Perspectives on Hilbert's 13th Problem

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Goals for Talk

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2. Entice more people to think about and work on H13 and surrounding problems.

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2. Entice more people to think about and work on H13 and surrounding problems. (more people, more progress)

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Definition (Kronecker, Chebotarev, Buhler-Reichstein)
Let $K \hookrightarrow L$ be a finite extension of k-fields. The *essential dimension* $ed_k(L/K)$ is the least d for which there exists:

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2. an inclusion of k-fields $K_0 \hookrightarrow K$,
3. a finite extension $K_0 \hookrightarrow L_0$, and
4. an isomorphism $K \otimes_{K_0} L_0 \cong L$ over K.

Theorem (Babylonians)
Let k be a field of characteristic not 2. Let $K \hookrightarrow L$ be a quadratic extension of k-fields. Then $ed_k(L/K) = 1$.

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(quadratic formula) ∎

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Let $K \hookrightarrow L$ be a finite extension of k-fields. The *resolvent degree* $rd_k(L/K)$ is the least d for which there exists:

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2. a finite sequence of subfields $K = K_0 \hookrightarrow \cdots \hookrightarrow K_r = K'$ such that $ed_k(K_{i+1}/K_i) \leq d$ for all i. ("Accessory irrationalities")

Theorem (Italians)
Let k be a field of characteristic not 2 or 3. Let $K \hookrightarrow L$ be an extension of k-fields with $[L : K] \leq 4$. Then $rd_k(L/K) = 1$.

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Theorem (Hamilton, Klein)
Let k be a field of characteristic not 2, 3. Let $K \hookrightarrow L$ be a sextic extension of k-fields. Then $rd_k(L/K) \leq 2$.

Essence of H13

Problem (Hilbert)

Prove there exists an extension of k-fields $K \hookrightarrow L$ with $rd_k(L/K) > 1$.

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Prove there exists an extension of k-fields $K \hookrightarrow L$ with $rd_k(L/K) > 1$. (for at least one k)

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Problem (Hilbert)

Prove there exists an extension of k-fields $K \hookrightarrow L$ with $rd_k(L/K) > 1$.

Problem (Hilbert, Segre)

For each $n > 0$, prove there exists an extension of k-fields $K \hookrightarrow L$ with $rd_k(L/K) > n$.

Essence of H13

Problem (Hilbert)
Prove there exists an extension of k-fields $K \hookrightarrow L$ with $rd_k(L/K) > 1$.

What kind of problem is this?

