\mathcal{P}$ in the diagram*

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{Y} \arrow[d, "h"] \\ \mathcal{X} \times [0,1] \arrow[r, "H"'] & M \end{tikzcd}$$

is a DMB with boundary equal to the fiber over $(\mathcal{X} \times \{0\}) \amalg (\mathcal{X} \times \{1\})$.

Proof. The interval $[0,1]$ is a manifold with boundary, hence a DMB. The result now follows by Lemma 6.2.9.

□

We now give a corollary of the imbedding Theorem 6.1.5 which states that a compact derived manifold with boundary $\overline{\mathcal{P}}$ can be imbedded into the half space $\mathbb{R}^N_{\geq 0}$ such that the boundary $\partial\overline{\mathcal{P}}$ is contained in the hyperplane $\mathbb{R}^N_{=0}$.

Corollary 6.2.11. *Let $\overline{\mathcal{P}} = (\mathcal{P}, \mathcal{Z}, j)$ be a compact DMB. There is a coordinate projection $x_N : \mathbb{R}^N \to \mathbb{R}$ and an imbedding $i : \overline{\mathcal{P}} \to \mathbb{R}^N_{\geq 0}$ into the half-space such that*

$$\mathcal{Z} \cong i^{-1}\mathbb{R}^N_{=0}.$$

Proof. Suppose that $(\mathcal{X}, f)$ represents $\overline{\mathcal{P}}$; in particular $\mathcal{X}/f \cong \mathcal{Z}$. Let $W : \mathcal{X} \to \mathbb{R}^n$ be an imbedding. The result follows from Remark A.3.2 and the following all-Cartesian diagram:

$$\begin{tikzcd} \mathcal{Z} \arrow[r, "j"] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{P} \arrow[r] \arrow[d, "i"] \arrow[dr, phantom, "\diamond"] & \mathcal{X} \arrow[d, "{(W,f)}"] \\ \mathbb{R}^N_{=0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathbb{R}^N_{\geq 0} \arrow[r] \arrow[d, "x_N"] \arrow[dr, phantom, "\diamond"] & \mathbb{R}^N \arrow[d, "x_N"] \\ \bullet \arrow[r, "0"] & \mathbb{R}_{\geq 0} \arrow[r, "h_{\geq 0}"] & \mathbb{R}. \end{tikzcd}$$

□

As mentioned above, the so-called "collar neighborhood theorem" fails for derived manifolds. That is, if $\overline{\mathcal{P}}$ is a DMB, and $\mathcal{Z}_0$ is a boundary component of $\overline{\mathcal{P}}$, then there is *not* necessarily a subspace $\overline{\mathcal{P}'} \subset \overline{\mathcal{P}}$ such that $\overline{\mathcal{P}'} \cong \mathcal{Z}_0 \times [0,1]$. However, the following proposition accomplishes practically the same goal. By stretching out a boundary component, we can replace a DMB with a collared one.

Proposition 6.2.12. *Suppose that $\overline{\mathcal{P}} = (\mathcal{P}, \mathcal{Z}, j)$ is a derived manifold with boundary, and suppose that the boundary is a disjoint union, $\mathcal{Z} = \mathcal{Z}_0 \amalg \mathcal{Z}_1$. Then there exists a* DMB $\overline{\mathcal{P}'}$*, with boundary $\mathcal{Z}_0 \amalg \mathcal{Z}_1$, which contains a subspace which is isomorphic to $\mathcal{Z}_1 \times [-1,0]$.*

If $\overline{\mathcal{P}}$ is compact, then we can choose $\overline{\mathcal{P}'}$ to be compact.

Proof. Let $(\mathcal{X}, f)$ be a presentation of $\overline{\mathcal{P}}$, so that in particular, $\mathcal{Z} = \mathcal{X}/f$. Let $\mathcal{U}$ be an open neighborhood of $\mathcal{Z}_1$ in $\mathcal{X}$, such that $\mathcal{U} \cap \mathcal{Z}_0 = \emptyset$; in particular, $\mathcal{Z}_1 = \mathcal{U}/f$. Let $\beta(x) : \mathbb{R} \to \mathbb{R}$ be the "bump function" that is smooth, positive for $x > 0$, and 0 for $x \leq 0$. Define $\mathcal{V}$ as the pullback in the diagram

$$\begin{tikzcd}
\mathcal{V} \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\diamond"] & \mathcal{U} \arrow[d, "f"] \\
\mathbb{R} \arrow[r, "\beta"'] & \mathbb{R}.
\end{tikzcd}$$

Consider the diagram

$$\begin{tikzcd}
\mathcal{Z}_1 \times [-1,0] \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{Z}_1 \arrow[dr] \arrow[dd] & \\
& \mathcal{V} \arrow[rr] \arrow[dd] & & \mathcal{U} \arrow[dd, "f"] \\
{[-1,0]} \arrow[rr] \arrow[dr] & & \bullet \arrow[dr, "0"] & \\
& \mathbb{R} \arrow[rr, "\beta"'] & & \mathbb{R}.
\end{tikzcd}$$

The front, back, and right-hand faces are pullbacks, so the left-hand face is too. In other words, $\mathcal{V}$ contains a local smooth-ringed space which is equivalent to $\mathcal{Z}_1 \times [-1,0]$.

The open subset $\mathcal{U}_{f>0} \subset \mathcal{P}$ is isomorphic to $\mathcal{V}_{g>0} \subset \mathcal{V}$, by construction of β. Define $\mathcal{P}'$ to be the union of $\mathcal{V}_{g \geq -1}$ and $\mathcal{P} - \mathcal{Z}_1$ along $\mathcal{U}_{f>0}$. If $\overline{\mathcal{P}}$ is compact, then $\mathcal{Z}_1$ is compact and so is $\mathcal{V}_{[-1,0]}$. Therefore, $\overline{\mathcal{P}'}$ is compact as well.

□

Definition 6.2.13. Let T be a topological space, considered as a local smooth-ringed space (see Remark 3.2.7). A *derived manifold over T* is a map of local smooth-ringed spaces $s : \mathcal{X} \to T$, where $\mathcal{X}$ is a derived manifold. We form the category $\mathbf{dMan}_{/T}$ of derived manifolds over T in the usual way.

A *DMB over T* is defined to be a sequence $(\overline{\mathcal{P}}, s_P, s_Z)$, in which $\overline{\mathcal{P}} = (\mathcal{P}, \mathcal{Z}, j)$ is a DMB, and $s_P : \mathcal{P} \to T$ and $s_Z : \mathcal{Z} \to T$ are morphisms of local smooth-ringed spaces such that $s_P \circ j = s_Z$. Again we define the category $\mathbf{DMB}_{/T}$ of DMBs over T in the usual way.

Definition 6.2.14. We say that derived manifolds $\mathcal{X}$ and $\mathcal{Y}$ are *derived cobordant* if there exists a compact DMB $\overline{\mathcal{P}}$ such that

$$\partial \overline{\mathcal{P}} \cong \mathcal{X} \amalg \mathcal{Y}.$$

We say that $\overline{\mathcal{P}}$ *is a cobordism* between $\mathcal{X}$ and $\mathcal{Y}$.

Let T be a topological space, considered as a local smooth-ringed space (see Remark 3.2.7). Suppose that $f : \mathcal{X} \to T$ and $g : \mathcal{Y} \to T$ are derived manifolds over T. We say that $\mathcal{X}$ and $\mathcal{Y}$ are *cobordant over* T if there exists a cobordism $(\mathcal{P}, \mathcal{X} \amalg \mathcal{Y}, j)$ between $\mathcal{X}$ and $\mathcal{Y}$ and a morphism of local smooth-ringed spaces $h : \mathcal{P} \to T$ such that $h \circ j = (f \amalg g)$.

If M and N are (smooth) manifolds, then we say that M and N are *derived cobordant* if they are derived cobordant as derived manifolds, and we say that they are *smoothly cobordant* (or simply *cobordant*) if there is a (smooth) compact manifold with boundary cobording them.

