Derived smooth manifolds

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Goal for Derived Manifolds

- Given submanifolds $A, B \subset M$, obtain a "transverse intersection" $A \pitchfork B$ by perturbation.
  - One has $[A] \smile [B] = [A \pitchfork B]$ in cobordism $\Omega(M)$ (or integral cohomology, $H^*(M, \mathbb{Z})$, etc.)
- This $A \pitchfork B$ isn't unique ($\pitchfork$ is not functorial).
- Goal: enlarge **Man** to include non-transverse intersections ... (without losing meaningful topological structure,) and have

$$[A] \smile [B] = [A \cap B]$$

in full generality.

Methodology

- Mimic the construction of schemes. Except:
  - Use smooth version of rings, and
  - work homotopically.
  - Local models are zero-sets of smooth functions $f : \mathbb{R}^n \to \mathbb{R}^k$.
- Prove an imbedding theorem for derived manifolds.
- Use Thom-Pontrjagin construction to get a fundamental class.

Lax simplicial C^∞-rings

- Roughly, a C^∞-ring is a ring A with extra structure:
  One can apply C^∞-functions to A.
- For example, given a smooth function $f\colon \mathbb{R}^3 \to \mathbb{R}$,
  one has $f\colon A^3 \to A$.
- Use Lawvere's "algebraic theories,"
  but do so in a "homotopical way."

- Explicitly...
  - let $\mathbb{E} = \{\mathbb{R}^n | n \in \mathbb{N}\}$ denote Euclidean category,
  - consider the simplicial model category $\mathbf{sSets}^{\mathbb{E}}$, and
  - localize $\mathbf{sSets}^{\mathbb{E}}$ at the set of maps

$$C^\infty(\mathbb{R}^i) \amalg C^\infty(\mathbb{R}^j) \to C^\infty(\mathbb{R}^{i+j}).$$

  - Now F fibrant implies $F(\mathbb{R}^{i+j}) \xrightarrow{\simeq} F(\mathbb{R}^i) \times F(\mathbb{R}^j)$.

Local C^∞-ringed spaces

- The above "lax" model of C^∞-rings is left proper.
- A C^∞*-ringed space* is a pair $(X, \mathcal{O}_X)$
  - where X is a space, and
  - $\mathcal{O}_X$ is a homotopy sheaf of C^∞-rings on X.
- A local C^∞-ringed space is one with "local" stalks.
- This locality condition is closely related to Jacob Lurie's theory of "Geometries" and "Structured Spaces."

Derived manifolds

- A *principle derived manifold* is a homotopy limit:

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \longrightarrow & \mathbb{R}^0 \\ \downarrow & & \downarrow 0 \\ \mathbb{R}^n & \xrightarrow[f]{} & \mathbb{R}^k, \end{array}$$

  the "homotopy zero-set" of f.
- In particular,
  - $X = \{x \in \mathbb{R}^n | f(x) = 0\}$ is the zero-set of f, as a space, and
  - $\mathcal{O}_X = \mathcal{O}_{\mathbb{R}^0} \amalg_{\mathcal{O}_{\mathbb{R}^k}} \mathcal{O}_{\mathbb{R}^n}$ is a homotopy colimit of sheaves on X.

- A *derived manifold* is a local C^∞-ringed space $(X, \mathcal{O}_X)$ that can be covered by principle derived manifolds.

- **dMan** contains non-transverse intersections of submanifolds.
- Proposition: Given
  - a smooth manifold M,
  - two derived manifolds $\mathcal{X}$ and $\mathcal{Y}$, and
  - maps $\mathcal{X} \to M$ and $\mathcal{Y} \to M$,

  the homotopy fiber product $\mathcal{X} \times_M \mathcal{Y}$ is a derived manifold.

Example 1: Smooth manifolds

- Let M be a smooth manifold.
- Then C^∞_M is a sheaf of C^∞-rings on M.
- The pair (M, C^∞_M) is a local C^∞-ringed space, in fact a derived manifold.

- The map $\mathbf{i}\colon \mathbf{Man} \to \mathbf{dMan}$
  - is fully faithful, and
  - preserves products and transverse intersections.

Example 2: Line intersecting itself

- Let $x\colon \mathbb{R} \to \mathbb{R}^2$ and $y\colon \mathbb{R} \to \mathbb{R}^2$ be the x- and y-axis.
  Their intersection is a point (isomorphic to $\mathbb{R}^0$).
- What if we intersect the x-axis with itself? This is the homotopy limit $\mathcal{X} := (X, \mathcal{O}_X)$ in the diagram

$$\begin{array}{ccc} (X, \mathcal{O}_X) & \longrightarrow & \mathbb{R} \\ \big\downarrow & & \big\downarrow{\scriptstyle x} \\ \mathbb{R} & \xrightarrow[x]{} & \mathbb{R}^2. \end{array}$$

- Underlying spaces: $X = \mathbb{R}$. However on sheaves: $\mathcal{O}_X \neq C^\infty_{\mathbb{R}}$.
- Turns out: $[\mathcal{X}] = [\mathbb{R}^0]$. Fundamental class is a point.