Arnold's Mushroom

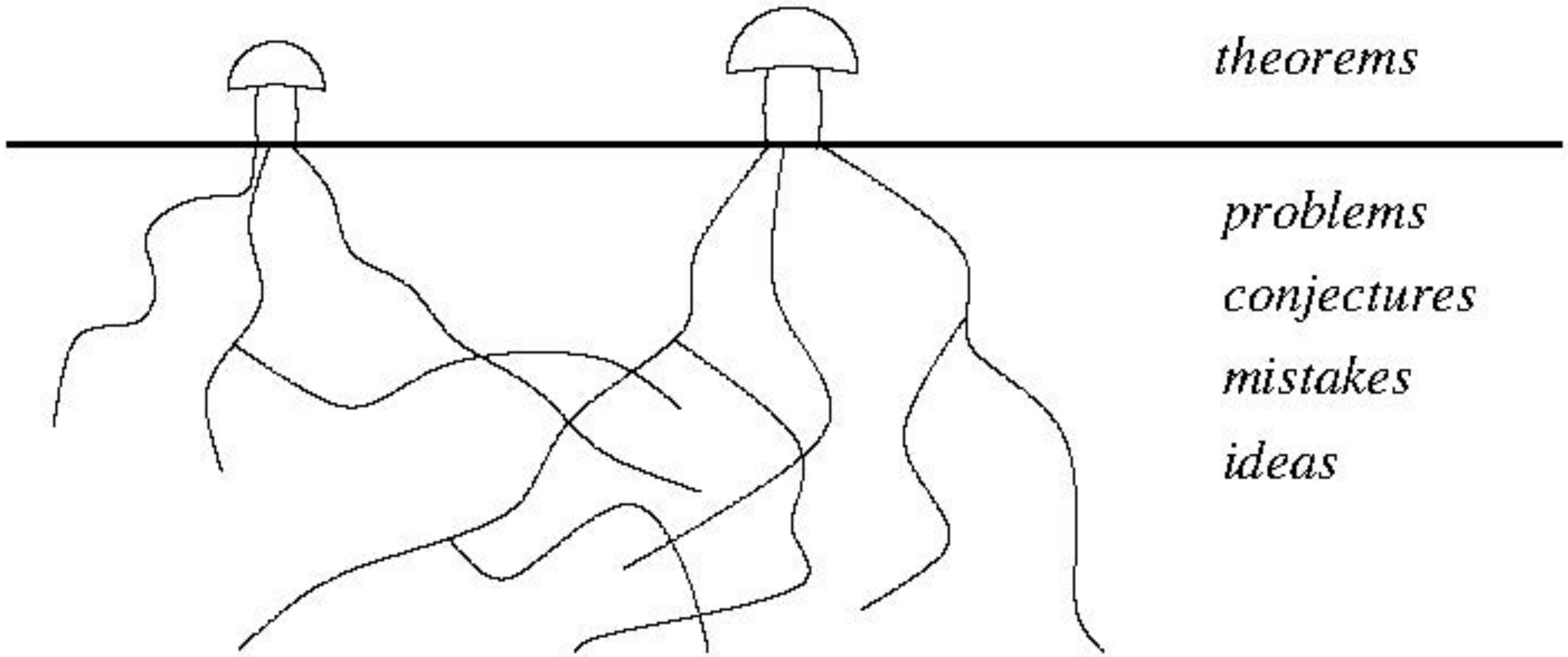

Figure 1: The Mathematical Mushroom

H13 as Galois theory

K - finitely generated k-field.

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$K \hookrightarrow \bar{K}$ an algebraic closure.

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$K \hookrightarrow \bar{K}$ an algebraic closure.

$\exists$ *resolvent filtration* $K \hookrightarrow K^0 \hookrightarrow K^1 \hookrightarrow \cdots \hookrightarrow K^d = \bar{K}$ where $d = tr.deg_k(K)$.

H13 as Galois theory

Resolvent filtration $K \hookrightarrow K^0 \hookrightarrow K^1 \hookrightarrow \cdots \hookrightarrow K^d = \bar{K}$

Properties

H13 as Galois theory

Resolvent filtration $K \hookrightarrow K^0 \hookrightarrow K^1 \hookrightarrow \cdots \hookrightarrow K^d = \bar{K}$

Properties

1. $K \hookrightarrow L$ factors through $K^i \Leftrightarrow rd_k(L/K) \leq i$.

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Properties

1. $K \hookrightarrow L$ factors through $K^i \Leftrightarrow rd_k(L/K) \leq i$.
2. $K \hookrightarrow K^i$ is Galois for all i.

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Properties

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2. $K \hookrightarrow K^i$ is Galois for all i.
3. Let $\varphi \colon \bar{K} \to \bar{K}'$ be a map of k-fields with $\varphi(K) \subset K'$. Then $\varphi(K^i) \subset (K')^i$.

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3. Let $\varphi : \bar{K} \to \bar{K}'$ be a map of k-fields with $\varphi(K) \subset K'$.
   Then $\varphi(K^i) \subset (K')^i$. ($\Rightarrow Aut_k(K) \cdot K^i = K^i$)

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4. K^i is solvably closed for $i \geq 1$.

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5. $K^0 = K \otimes_k \bar{k}$.