We will not work over general topological spaces T because collared replacements (as in Proposition 6.2.12) may not exist over general T. We instead work only over ANRs. Lemma 6.2.16 shows that this is not too much of a restriction.

Definition 6.2.15. A metric space X is called an *absolute neighborhood retract (or ANR)* if, for every metric space Z, every closed subspace $C \subset Z$, and every map $f : C \to X$, there is an open neighborhood $C \subset U \subset Z$ and a map $f' : U \to X$ such that $f'|_C = f$.

The following lemma is proven in [8, 5.2.1].

Lemma 6.2.16. *If X is a topological space with the homotopy type of a CW complex, then X has the homotopy type of an ANR.*

Lemma 6.2.17. *Suppose that $\mathcal{X}$ is a local smooth-ringed space, $f : \mathcal{X} \to \mathbb{R}^n$ is a map, and $\mathcal{A} = \mathcal{X}/f$ is the zeroset of f. If T is an ANR and $g : \mathcal{A} \to T$ is a map of local smooth-ringed spaces, then there exists an open subspace $\mathcal{X}' \subset \mathcal{X}$ with $\mathcal{A} \subset \mathcal{X}$, a map $g' : \mathcal{X}' \to T$ such that $g'|_{\mathcal{A}} = g$, and a map $f' : \mathcal{X}' \to \mathbb{R}^n$ such that $\mathcal{A} = \mathcal{X}'/f'$.*

Proof. Since $\mathcal{A}$ is closed in $\mathcal{X}$, there exists an open neighborhood $\mathcal{X}' \subset \mathcal{X}$ of $\mathcal{A}$ and an extension $g' : \mathcal{X}' \to T$ of g. The result follows from Lemma 4.1.15.

□

Corollary 6.2.18. *Let T be an ANR. Suppose that $g : \overline{\mathcal{P}} = (\mathcal{P}, \mathcal{Z}, j) \to T$ is a derived manifold with boundary over T, and suppose that the boundary is a disjoint union, $\mathcal{Z} = \mathcal{Z}_0 \amalg \mathcal{Z}_1$. Then there exists a* DMB *$\overline{\mathcal{P}'}$ over T, with boundary $\mathcal{Z}_0 \amalg \mathcal{Z}_1$, such that $\overline{\mathcal{P}'}$ contains a subspace which is equivalent to $\mathcal{Z}_1 \times [-1, 0]$.*

If $\overline{\mathcal{P}}$ is compact, then we can choose $\overline{\mathcal{P}'}$ to be compact.

Proof. Let $(\mathcal{X}, f)$ be a presentation of $\overline{\mathcal{P}}$, and let $i : \mathcal{P} \to \mathcal{X}$ be the inclusion. Since T is an ANR, there exists a map $g' : \mathcal{X} \to \mathbb{R}^n$ such that $g'i = g$. The proof of Proposition 6.2.12 is easily adapted to this more general setting.

□

Theorem 6.2.19. *Let T be an ANR. The relation of derived cobordism over T is an equivalence relation on the category of derived manifolds over T.*

Proof. Clearly the derived cobordism relation is reflexive and symmetric. We show that it is transitive. Suppose that $\mathcal{A}$ is cobordant to $\mathcal{B}$ over T and $\mathcal{B}$ is cobordant to $\mathcal{C}$ over T. Suppose that $(\mathcal{X}, f)$ represents the first cobordism and $(\mathcal{X}', f')$ represents the second cobordism. The component $\mathcal{B} \subset \mathcal{X}/f$ is equivalent to the component $\mathcal{B} \subset \mathcal{X}'/f'$.

By Corollary 6.2.18, we can find compact DMBs $\mathcal{Y}$ and $\mathcal{Z}$ (over T) with boundaries $\mathcal{A} \amalg \mathcal{B}$ and $\mathcal{B} \amalg \mathcal{C}$ respectively, and each with a derived submanifold that is equivalent to $\mathcal{B} \times [0, 1)$. Let $\mathcal{Y}'$ and $\mathcal{Z}'$ denote the compliment of $\mathcal{B} \times \{0\}$ in $\mathcal{Y}$ and $\mathcal{Z}$, respectively. We conclude by gluing $\mathcal{Y}'$ and $\mathcal{Z}'$ together along their common open derived submanifold $\mathcal{B} \times (0, 1)$ to produce a cobordism between $\mathcal{A}$ and $\mathcal{C}$ over T.

□

Definition 6.2.20. Let T be an ANR. We define the (unoriented) *derived cobordism group over* T to be the group whose underlying set is the set of derived cobordism classes of compact derived manifolds, whose additive structure is defined by disjoint union; the group is graded by the dimension of manifolds. We denote this group by $\mathcal{N}_T^{der}$. We denote its smooth counterpart by $\mathcal{N}_T$.

Remark 6.2.21. We have used the symbol $\mathcal{N}$ before when discussing normal and conormal bundles. We will not have occasion to discuss either of these again in what follows, so the notation should not be confusing.

Let $i : \mathbf{Man} \to \mathbf{dMan}$ denote the inclusion of categories. Clearly, if two compact manifolds are (smoothly) cobordant then they are derived cobordant. Thus we have a group homomorphism $\iota_T : \mathcal{N}_T \to \mathcal{N}_T^{der}$. We will prove that if T is an ANR then ι_T is in fact an isomorphism. This can be thought of as a Thom-Pontrjagin theorem.

The following Lemma is proved in [26, App 2, p.24], and we will review the terminology after stating it.

Lemma 6.2.22. *Let M and N be manifolds, $f : M \to N$ a smooth map, and $i : N' \subset N$ a closed submanifold. Let $j : A \subset M$ be a closed submanifold such that the transverse regularity condition for f on N' is satisfied for all points of $A \cap f^{-1}(N')$. Then there is a smooth homotopy $H : M \times [0, 1] \to N$ such that $H(-, 0) = f(-)$ and $H(-, 1) = g(-)$, where $g : M \to N$ is a smooth map that is transverse regular on N', and $g|_A = f|_A$.*

The "transverse regularity condition," which is defined in [26, App 2, p.23], can be reformulated in our terminology (see Definition 4.1.16) by saying that $i : N' \to N$ and $f \circ j : A \to N$ are transverse.

Lemma 6.2.23. *Let M and N be manifolds, $f : M \to N$ a smooth map, and $i : N' \subset N$ a closed submanifold. Let $j : A \subset M$ be a closed submanifold such that the transverse regularity condition for f on N' is satisfied for all points of $A \cap f^{-1}(N')$. Suppose that the fiber product Z in the category of topological* *spaces*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & N' \arrow[d, "i"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

is compact. Then there exists a smooth homotopy $H : M \times [0,1] \to N$ such that the preimage $H^{-1}(N')$ is compact and such that $H(-,0) = f(-)$ and $H(-,1) = g(-)$, where $g : M \to N$ is a smooth map which is transverse regular on N', and $g|_A = f|_A$.

Proof. The only difference between this Lemma and Lemma 6.2.22 is the condition that Z be compact and the conclusion that $H^{-1}(N')$ is compact. Since M is a manifold, it is metrizable. Cover $f^{-1}(N') \cup A$ with finitely many closed balls, and let M_1 be there union, which is a compact manifold with boundary. Let M_2 be an open neighborhood of $f^{-1}(N') \cup A$ inside M_1. Apply Lemma 6.2.22 with M_1 in place of M to get a smooth homotopy $H_1 : M_1 \times I \to N$ satisfying the conclusions of that lemma. Construct an all-Cartesian diagram of topological spaces

$$\begin{tikzcd} P_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & P_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & N' \arrow[d] \\ M_2 \times I \arrow[r] & M_1 \times I \arrow[r] & N. \end{tikzcd}$$

Since M_2 contains $f^{-1}(N')$, we have $P_2 = P_1$. Since $N' \to N$ is a closed imbedding, it is proper. Therefore, since $M_1 \times I$ is compact, so is P_1. Let $H : M_2 \times I \to N$ be the composition in the diagram; it satisfies the conclusions of the Lemma.