Example 3: Section of a vector bundle

- Let $p\colon E \to B$ be a vector bundle, $z\colon B \to E$ the zero section.
- If $s\colon B \to E$ is a section transverse to z, then $z \cap s$ is smooth manifold.
- If not transverse, $z \cap s$ is still a derived manifold.

- Theorem: All derived manifolds are obtainable in this way.

Failure of nullstellensatz

- Recall: the line intersect itself in $\mathbb{R}^2$ is "like" a point.
- That is, take $C^\infty[x,y]$ and mod out by y twice.
  Doing so is *different* than modding by y once:

$$C^\infty[x,y]/(y,y) \not\cong C^\infty[x,y]/(y).$$

- In fact, $C^\infty[x,y]/(y,2y,y^2)$ has virtual dimension $2-3=-1$.
- The Nullstellensatz fails:
  - Given a submanifold $X \subset M$,
  - if we mod out $C^\infty(M)$ by all functions which vanish on X,
  - the result is a $-\infty$-dimensional derived manifold,
  - which is certainly not X!
- What does this mean to us?

Thom-Pontrjagin construction

The main idea:

- Imbed your compact manifold in $\mathbb{R}^n$.
- Use the tubular neighborhood theorem to find a map $S^n \to MO$.
- Connects the geometric world, "bordism,"
  to homotopic world, "Thom Spectrum."

Imbedding theorem for manifolds

- Tubular neighborhood theorem for manifolds M:
  - E is a vector bundle,
  - $i\colon M \to U \subset \mathbb{R}^n$ is an imbedding,
  - $U \subset \mathbb{R}^n$ is an open subset,
  - $r\colon U \to M$ is the retraction.

$$\begin{array}{ccc} M & & \\ \downarrow & \searrow^{i} & \\ E & \xleftarrow{\cong} & U \\ {\scriptstyle p}\downarrow & \swarrow_{r} & \\ M & & \end{array}$$

- Recover M as the zero-set of the canonical section

$$\begin{array}{ccc} M & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow{\scriptstyle z} \\ U & \xrightarrow[s]{} & r^*E. \end{array}$$

Imbedding theorem for derived manifolds

- Theorem: If $\mathcal{X}$ is a compact derived manifold, there exists
  - an open $U \subset \mathbb{R}^n$,
  - an imbedding $i\colon \mathcal{X} \to U$,
  - a vector bundle $E \to U$, and
  - a section $s\colon U \to E$,

  such that $\mathcal{X}$ is the zeroset of s:

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & U \\ \big\downarrow & \lrcorner & \big\downarrow z \\ U & \xrightarrow[s]{} & E. \end{array}$$

Changing the section

- $$\begin{array}{ccc} M & \longrightarrow & U \\ \big\downarrow & \lrcorner & \big\downarrow z \\ U & \xrightarrow[s]{} & E \end{array}$$

  Suppose s is transverse to z. Then acting on s by $\sigma \in GL(E)$ doesn't change manifold M.

- $$\begin{array}{ccc} \mathcal{X} & \longrightarrow & U \\ \big\downarrow & \lrcorner & \big\downarrow z \\ U & \xrightarrow[s]{} & E \end{array}$$

  In fact, regardless of transversality, acting on s by $\sigma \in GL(E)$ doesn't change derived manifold $\mathcal{X}$.

- Any two sections $s, t \colon U \to E$ have zero-sets which are *derived cobordant.*

Maps to MO

- Given a compact derived manifold $\mathcal{X}$, we get
  - an open set $U \subset \mathbb{R}^n$,
  - a vector bundle $E \to U$, and
  - a section $s\colon U \to E$.
- This is exactly what is needed for a map $S^n \to MO$.
- Derived cobordisms give rise to homotopies $S^n \times [0,1] \to MO$, and vice versa.
- So, we have the Thom-Pontrjagin construction *without the transversality requirement*.

An isomorphism

- The functor $\mathbf{i}\colon \mathbf{Man} \to \mathbf{dMan}$ induces

$$\mathbf{i}_*\colon \Omega \to \Omega^{der}.$$

- Every derived manifold is derived cobordant to a smooth manifold.
- Every two manifolds which are derived cobordant are (smoothly) cobordant.
- Theorem: $\mathbf{i}_*$ is an isomorphism.

- Punchline: derived manifolds give *no new classes*,
  but you don't have to worry about transversality,
  and we have an intersection theory at the space level.

Fundamental class

- For a compact $\mathcal{X}$, define $[\mathcal{X}]$ using the induced map $S^n \to MO$.
- If $A, B \subset M$ are compact submanifolds,
  then $\mathcal{X} := A \times_M B$ is derived cobordant to any $A \pitchfork B$,
  so $[\mathcal{X}] = [A \pitchfork B]$.
- In particular, write $\mathcal{X}$ as $A \cap B$, and we have

$$[A] \smile [B] = [A \cap B],$$

as desired. Goal achieved!

Applications

The theory of Derived Manifolds may be useful....

- Non-functoriality of transversality is sometimes an issue.
- Equivariant Thom-Pontrjagin theory.
- Floer Homology: perturbing defining equations is ugly.
- String Topology?

- Possibly gaining insight into Lurie's "Derived Algebraic Geometry."