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5. $K^0 = K \otimes_k \bar{k}$. ($rd_k = rd_{\bar{k}}$)

H13 as Galois theory

Resolvent filtration $K \hookrightarrow K^0 \hookrightarrow K^1 \hookrightarrow \cdots \hookrightarrow K^d = \bar{K}$

Resolvent filtration $\Gamma_K \trianglerighteq \Gamma_{K\bar{k}} \trianglerighteq \Gamma_K^1 \trianglerighteq \cdots \trianglerighteq \Gamma_K^d = 1.$

Properties

1. Maps of k-fields preserve the filtration.

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Properties

1. Maps of k-fields preserve the filtration. (Is filtration characteristic?)

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Properties

1. Maps of k-fields preserve the filtration.
2. $rd_k > 1 \Leftrightarrow \exists K$ s.t. $\Gamma_K^1 \neq 1.$

H13 as Galois theory

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X - k-scheme, $\dim_k X = d$, $\bar{\eta} \to X$ geometric point.

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Resolvent filtration:

$$\pi_1^{et}(X, \bar{\eta}) \trianglerighteq \pi_1^{et}(X_{\bar{k}}, \bar{\eta}) \trianglerighteq \pi_1^{et}(X, \bar{\eta})^1 \trianglerighteq \cdots \trianglerighteq \pi_1^{et}(X, \bar{\eta})^d = 1.$$

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For $f\colon (X,\bar{\eta}) \to (Y,\bar{\eta})$ over k,

$$f_*(\pi_1^{et}(X,\bar{\eta})^i) \subset \pi_1^{et}(Y,\bar{\eta})^i .$$

H13 as Galois theory

Theorem (Farb-Kisin-W)

Let $N \geq 3$, and let $A_{g,N}$ be the (fine) moduli space of PPAVs.

Let $\bar{\eta} \to A_{g,N}$ be a geometric point. Let $p \nmid N$ be a prime.

Then $\pi_1^{et}(A_{g,N/\mathbb{C}}, \bar{\eta})^{\binom{g+1}{2}-1} \neq 1.$

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Proof:

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Proof: 1) Reduce to showing that for any étale $E \to A_{g,N/\mathbb{C}}$ such that $\pi_1^{et}(E, \bar{\eta}) \to \pi_1^{et}(A_{g,N/\mathbb{C}}, \bar{\eta}) \to Sp_{2g}(\mathbb{F}_p)$ is surjective,

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2) Congruence Subgroup Property $\Rightarrow \exists \ell$ with $p \nmid \ell$ s.t. $A_{g,\ell N/\mathbb{C}} \to E \to A_{g,N/\mathbb{C}}$.

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3) FKW2021 $\Rightarrow \nexists\, f$: $A_{g,\ell N/\mathbb{C}} \to Z$, $\tilde{Z} \to Z$ étale, and $f^*\tilde{Z} \simeq A_{g,pN}|_{A_{g,\ell N/\mathbb{C}}}$ if $\dim Z < A_{g,N}$. ∎

H13 as Galois theory

Theorem (Farb-Kisin-W)

Let $N \geq 3$, and $p \nmid N$. Then $\pi_1^{et}(A_{g,N}, \bar{\eta})^{\binom{g+1}{2}-1} \neq 1$.

$\therefore$ Core challenge is birational!

H13 as Galois theory

Counterpoint: rd_k minimizes over accessory irrationalities. Disallow these but allow birational trans.

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Definition (Buhler-Reichstein)

Let $Y \to X$ be a finite map of k-schemes. The *essential dimension* $ed_k(Y/X)$ is the least d for which there exists:

1. a rational map $f\colon X \dashrightarrow Z$ with $\dim_k Z \leq d$. ,
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2. a finite map $\tilde{Z} \to Z$, and
3. a birational isomorphism $f^*\tilde{Z} \simeq Y$ over X.

$\therefore$ For every n, $\exists Y \to X$ with $ed_k(Y/X) > n$.

H13 as Galois theory

∴ Challenge is interaction of birationality and accessory irrationalities.

H13 as Galois theory

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Problem Does there exist a local ring A such that $\pi_1^{et}(Spec(A))^1 \neq 1$?

H13 as Group Theory

Definition

Let k be a field. Define the *resolvent degree* of a finite group G by

$$rd_k(G) := \sup_{L/K \text{ Galois } G\text{-ext.}} rd_k(L/K)$$

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Properties

1. $H \subset G \Rightarrow rd_k(H) \leq rd_k(G)$
2. $rd_k(G) \leq \max\{rd_k(G_i) \mid G_i \textit{ simple factor of } G\}$
3. $rd_k(G) \leq \min\{\dim_k V \mid \exists\ \rho : G \to GL(V)\}$.