□

Theorem 6.2.24. *Suppose that T is an ANR. The map $\iota_T : \mathcal{N}_T \to \mathcal{N}_T^{der}$, from the cobordism group of manifolds to the derived cobordism group of derived manifolds, is an isomorphism.*

Proof. We first show that ι_T is surjective. Suppose that $g : \mathcal{Z}_0 \to T$ is a compact derived manifold over T; we want to show that there exists a smooth manifold $\mathcal{Z}_1$ that is derived cobordant to $\mathcal{Z}_0$ over T. Imbed $\mathcal{Z}_0$ into $\mathbb{R}^N$. Let $M \subset \mathbb{R}^N$, $\pi : E \to M$ and $s : M \to E$ be the open neighborhood, the vector bundle, and the section from Theorem 6.1.12, so that $\mathcal{Z}_0$ is the fiber product in the diagram

$$\begin{tikzcd} \mathcal{Z}_0 \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\ulcorner", very near start] & M \arrow[d, "s"] \\ M \arrow[r, "z"'] & E. \end{tikzcd}$$

By Lemma 6.2.17, we may shrink M and extend g to a map $g' : M \to T$. The zerosection $z : M \to E$ can be considered a closed submanifold of E; note also that z is proper. Thus we apply Lemma 6.2.23, to obtain a smooth homotopy $H : M \times [0,1] \to E$ between s and a section s', which is transverse to z and for which $H^{-1}(z)$ is compact. Let $\mathcal{Z}_1$ be the zeroset of s'. Since $H^{-1}(Z)$ is compact, it provides a derived cobordism between $\mathcal{Z}_0$ and $\mathcal{Z}_1$, and the latter is a smooth manifold by Corollary 4.1.18.

Showing that ι_T is injective is similar but now uses the full strength of Lemma 6.2.23. Suppose that $\mathcal{Z}_0$ and $\mathcal{Z}_1$ are smooth manifolds which are derived cobordant over T; we want to show that they are smoothly cobordant over T. Although we will not mention it explicitly, all constructions will be suitably "over T" as above.

Let $\mathcal{Z} = \mathcal{Z}_0 \amalg \mathcal{Z}_1$, let $\overline{\mathcal{P}} = (\mathcal{P}, \mathcal{Z}, j)$ be a DMB, and let $(\mathcal{X}, g)$ be a presentation of $\overline{\mathcal{P}}$. By Corollary 6.1.6, there is an imbedding $i : \mathcal{X} \to \mathbb{R}^N$, with $\mathcal{Z} \cong i^{-1}(\mathbb{R}^N_{=0})$, and extension of g to $\mathbb{R}^N$. Let $(M, p : E \to M, s)$ be a manifold, vector bundle, and section as in Theorem 6.1.12. Let $g' : M \to \mathbb{R}$ be the extension of g, restricted to M. Then the diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \bullet \arrow[d, "0"] \\
\mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & Q' \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathbb{R}_{\geq 0} \arrow[d, "h_{\geq 0}"] \\
\mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & M \arrow[r, "g'"] \arrow[d, "s"] & \mathbb{R} \\
M \arrow[r, "z"] & E &
\end{tikzcd}$$

is all-Cartesian. By Lemma A.3.3 one finds that the diagram

$$\begin{tikzcd}
\mathcal{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & \mathcal{P} \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & M \arrow[d, "{(s,g')}"] \\
M \times \{0\} \arrow[r] & M \times \mathbb{R}_{\geq 0} \arrow[r, "{(z,h_{\geq 0})}"] & E \times \mathbb{R}
\end{tikzcd}$$

is all-Cartesian as well.

The manifold $\mathcal{Z}$ is the fiber product of manifolds taken in the category of derived manifolds. By Theorem 4.2.1, the maps $(z, h_{=0}) : M \times \{0\} \to E \times \mathbb{R}$ and $(s, g') : M \to E \times \mathbb{R}$ are transverse. Considering $\mathcal{Z}$ as a closed submanifold of M, we can find a smooth homotopy $H : M \times [0,1] \to E \times \mathbb{R}$ such that

- H is a homotopy relative to $\mathcal{Z}$,
- $H(-,0) = (s, g')$, and
- $H(-,1)$ is transverse to $(z, h_{\geq 0})$.

If $\mathcal{P}'$ is the fiber product in the diagram

$$\begin{tikzcd} \mathcal{P}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond"] & M \arrow[d, "H(-,1)"] \\ M \times \mathbb{R}_{\geq 0} \arrow[r, "(z,h_{\geq 0})"'] & E \times \mathbb{R} \end{tikzcd}$$

then it is a smooth manifold with boundary (see Remark 6.2.4) and $\partial\mathcal{P}' \cong \mathcal{Z}$. Thus $\mathcal{Z}_0$ and $\mathcal{Z}_1$ are smoothly cobordant over T.

□

Corollary 6.2.25. *If $\mathcal{X}$ and $\mathcal{Y}$ are equidimensional and cobordant then they have the same dimension.*

Proof. This follows from Proposition 5.2.14.

□

6.3 Fundamental Class

After defining cohomology and homology for derived manifolds, we show that for every compact derived manifold $\mathcal{X}$ (without boundary) of dimension n, there exists a well-defined class $[\mathcal{X}] \in H_n(\mathcal{X}, \mathbb{Z}/2)$. This class agrees with the fundamental class when $\mathcal{X}$ is a manifold, and behaves as expected with respect to cobordism, intersection product, and Euler classes.

Definition 6.3.1. Suppose that $\mathcal{X} = (X, \mathcal{O}_X)$ is a local smooth-ringed space and R is a commutative ring. The *cohomology ring of $\mathcal{X}$ with coefficients in R* is defined to be the graded R-module $H^*(\mathcal{X}; R)$ whose mth graded piece is

$$H^m(\mathcal{X}; R) = [X, K(R, m)],$$

where $[\,,\,]$ denotes homotopy classes of maps, and $K(R, m)$ denotes an ANR with the homotopy type of an Eilenberg-Maclane space of type (R, m) (see Lemma 6.2.16).

If R is a field, define *the mth homology R-module of $\mathcal{X}$*, $H_m(\mathcal{X}; R)$, as the R-linear dual,

$$H_m(\mathcal{X}, R) = \mathrm{Hom}_{R\text{-mod}}(H^m(\mathcal{X}, R), R).$$

If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of local smooth-ringed spaces, composition with f defines maps

$$f^* : H^m(\mathcal{Y}, R) \to H^m(\mathcal{X}, R) \text{ and } f_* : H_m(\mathcal{X}, R) \to H_m(\mathcal{Y}, R).$$

Remark 6.3.2. The R-cohomology (and homology) of a local smooth-ringed space $\mathcal{X} = (X, \mathcal{O}_X)$ *does not* take into account any of the additional (sheaf-theoretic) structure of $\mathcal{X}$. It is defined only in terms of the topological space X.

Note that the above definitions, when restricted to the subcategory of compact manifolds, gives the usual notion of cohomology and, in the case that R is a field, the usual notion of homology. In this paper we work only with $R = \mathbb{Z}/2$ homology and cohomology. Let $K_n = K(\mathbb{Z}/2, n)$.

Let $\mathcal{X}$ be a compact derived manifold of dimension n. We define the class $[\mathcal{X}] \in H_n(\mathcal{X}) = H_n(\mathcal{X}, \mathbb{Z}/2)$ as follows. Let $[\phi] \in [\mathcal{X}, K_n]$ be a homotopy classes of maps. Let $\phi : X \to K_n$ be a map of topological spaces that represents $[\phi]$. Then use Theorem 6.2.24 to find a compact manifold M and a derived cobordism $\overline{\mathcal{P}} \to K_n$ between $\mathcal{X}$ and M. Define

$$\langle [\phi], [\mathcal{X}] \rangle := \langle [\phi], [M] \rangle \in \mathbb{Z}/2.$$

Then $[\mathcal{X}]$ is a well-defined element of $H_n(\mathcal{X})$; indeed if M' is also derived cobordant to $\mathcal{X}$ over K_n, then M and M' are derived cobordant, hence cobordant, which implies that

$$\langle [\phi], [M] \rangle + \langle [\phi], [M'] \rangle = 0.$$

Proposition 6.3.3. *Suppose that M, N, and Q are compact manifolds of dimension m, n, and q, that $g : Q \to M$ is a smooth map, and that $H : N \times [0,1] \to M$ is a smooth homotopy such that $H(-,0) = f$ and $H(-,1) = f'$. Let $\mathcal{X}$ and $\mathcal{X}'$ be the pullbacks*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "\diamond"] & Q \arrow[d, "g"] \\ N \arrow[r, "f"'] & M \end{tikzcd} \qquad \begin{tikzcd} \mathcal{X}' \arrow[r, "j'"] \arrow[d, "i'"'] \arrow[dr, phantom, "\diamond"] & Q \arrow[d, "g"] \\ N \arrow[r, "f'"'] & M. \end{tikzcd}$$

Let $d = n + q - m$ be the dimension of $\mathcal{X}$ and $\mathcal{X}'$. Then

$$i_*[\mathcal{X}] = (i')_*[\mathcal{X}'] \in H_d(N)$$

and

$$j_*[\mathcal{X}] = (j')_*[\mathcal{X}'] \in H_d(Q).$$

Proof. Let $\overline{\mathcal{P}}$ be the pullback DMB (see Lemma 6.2.9) in the diagram

$$\begin{tikzcd} \overline{\mathcal{P}} \arrow[r, "a"] \arrow[d, "b"'] \arrow[dr, phantom, "\diamond"] & Q \arrow[d, "g"] \\ N \times [0,1] \arrow[r, "H"'] & M. \end{tikzcd}$$

So $\overline{\mathcal{P}}$ is a cobordism between $\mathcal{X}$ and $\mathcal{X}'$. Any map $\phi : N \to K_d$ (resp. $\psi : Q \to K_d$) induces a map $\overline{\mathcal{P}} \to K_d$ which in turn induces maps $\mathcal{X} \to K_d$ and $\mathcal{X}' \to K_d$. Thus $\mathcal{X}$ and $\mathcal{X}'$ are derived cobordant over K_d, so

$$\langle [\phi], [\mathcal{X}] \rangle = \langle [\phi], [\mathcal{X}'] \rangle.$$

∎

Let M be a compact connected manifold of dimension n. Following Bredon [5, VI.11], let $D : H_i(M) \to H^{n-i}(M)$ be the Poincaré dual isomorphism, and define the intersection product

$$\cdot : H_i(M) \otimes H_j(M) \to H_{i+j-n}(M); \;\; a \cdot b := D(D(a) \cup D(b)).$$

Corollary 6.3.4. *Suppose that M_0 and M_1 are closed submanifolds of a compact manifold N, and let $\mathcal{X}$ be their intersection (i.e. $\mathcal{X}$ is the homotopy fiber product of $M_0 \to N \leftarrow M_1$). Then the image of $[\mathcal{X}]$ in $H_*(N)$ is equal to the intersection product $[M_0] \cdot [M_1]$.*

Proof. We can perturb M_0 to get a submanifold $M_0' \subset N$ that is transverse to M_1. The intersection product $[M_0] \cdot [M_1]$ is equal to the pushforward of $[M]$, where $M = M_0' \cup M_1$. The pushforward of $[M]$ is equal to the pushforward of $\mathcal{X}$ by Proposition 6.3.3.

□

The above corollary says that the fiber product in the category of derived manifolds is intersection-theoretically correct in homology.

Corollary 6.3.5. *Suppose that M is a compact manifold, $f : E \to M$ is a vector bundle of rank n, and $s : M \to E$ is any section. Let $\mathcal{X}$ be the zeroset of s; that is $\mathcal{X}$ is the fiber product in the diagram*

$$\begin{tikzcd} \mathcal{X} \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\diamond"] & M \arrow[d, "z"] \\ M \arrow[r, "s"'] & E, \end{tikzcd}$$

where z is the zerosection. The Euler class $e(E) \in H^n(M)$ is Poincaré dual to $j_*[\mathcal{X}]$, where $j = zi = si$.

Proof. We can perturb s so that it is transverse to z. It is well known that the Euler class is the Poincaré dual of the transverse zero section. The perturbation amounts to a cobordism connecting $\mathcal{X}$ to the transverse zero section. The result follows from Proposition 6.3.3.

□

Appendix A

Appendix

A.1 The Joyal model structure

In [20, 1.1.5.1], Lurie constructs a functor $\mathfrak{C}$ which takes simplicial sets to simplicial categories, and which is left adjoint to the simplicial nerve functor N. We use this to define the Joyal model structure.

Definition A.1.1. The Joyal model structure on the category of simplicial sets defines a morphism $f : S \to T$ as

1. a Joyal-cofibration if it is a monomorphism of simplicial sets,
2. a Joyal-weak equivalence if the induced functor $h\mathfrak{C}(f) : h\mathfrak{C}(S) \to h\mathfrak{C}(T)$ on $\mathcal{H}$-enriched homotopy categories is an equivalence,
3. a Joyal-fibration if it has the RLP with respect to acyclic cofibrations.

In [20, 1.3.4.1], Lurie show that these designations give **sSets** the structure of a left proper combinatorial model category. We will not need to know much about the Joyal model structure, except how it relates to the usual model structure and how it behaves with respect to colimits. We will use the words cofibration, fibration, and weak equivalence to refer to these designations under the usual model structure.

Lemma A.1.2. *The identity functor* **sSets** $\to$ **sSets** *is a left Quillen functor from the Joyal model structure to the usual model structure on* **sSets**. *It preserves weak equivalences.*

Proof. By [20, 2.3.6.4] the fibrant objects in the Joyal model structure are the simplicial sets which satisfy the inner Kan condition. Thus every fibrant simplicial set is Joyal-fibrant. In a simplicial model category, a map $f : S \to T$ is a weak equivalence if and only if for all fibrant objects F, the map of simplicial sets

$$\mathrm{Map}(T, F) \to \mathrm{Map}(S, F)$$

is a weak equivalence (in the usual model structure). Therefore, the set of Joyal-weak equivalences is a subset of the set of weak equivalence, and the result follows.

□

Lemma A.1.3. *Let I be a filtered category and $X : I \to$ **sSets** a diagram of simplicial sets. Then the colimit of X is weakly equivalent to the homotopy colimit of X, in both the Joyal model structure and the usual model structure on* **sSets***.*

Proof. Since the left Quillen functor from the Joyal structure to the usual structure preserves colimits, homotopy colimits, and weak equivalences, it suffices to prove the result in the Joyal structure, by Lemma A.1.2.

Let $X' : I \to \mathbf{sSets}$ be a diagram of Joyal-cofibrations such that $X \simeq_J X'$, so that

$$\operatorname*{hocolim}_I X \simeq_{Joy} \operatorname*{hocolim}_I X' \simeq_{Joy} \operatorname{colim} X'.$$

It suffices to show that $\operatorname{colim} X \simeq_{Joy} \operatorname{colim} X'$; i.e. that weak equivalences are stable under filtered colimits in the Joyal structure. This is part of the definition of a model category being combinatorial, and the fact that the Joyal model structure is combinatorial is proven in [20, 1.3.4.1].