H13 as Group Theory

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5. Upper bounds for sporadic groups.

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6. $\lim_n (n - rd_k(A_n)) = \infty$.

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Conjectures (Hilbert)

1. $rd_{\mathbb{C}}(A_6) = 2.$ (Sextic conj.)
2. $rd_{\mathbb{C}}(A_7) = 3.$ (H13)
3. $rd_{\mathbb{C}}(A_8) = rd_{\mathbb{C}}(A_9) = 4.$ (Octic and Nonic conj.)

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6. $\lim_n (n - rd_k(A_n)) = \infty$.

Problem Find *any* finite group G with $rd_k(G) > 1.$

H13 as Group Theory

Idea expand the context!

H13 as Group Theory

Definition

Let $F\colon \mathit{Fields}_k \to \mathit{Set}$ be a functor. Let $\alpha \in F(K)$.

H13 as Group Theory

Definition

Let $F\colon\ Fields_k \to Set$ be a functor. Let $\alpha \in F(K)$.

1. The *essential dimension* of α, $ed_k(\alpha)$, is the least d such that $\exists L$ with $tr.deg_k(L) \leq d$ and $\alpha \in Im(F(L) \to F(K))$.

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1. The *essential dimension* of α, $ed_k(\alpha)$, is the least d such that $\exists L$ with $tr.deg_k(L) \leq d$ and $\alpha \in Im(F(L) \to F(K))$.
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H13 as Group Theory

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H13 as Group Theory

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H13 as Group Theory

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1. $Fin(K) := \{\alpha\colon K \hookrightarrow A \mid \alpha$ fin. s.s. comm.$\}/\cong$. Then $rd_k(\alpha) = rd_k(A/K)$ when A is a field.

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3. Γ arithmetic group. $rd_k(\Gamma) := rd_k(H^1(-;\hat{\Gamma}))$.

H13 as Group Theory

Theorem (Reichstein)
Let G be a connected algebraic group. Then $rd_k(G) \leq 5$, and $rd_k(G) = 1$ if G has no simple factors of type E_8.

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Conjecture $\Rightarrow rd_k(G) = 1$ for G connected.

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Let G be a reductive group over $\mathbb{Q}$ with hermitian symmetric domain X. Let $\Gamma \subset G(\mathbb{R})$ be a cocompact arithmetic group.