□

Proposition A.1.4. *Consider the category* **sSets** *of simplicial sets with the Joyal model structure. The smallest subcategory $\mathcal{C} \subset$ **sSets** that contains Δ^0 and Δ^1 and that is closed under taking homotopy colimit is all of* **sSets***.*

Proof. Let $[n]$ denote the simplicial set with vertices $\{0, 1, \ldots, n\}$, a single edge connecting the ith vertex to the $(i+1)$th vertex for $0 \leq i < n$, and no other non-degenerate simplices. It is clear that $\mathfrak{C}(\Delta^n)$ is equivalent to $\mathfrak{C}([n])$, so Δ^n is categorically equivalent to $[n]$. Since $[n]$ is a (homotopy) colimit of 0- and 1-simplices, we find that every simplex Δ^n is in $\mathcal{C}$.

Let T denote a finite simplicial set; we will show that T is an object in $\mathcal{C}$. We do so by simultaneously inducting on the dimension of T and the number of simplices in the top-dimensional component of T. Write T as a colimit

$$\begin{tikzcd} \partial\Delta^n \arrow[r, "i"] \arrow[d] & \Delta^n \arrow[d] \\ T' \arrow[r] \arrow[ur, phantom, "\lrcorner"] & T, \end{tikzcd} \tag{A.1}$$

where T' has one fewer simplex in dimension n than T has. By induction, we may assume that $\partial\Delta^n$ and T' are in $\mathcal{C}$. Diagram A.1 is also a homotopy colimit diagram because i is a cofibration and the Joyal model structure is left proper ([20, 1.3.4.1]). Since Δ^n is in $\mathcal{C}$, so is T by [20, 4.2.3.10] and [20, 4.2.4.1].

The result now follows because every simplicial set can be written as the filtered homotopy colimit of a diagram of finite simplicial sets.

□

Corollary A.1.5. *Suppose that $R : \mathbf{sSets} \to \mathbf{sSets}$ is a functor which takes homotopy colimits in the Joyal model structure to homotopy limits. If $R(\Delta^0)$ and $R(\Delta^1)$ are both contractible, then $R(X)$ is contractible for all $X \in \mathbf{sSets}$.*

Proof. Obvious, given Proposition A.1.4.

□

Lemma A.1.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories. Let $F, G \in \mathrm{Fun}(\mathcal{C}, \mathcal{D})_0$ be functors. For any map of ∞-categories $i : \mathcal{S} \to \mathcal{C}$, let*

$$R(i) = \mathrm{Map}_{\mathrm{Fun}(\mathcal{S},\mathcal{D})}(F \circ i, G \circ i)$$

be the space of natural transformations between the restrictions of F and G. Then $R : (\mathbf{sSets} \downarrow \mathcal{C}) \to \mathbf{sSets}$ takes homotopy colimits in the Joyal model structure to homotopy limits of simplicial sets.

Proof. By [20, 4.2.1.8], we may write $R(i)$ as the homotopy limit in the diagram

$$\begin{tikzcd}
R(i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\diamond", very near start] & \Delta^0 \arrow[d] \\
\mathrm{Map}(mcS, \mathcal{D}^{\Delta^1}) \arrow[r, "(F\circ i, G\circ i)"] & \mathrm{Map}(\mathcal{S}, \mathcal{D}^{\partial\Delta^1}).
\end{tikzcd}$$

The forgetful functor $(\mathbf{sSets} \downarrow \mathcal{C}) \to \mathbf{sSets}$ preserves homotopy colimits. Therefore the bottom two objects in the diagram, thought of as functors from $\mathbf{sSets} \downarrow \mathcal{C}$ take homotopy colimits in i to homotopy limits. Using [20, 4.2.3.10], one checks that the same is true for R.

□

A.2 Limit-preserving functors

Let $\mathbf{PP}$ denote the full subcategory of $\mathbf{Sets}^{\mathbf{Man}}$ consisting of product-preserving functors. Let $\mathbf{sPP}$ denote the full subcategory of $\mathbf{sSets}^{\mathbf{Man}}$ consisting of product preserving functors. Our goal is to prove that $\mathbf{sPP}$ is a simplicial model category. We actually prove something more general: that for any type of limit, the full subcategory of functors preserving those limits is a simplicial model category.

Our main obstacle is showing that this subcategory is cocomplete, and to do so we prove that the inclusion of categories has a left adjoint. This could be done using Freyd's adjoint functor theorem, but we prove it explicitly in Theorem A.2.7 because we will need more information than Freyd's theorem provides.

Definition A.2.1. Let $\mathcal{C}$ and $\mathcal{M}$ be categories Suppose that $\mathcal{L}$ is a set of diagrams $L : I_L \to \mathcal{C}$. We say that a functor $F : \mathcal{C} \to \mathcal{M}$ *preserves $\mathcal{L}$-limits* if for every functor $L : I_L \to \mathcal{C}$ which has a limit $\overline{L}$ in $\mathcal{C}$, the functor $L' = F \circ L : I_L \to \mathcal{M}$ has a limit $\overline{L'}$ in $\mathcal{M}$, and $F(\overline{L}) \cong \overline{L'}$.

Definition A.2.2. Let $\mathcal{M}$ be a closed monoidal category which is closed under colimits. Let $\mathcal{C}$ be a category that is enriched over $\mathcal{M}$, and let $\mathrm{Map} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{M}$ denote the mapping functor. For an object $C \in \mathcal{C}$, let $H_C \in \mathcal{M}^{\mathcal{C}}$ denote the covariant functor

$$H_C(C') = \mathrm{Map}(C, C') \in \mathcal{M}$$

(so $H : \mathcal{C} \to \mathcal{M}^{\mathcal{C}}$ is contravariant). Suppose that $L : I \to \mathcal{C}$ is a diagram for which a limit $\overline{L} \in \mathcal{C}$ exists. Let $H(L) : I^{\mathrm{op}} \to \mathcal{M}^{\mathcal{C}}$ denote the result of applying H to L. The *morphism associated to L* is defined to be the natural map

$$\gamma_L : \operatorname*{colim}_I(H(L)) \longrightarrow H_{\overline{L}}$$

Example A.2.3. Definition A.2.2 is a bit abstract. In this work, $\mathcal{M}$ will always be either **Sets** or **sSets**; let us say $\mathcal{M} = \mathbf{Sets}$ for this discussion. Suppose that $\mathcal{C} = \mathbf{Man}$ is the category of manifolds. Suppose that I is the category consisting of two objects and no non-identity morphisms. A functor $L : I \to \mathbf{Man}$ is just a pair of manifolds L_1, L_2; the limit of L is $L_1 \times L_2$. Clearly, $H(L)$ is the coproduct $H_{L_1} \amalg H_{L_2} : \mathbf{Man} \to \mathbf{Sets}$. The morphism associated to L is

$$H_{L_1} \amalg H_{L_2} \longrightarrow H_{L_1 \times L_2}.$$

Lemma A.2.4. *Let $\mathcal{M}$ be a closed monoidal category which is closed under colimits, and let $\mathcal{C}$ be a category that is enriched over $\mathcal{M}$. Let $\mathcal{L}$ be a set of diagrams in $\mathcal{C}$, such that $\mathcal{C}$ is closed under $\mathcal{L}$-limits. Then a functor $F : \mathcal{C} \to \mathcal{M}$ preserves $\mathcal{L}$-limits if and only if, for each $L : I \to \mathcal{C}$ in $\mathcal{L}$ the map*

$$\mathrm{Map}(\gamma_L, F) : \mathrm{Map}(H_{\overline{L}}, F) \longrightarrow \mathrm{Map}(\mathrm{colim}(H(L)), F)$$

is an isomorphism in $\mathcal{M}$, where γ_L is the morphism associated to L.

Proof. This follows from Yoneda's lemma. □

Example A.2.5. Continuing from where we left off in Example A.2.3, let $\mathcal{L}$ denote the set of all I-shaped diagrams in **Man** (i.e. $\mathcal{L}$ is the set of all pairs of manifolds). A functor $F : \mathbf{Man} \to \mathbf{Sets}$ is "pairwise product-preserving" (i.e. preserves $\mathcal{L}$-limits) if and only if for each $L : I \to \mathbf{Man}$, the map $F(L_1 \times L_2) \to F(L_1) \times F(L_2)$ is an isomorphism. By Yoneda's lemma, this holds if and only if the map

$$\mathrm{Map}(\gamma_L, F) : \mathrm{Map}(H_{L_1 \times L_2}, F) \to \mathrm{Map}(H_{L_1} \amalg H_{L_2}, F)$$

is an isomorphism.

Lemma A.2.6. *Let $\mathcal{C}, \mathcal{M}$ be as in Definition A.2.2. Suppose that I is a finite category, that $L : I \to \mathcal{C}$ is a diagram, and that γ_L is the associated morphism. Then the source and target of γ_L are small.*

Proof. Representable functors are small. Finite colimits of small objects are small, essentially because filtered colimits commute with finite limits.

□

Theorem A.2.7. *Let $\mathcal{M}$ be a proper closed monoidal model category. Let $\mathcal{C}$ be a category that is enriched over $\mathcal{M}$, and let $\mathrm{Map} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{M}$ denote the mapping functor.*

Let Γ be a set of maps in $\mathcal{M}^{\mathcal{C}}$ such that the source and target of each map is a small object in $\mathcal{M}^{\mathcal{C}}$. Suppose that $\mathcal{S}_\Gamma$ is the full subcategory of $\mathcal{M}^{\mathcal{C}}$ consisting of those functors $F : \mathcal{C} \to \mathcal{M}$ such that for any $(\gamma : G \to H) \in \Gamma$, the natural map of simplicial sets

$$\gamma^* : \mathrm{Map}(H, F) \to \mathrm{Map}(G, F)$$

is an isomorphism. The inclusion $U : \mathcal{S}_\Gamma \to \mathcal{M}^{\mathcal{C}}$ has a left adjoint $T : \mathcal{M}^{\mathcal{C}} \to \mathcal{S}_\Gamma$ with the following properties:

1. *the counit of the adjunction $\epsilon : TU \to \mathrm{id}_{\mathcal{S}_\Gamma}$ is a natural isomorphism, and*
2. *in the injective model structure on $\mathcal{M}^{\mathcal{C}}$, if F is a functor such that for all $\gamma \in \Gamma$, the map γ^* is a weak equivalence of simplicial sets and γ itself is a cofibration, then $UT(F)$ is weakly equivalent to F.*

Proof. The following is essentially a small object argument.

Let $T_0 = F$. We will construct T as the (directed) colimit of a diagram

$$T_0 \to T_1 \to \cdots$$

in which each T_{m+1} is defined in terms of T_m. Suppose that T_m has already been constructed.

For each $(\gamma : G^\gamma \to H^\gamma) \in \Gamma$, let

$$J_m^\gamma = \mathrm{Map}(G^\gamma, T_m); \text{ and } I_m^\gamma = \mathrm{Map}(H^\gamma, T_m).$$

We have $\gamma_m^* : I_m^\gamma \to J_m^\gamma$.

We define four functors in $\mathcal{M}^{\mathcal{C}}$:

$$A_m = \coprod_{\gamma \in \Gamma} G^\gamma \otimes I_m^\gamma$$

$$B_m = \coprod_{\gamma \in \Gamma} G^\gamma \otimes J_m^\gamma$$

$$C_m = \coprod_{\gamma \in \Gamma} H^\gamma \otimes I_m^\gamma$$

$$D_m = \coprod_{\gamma \in \Gamma} H^\gamma \otimes J_m^\gamma.$$

If $\gamma \in \Gamma$, we will let ι_γ denote the inclusion of the γ component of any of these four functors (e.g. $\iota_\gamma : H^\gamma \otimes J_m^\gamma \to D_m$).

The diagram

$$\begin{tikzcd} A_m \arrow[r, "w_m"] \arrow[d, "x_m"'] & B_m \arrow[d, "y_m"] \\ C_m \arrow[r, "z_m"'] & D_m \end{tikzcd}$$

commutes, where the maps are various compositions with γ and γ^*. Let $b_m : B_m \to T_m$ be the map defined on a component $G^\gamma \otimes J_m^\gamma$ by the adjoint of the identity map $J_m^\gamma \to \mathrm{Map}(G^\gamma, T_m)$. Similarly, define $c_m : C_m \to T_m$ as the adjoint to an identity map. The diagram

$$\begin{tikzcd} A_m \arrow[r, "w_m"] \arrow[d, "x_m"'] & B_m \arrow[d, "b_m"] \\ C_m \arrow[r, "c_m"'] & T_m \end{tikzcd}$$

commutes. We let L_m denote the pushout in the diagram

$$\begin{tikzcd} A_m \arrow[r, "w_m"] \arrow[d, "x_m"'] \arrow[dr, phantom, "\lrcorner", very near end] & B_m \arrow[d, "r_m"] \\ C_m \arrow[r, "s_m"'] & L_m, \end{tikzcd}$$

and use it to define T_{m+1} as the pushout in the diagram

$$\begin{tikzcd} L_m \arrow[r, "d_m"] \arrow[d, "\phi_m"'] \arrow[dr, phantom, "\lrcorner", very near end] & T_m \arrow[d, "e_m"] \\ D_m \arrow[r, "\psi_m"'] & T_{m+1}. \end{tikzcd}$$

It helps to see all of this in one diagram (we have suppressed the subscript $-_m$):

$$\begin{tikzcd}
 & & D_m \arrow[dr] & & \\
 & D_m \arrow[ur] \arrow[dl] & & D_m \arrow[ddr, "\psi"] & \\
D_m \arrow[urrr] & & B_m \arrow[uu, "y"] \arrow[dr, "r"] \arrow[dd, "b"] & & \\
 & A_m \arrow[uu] \arrow[ur, "w"] \arrow[dl, "x"] \arrow[dd] & & L_m \arrow[uu, "d"] \arrow[dd, "\phi"] & T_{m+1} \\
C_m \arrow[uu, "z"] \arrow[urrr, "s"] \arrow[dd, "c"] & & T_m \arrow[dr] & & \\
 & T_m \arrow[ur] \arrow[dl] & & T_m \arrow[uur, "e"] & \\
T_m \arrow[urrr] & & & &
\end{tikzcd}$$

We repeat this construction recursively and define

$$T = T_\infty(F) = \operatorname{colim}(T_0 \xrightarrow{e_0} T_1 \xrightarrow{e_1} \cdots).$$

We now must show that for each $\gamma \in \Gamma$, the map

$$\gamma^* : \operatorname{Map}(H^\gamma, T) \to \operatorname{Map}(G^\gamma, T)$$

is an isomorphism. We will provide a natural map

$$K^\gamma : \operatorname{Map}(G^\gamma, T) \to \operatorname{Map}(H^\gamma, T)$$

and show that it is inverse to γ^*. Since G^γ is small by hypothesis,

$$\operatorname{Map}(G^\gamma, T) \cong \operatorname*{colim}_m \operatorname{Map}(G^\gamma, T_m).$$

Thus it suffices to find a map

$$K^\gamma_m : \operatorname{Map}(G^\gamma, T_m) \to \operatorname{Map}(H^\gamma, T_{m+1}).$$

The composition

$$H^\gamma \otimes J^\gamma_m \xrightarrow{\iota_\gamma} D_m \xrightarrow{\psi} T_{m+1},$$

in which ι_γ is the inclusion of a summand, gives the desired map $K^\gamma_m : J^\gamma_m \to I^\gamma_m$. We must show that K^γ and γ^* are mutually inverse.