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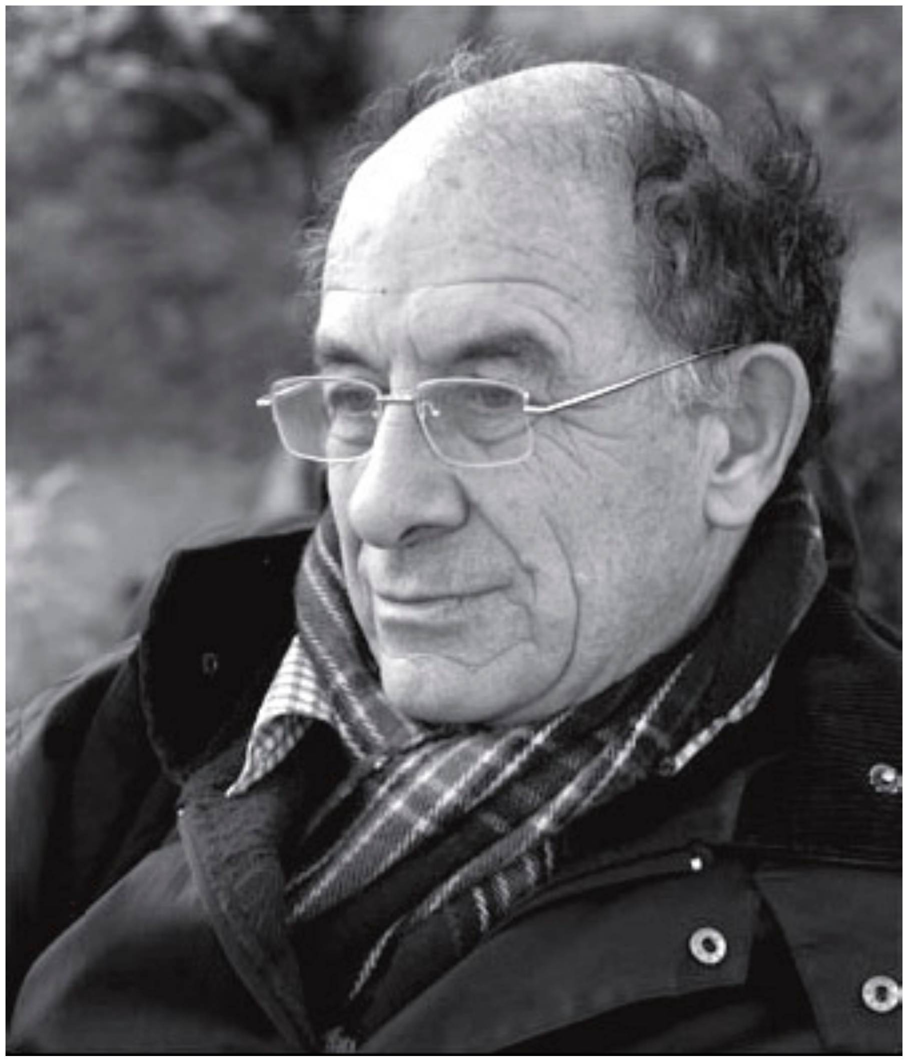
Question Is a finite group more like a connected algebraic group or an arithmetic lattice?

H13 via Hodge Theory?



"Perhaps there is some kind of a mixed Hodge structure whose weight filtration provides the information . . ."

H13 via Hodge Theory?

Theorem (Farb-Kisin-W)

Let X be a smooth complex variety.

H13 via Hodge Theory?

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In fact, theorem of the fixed part implies:

Theorem (Farb-Kisin-W)
Let X be a smooth complex variety. Let $H_{\mathbb{Z}} \to X$ be an integral VHS with period map $\pi\colon X \to \Gamma\backslash D$. Let $E \to X$ be any quasifinite map. Then $rd_{\mathbb{C}}(H_{\mathbb{Z}}|_E) \geq \dim_{\mathbb{C}} \pi(X)$.

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Idea use p-adic Hodge theory to adapt this argument to finite covers, e.g. $A_{g,p} \to A_g$.

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Problem Heuristic is too strong :(

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Proposition
Let G be a finite group, and $\tilde{X} \to X$ a connected G-cover. There exists $E \to X$ gen. finite, such that $\tilde{X}|_E \to E$ is connected, and $rd_{\mathbb{C}}(\tilde{X}|_E/E) = 1$.

H13 via Hodge Theory?

Theorem (Farb-Kisin-W)
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Idea use p-adic Hodge theory to adapt this argument to finite covers, e.g. $A_{g,p} \to A_g$.

Theorem (Farb-Kisin-W)
Let $N \geq 3$ and $p \nmid N$. Let $E \to A_{g,N}$ be quasifinite of degree prime to p. Then $ed_{\mathbb{C}}(A_{g,pN}|_E/E) = \dim_{\mathbb{C}} A_{g,N}$.