It suffices to show that the map

$$K^\gamma_m \circ \gamma^* : I^\gamma_m \to I^\gamma_{m+1})$$

is equal to e_*. This follows by taking adjoints of the top and bottom compositions in the commutative diagram

$$\begin{tikzcd}
H^\gamma \otimes I^\gamma_m \arrow[r, "\gamma^*"] \arrow[dr, "\iota_\gamma"'] & H^\gamma \otimes J^\gamma_m \arrow[r, "\iota_\gamma"] & D_m \arrow[r, "\psi"] & T_{m+1}. \\
 & C_m \arrow[ur, "z"] \arrow[r, "c"] & T_m \arrow[ur, "e"] &
\end{tikzcd}$$

Indeed, the top composition $\psi \circ \iota_\gamma \circ \gamma^*$ is adjoint to $K^\gamma_m \circ \gamma^*$ and the composition $c \circ \iota_\gamma$ is adjoint to the identity map $I_m \to I_m$.

We must also show that the map

$$\gamma^* \circ K^\gamma_m : \mathrm{Map}(G^\gamma, T_m) \to \mathrm{Map}(G^\gamma, T_{m+1})$$

is equal to e_*. This follows similarly by taking adjoints of the top and bottom compositions in the commutative diagram

$$\begin{tikzcd}
G^\gamma \otimes J^\gamma_m \arrow[r, "\gamma"] \arrow[dr, "\iota_\gamma"'] & H^\gamma \otimes J^\gamma_m \arrow[r, "\iota_\gamma"] & D_m \arrow[r, "\psi"] & T_{m+1}. \\
& B_m \arrow[ur, "y"] \arrow[r, "b"] & T_m \arrow[ur, "e"] &
\end{tikzcd}$$

We will prove that the transformation $\epsilon : TU \to \mathrm{id}_{\mathcal{S}_\Gamma}$ is a natural isomorphism. Suppose that there exists $G \in \mathcal{S}_\Gamma$ such that $UG = F = T_0(F)$. Then one easily checks that the map $\gamma^* : I^\gamma_0 \to J^\gamma_0$ is a bijection for every γ. Hence, $A_0 = B_0, C_0 = D_0$, so $L_0 = D_0$, so $T_0 = T_1$. Repeating, we find that $T = F$, which proves property 1.

Now, we go back to prove adjointness. For each $F : \mathcal{C} \to \mathcal{M}$, define the unit map $\eta_F : F \to UT(F)$ to be the structure map for the directed colimit

$$F = T_0(F) \to T_\infty(F) = UT(F).$$

It is immediate that $(U * \epsilon) \circ (\eta * U) = \mathrm{id}_U$, and it is clear from the construction that $(\epsilon * T) \circ (T * \eta) = 1_T$; thus T and U are adjoint.

Finally, we prove property 2; by induction it suffices to show that $e_0 : T_0 \to T_1$ is a weak equivalence. If for all $\gamma \in \Gamma$, the map $g^* : I_0 \to J_0$ is a weak equivalence, then $A_0 \to B_0$ is an acyclic cofibration and $C_0 \simeq D_0$ is a weak equivalence. Since $\mathcal{M}^\mathcal{C}$ is proper, it follows that $C_0 \to L_0$ is a weak equivalence, hence $L_0 \to D_0$ is a weak equivalence by two-out-of-three. It suffices to show that $L_0 \to D_0$ is a cofibration, for then e_0 will be an acyclic cofibration. The composition $B_0 \to L_0 \to D_0$ is a cofibration. In the injective model structure on $\mathcal{M}^\mathcal{C}$, a map f is a cofibration if and only if for all $X \in \mathcal{C}$, $f(X)$ is an injection of simplicial sets. But if the composition of two maps is an injection then so is the second map, completing the proof.

□

Corollary A.2.8. *Suppose that $\mathcal{C}$ is a small category, and that $\mathcal{L}$ is a set of diagrams in $\mathcal{C}$, such that $\mathcal{C}$ is closed under taking limits in $\mathcal{L}$. Let $\mathcal{S}$ denote the category of covariant functors $\mathcal{C} \to$ **Sets** which preserve limits in $\mathcal{L}$. Then $\mathcal{S}$ is cocomplete.*

Proof. For any element $L : I_L \to \mathcal{C}$ in $\mathcal{L}$ with limit $\overline{L}$, let γ_L denote the natural transformation

$$\gamma_L : \mathrm{Hom}(-, \overline{L}) \longrightarrow \lim(\mathrm{Hom}(-, L)).$$

Let $\Gamma = \{\gamma_L | \ L \in \mathcal{L}\}$. A functor $F : \mathcal{C} \to \mathbf{Sets}$ preserves $\mathcal{L}$-limits if and only if for all $(\gamma : A \to B) \in \Gamma$, the map $\gamma^* : \mathrm{Hom}(B, F) \to \mathrm{Hom}(A, F)$ is an isomorphism. Since representable objects are small, the source and target of every $\gamma \in \Gamma$ is small; thus we may apply Corollary A.2.7 and obtain an adjoint pair

$$F : \mathbf{Sets}^{\mathcal{C}} \rightleftarrows \mathcal{S} : U.$$

Colimits exist in $\mathbf{Sets}^{\mathcal{C}}$ (taken object-wise).

Let $Y : I \to \mathcal{S}$ be a diagram. If one sets

$$\overline{Y} = \mathrm{colim}(FUY) := F(\mathrm{colim}(UY)),$$

it is easy to see that $\overline{Y}$ is a colimit for Y.

□

For a manifold M, recall that $H_M : \mathbf{Man} \to \mathbf{Sets}$ is the functor $H_M(-) = \mathrm{Hom}_{\mathbf{Man}}(M, -)$. Recall that $\bullet = \mathbb{R}^0$. For manifolds M, N, there is a natural map $\gamma_{M,N} : H_M \amalg H_N \to H_{M\times N}$ induced by the two projections. Let $\Gamma_{\mathbf{PP}}$ consist of all such maps $\gamma_{M,N}$ and the map $\emptyset \to H_\bullet$, where $\emptyset : \mathbf{Man} \to \mathbf{Sets}$ is the initial functor.

Corollary A.2.9. *Let $\mathcal{C} = \mathbf{Man}$ and let $\Gamma = \Gamma_{\mathbf{PP}}$ be as above. Then the hypotheses of Theorem A.2.7 are satisfied, and $\mathcal{S}_\Gamma$ is the category of finite-product preserving functors from manifolds to sets. Thus we have an adjunction*

$$L : \mathbf{Sets}^{\mathbf{Man}} \rightleftarrows \mathbf{PP} : R$$

which satisfies the properties described in the theorem.

Corollary A.2.10. *The category* $\mathbf{PP}$ *is complete and cocomplete.*

Proof. Clearly $\mathbf{PP}$ is complete because the limit of product-preserving functors is product-preserving. If I is a small category and $X : I \to \mathbf{PP}$ is a diagram, then one has

$$\mathrm{colim}(X) \cong \mathrm{colim}(LU(X)) \cong L\,\mathrm{colim}(U(X))$$

by Corollary A.2.9. Thus $\mathbf{PP}$ is also cocomplete.

□

Theorem A.2.11. *The category* $\mathbf{sPP}$ *is a simplicial model category in which a map $A \to B$ is a weak equivalence (resp. fibration) if and only if the induced map $A(M) \to B(M)$ is a weak equivalence (resp. fibration) for every manifold M.*

Proof. Note that $A(M) \to B(M)$ is the same as

$$\mathrm{Map}(H_M, A) \to \mathrm{Map}(H_M, B).$$

Since $\mathbf{PP}$ is complete and cocomplete, we may apply theorem [9, II.5.9].

□

A.3 Lemmas from category theory

Lemma A.3.1. *Let $\mathcal{C}$ be a category (resp. simplicial category). Suppose that the right square in the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

is a limit (resp. homotopy limit). Then the left square is a limit (resp. homotopy limit) if and only if the big rectangle is.

Proof. This is simple category theory.

□

Remark A.3.2. In some proofs, we utilize Lemma A.3.1 in the following way. We have a diagram such as

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "w"] & \\ C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[d, "x"] \\ F \arrow[r, "y"] & G \arrow[r, "z"] & H \end{tikzcd} \tag{A.2}$$

for which we know that some combination of squares and rectangles are limits or homotopy limits, and we can conclude by repeated application of Lemma A.3.1 that the others are as well. We call such a diagram (in which every square and rectangle is Cartesian or homotopy Cartesian) an *all-Cartesian diagram.*

Suppose that Diagram A.2 is all-Cartesian. Note that, up to isomorphism, such a diagram can be constructed from the data (w, x, y, z).