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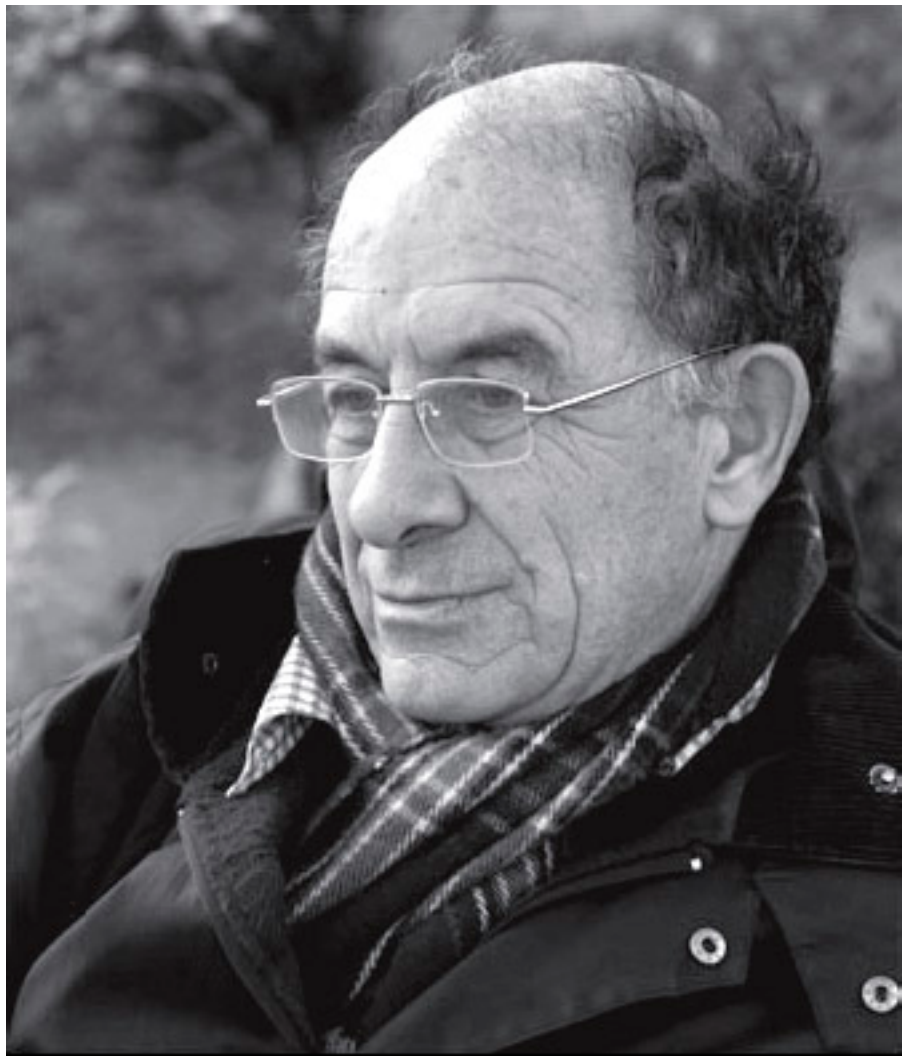
Question Is there a different Hodge theoretic approach?

H13 via Characteristic Classes?



Trudy Moskov. Mat. Obšč.
Tom 21 (1970)

Trans. Moscow Math. Soc.
Vol. 21 (1970)

ON SOME TOPOLOGICAL INVARIANTS OF ALGEBRAIC FUNCTIONS[1)]

UDC 513.83

V. I. ARNOL'D

CONTENTS

There are some interesting connections between the theory of algebraic functions and Artin's theory of braids. For instance, the space G_n of polynomials of

H13 via Characteristic Classes?

Arnold's Dictionary

Algebraic Function	Vector bundle
B_n	GL_n
$Conf_n(\mathbb{C})$	BGL_n
Fox-Neuwirth cells	Schubert cells
Characteristic classes	Characteristic classes

H13 via Characteristic Classes?

Can characteristic classes detect $rd > 1$?

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No :(

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Norm-Residue Isomorphism (Bloch-Kato)

$\Rightarrow rd_k(H^*(-;\mathbb{F}_p)) = 1.$

H13 via Characteristic Classes?

Can characteristic classes detect $rd > 1$?

No :(

Norm-Residue Isom. $\Rightarrow$ every mod p Galois cohomology class dies after after pulling back along a $\sqrt[p]{\ }$-cover.

H13 via Characteristic Classes?

Can characteristic classes detect $rd > 1$?

No :(

Work-in-progress Let X be a complex variety. For any p, there exists a p-power branched cover $E \to X$ such that $Im(H^*(X;\mathbb{F}_p) \to H^*(E;\mathbb{F}_p)) = 0$.

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∴ Characteristic classes can't obstruct (enough) accessory irrationalities.