Lemma A.3.3. *Consider the diagram*

$$\begin{tikzcd} A \arrow[r, "q"] \arrow[d, "s"] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r, "r"] \arrow[d, "t"] \arrow[dr, phantom, "\ulcorner", very near start] & C \arrow[d, "u"] \\ D \arrow[r, "v"] \arrow[d, "x"] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[r, "w"] \arrow[d, "y"] & F \\ G \arrow[r, "z"] & H. & \end{tikzcd}$$

It is all Cartesian (resp. all homotopy Cartesian) if and only if the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d] \\ C \times G \arrow[r] & F \times H \end{tikzcd}$$

is also Cartesian (resp. homotopy Cartesian).

Proof. Let X be the fiber product in the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d] \\ C \times G \arrow[r] & F \times H. \end{tikzcd}$$

One uses the universal properties held by A and X to find a unique map $A \to X$ and a unique map $X \to A$ making diagrams commute. These must be mutually inverse maps.

□

Corollary A.3.4. *The diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is Cartesian (resp. homotopy Cartesian), if and only if the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d] \\ B \times C \arrow[r] & D \times D \end{tikzcd}$$

is Cartesian (resp. homotopy Cartesian).

Proof. Apply Lemma A.3.3 to the all Cartesian diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & B \arrow[d] \\ C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[r] \arrow[d] & D \\ C \arrow[r] & D. & \end{tikzcd}$$

□

Lemma A.3.5. *Let $\mathcal{C}$ be a category, let L be a set of finite cardinality n, and for each $l \in L$, let A_l and B_l be objects of $\mathcal{C}$, let $f_l : A_l \to B_l$ be a morphism between them, and let $g_l : A_l \to C$ be a morphism to an object C. Let $A = \amalg_l A_l, B = \amalg_l B_l$, and $f : A \to B$ and $g : A \to C$ the induced morphisms. Then the colimit (resp. homotopy colimit) of the diagram Ψ:*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \\ B & \end{tikzcd}$$

is the colimit (resp. homotopy colimit)

$$(B_1 \amalg_{A_1} C) \amalg_C (B_2 \amalg_{A_2} C) \amalg_C \cdots \amalg_C (B_n \amalg_{A_n} C).$$

In particular, if $A_i \to B_i$ is an isomorphism (resp. weak equivalence) for each $i > 1$, then the colimit of Ψ is simply

$$B_1 \amalg_{A_1} C.$$

Proof. This follows by checking that both sides of the isomorphism (resp. weak equivalence) satisfy the necessary universal property.

□

We now discuss restrictions and left Kan extensions of functors. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a functor and $X \in \mathcal{D}$ is an object. Recall that $(F \downarrow X)$ is the category whose objects are pairs (c, ϕ) where $c \in \mathcal{C}$ and $\phi : F(c) \to X \in \mathrm{Mor}(\mathcal{D})$. The morphisms in $(F \downarrow X)$ are the obvious commutative diagrams.

Definition A.3.6. Suppose that $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor between small categories, and that $\mathcal{M}$ is a category. The functor $R_F : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$, induced by composition, is called *the restriction of functors.*

Example A.3.7. Suppose that C and D are topological spaces and $f : D \to C$ is a continuous map. A presheaf X on D gives rise to a presheaf f_*X on C by restriction of functors. Let $F : \mathrm{Op}(C)^{\mathrm{op}} \to \mathrm{Op}(D)^{\mathrm{op}}$ denote the functor taking an open set in C to its preimage. Note that for any open set $U \subset D$, $(F \downarrow U)$ is the category of open sets $V \subset C$ such that $f^{-1}(V) \supset U$.

Definition A.3.8. Let $F : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C}' \to \mathcal{D}$ be covariant functors. The category $(F \downarrow F')$ is defined to be the category whose objects are triples (c, g, c'), where $c \in \mathcal{C}$ and $c' \in \mathcal{C}'$ are objects, and $g : F(c) \to F'(c')$ is a morphism in $\mathcal{D}$. If $C_0 = (c_0, g_0, c_0')$ and $C_1 = (c_1, g_1, c_1')$, then we define $\mathrm{Hom}_{(F\downarrow F')}(C, C')$ to be the set of pairs (x, x'), where $x : c_0 \to c_1, x' : c_0' \to c_1'$ are morphisms such that $g_1 F(x) = F'(x') g_0$.

In particular, suppose that $X \in \mathcal{D}$ is an object and $F : \mathcal{C} \to \mathcal{D}$ is a functor. The notation $(F \downarrow X)$ takes for granted the identification of X with a functor $X : [0] \to \mathcal{D}$, where $[0]$ is the one-morphism category.

Lemma A.3.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor between mall categories, and let $\mathcal{M}$ be a category that has all small colimits. Let $R = R_F : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{C}}$ be the restriction of functors. The functor R has a left adjoint $L = L_F : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{D}}$. For $G : \mathcal{C} \to \mathcal{M}$ and $X \in \mathcal{D}$, L is defined by the formula*

$$L(G)(X) = \operatorname*{colim}_{(F(A)\to X)\in(F\downarrow X)} G(A).$$

Proof. It suffices to find a unit map η and a counit map ϵ such that the composition of natural transformations

$$R \xrightarrow{\eta} RLR \xrightarrow{\epsilon} R$$

and the composition of natural transformations

$$L \xrightarrow{\eta} LRL \xrightarrow{\epsilon} L$$

are both equal to the identity.

For a functor $G : \mathcal{C} \to \mathcal{M}$ and an object $B \in \mathcal{C}$, we need

$$\eta_G(B) : G(B) \to \operatorname*{colim}_{(F(A) \to F(B)) \in (F \downarrow F(B))} G(A).$$

But the category $(F \downarrow F(B))$ has a final object, $F(B)$, so

$$\operatorname*{colim}_{A \in (F \downarrow F(B))} G(A) = G(B).$$

We take $\eta_G(B)$ to be the identity functor. This is clearly natural in G and B.

For a functor $H : \mathcal{D} \to \mathcal{M}$ and an object $X \in \mathcal{D}$, we need

$$\epsilon_H(X) : \operatorname*{colim}_{(F(A) \to X) \in (F \downarrow X)} H(F(A)) \to H(X).$$

For this it suffices to find a map $H(F(A)) \to H(X)$ for every map $F(A) \to X$, and we take the one given by H. This is clearly natural in H and X.

Using a cofinality argument, one shows that $R \xrightarrow{\eta} RLR \xrightarrow{\epsilon} R$ and $L \xrightarrow{\eta} LRL \xrightarrow{\epsilon} L$ are indeed equal to the identity.

□

The following is a reformulation which is more useful for working with presheaves.

Lemma A.3.10. *Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor between small categories and let $\mathcal{M}$ be a category that has all small colimits. Then F induces a functor $R = R_F : \mathcal{M}^{\mathcal{D}^{\mathrm{op}}} \to \mathcal{M}^{\mathcal{C}^{\mathrm{op}}}$ by composition. The functor R has a left adjoint $L = L_F : \mathcal{M}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{M}^{\mathcal{D}^{\mathrm{op}}}$. For $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{M}$ and $X \in \mathcal{D}$, L is defined by the formula*

$$L(G)(X) = \operatorname*{colim}_{(X \to F(A)) \in (X \downarrow F)} G(A).$$

Proof. Follows directly from Lemma A.3.9

□

Definition A.3.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between small categories, and let $\mathcal{M}$ be a category. Let $L : \mathcal{M}^{\mathcal{C}} \to \mathcal{M}^{\mathcal{D}}$ be the left adjoint to the restriction of functors. Then L is called the *left Kan extension functor.*

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