H13 via “Special” Points

“[We must] fathom the nature and significance of the necessary accessory irrationalities.”

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Definition

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   $K \hookrightarrow K^{rd_k^{\leq d}} = K^d$ (resolvent filtration).

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Conjecture (Hilbert's Sextic)

Let $T \to Spec(\mathbb{C}(x, y))$ be the A_6-torsor associated to the Valentiner action $A_6 \circlearrowleft \mathbb{CP}^2$. Let $A_6 \circlearrowleft X$ be a faithful action on a smooth, irreducible curve. Then ${}^{T}X(\mathbb{C}(x, y)^1) = \varnothing$.

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Problem How do we obstruct solvable points?

Arnold's Mushroom

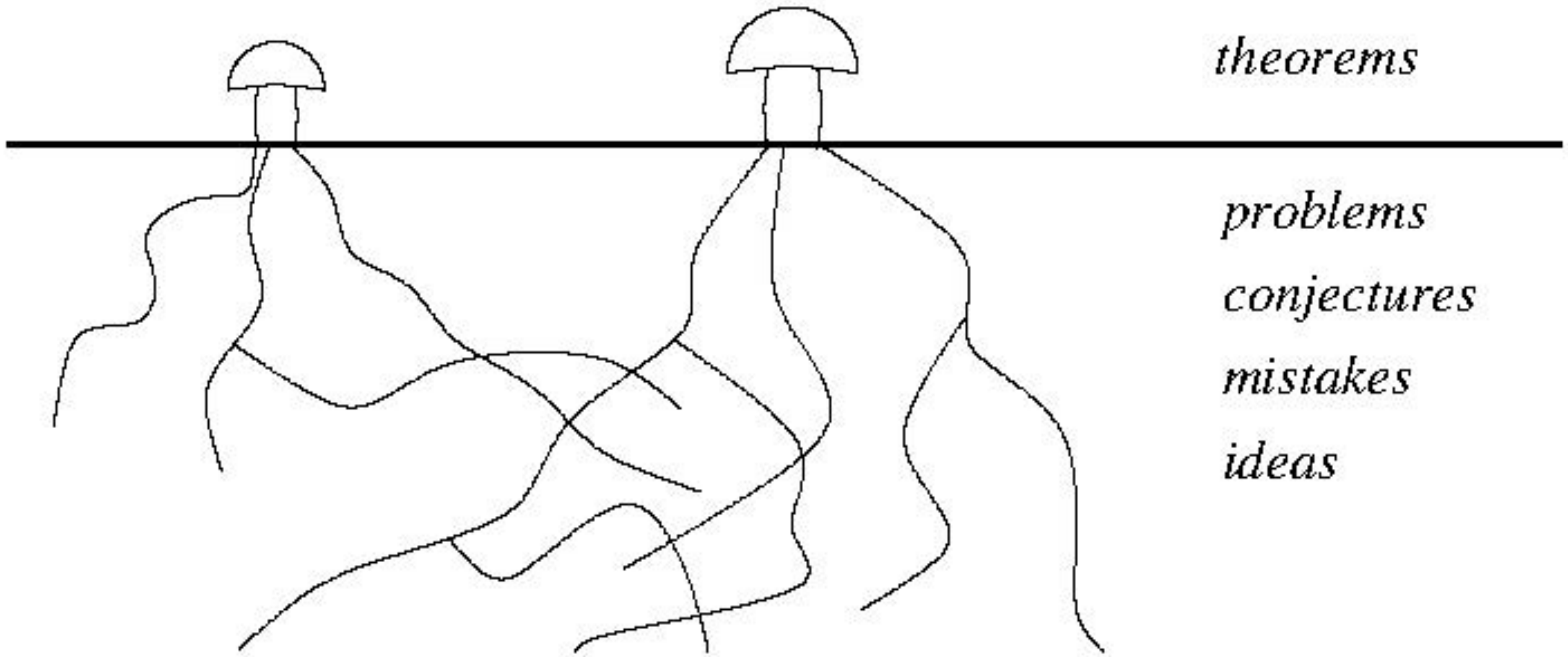

Figure 1: The Mathematical Mushroom

Thank you!