Minimal weights of mod p Galois representations

Hanneke Wiersema, University of Cambridge

April 12, 2022

VaNTAGe Seminar

The weight in Serre's conjecture

Invent. math. 109, 563–594 (1992)

Inventiones mathematicae

The weight in Serre's conjectures on modular forms

Bas Edixhoven*
Mathematisch Institut Budapestlaan, Postbus 80.010, NL 3508 TA Utrecht, The Netherlands

Oblatum 29-IV-1991 & 23-III-1992

Contents

1 Introduction 563
2 Modular forms mod p and Galois representations 565
3 Tate's θ-cycles 569
4 The weight in Serre's conjectures 571
5 Group schemes of type $(p, \ldots, p)$ when $e = p-1$ 574
6 The local representation in the supersingular case 578
7 Reduction to weight at most $p+1$ 585
8 Finiteness at p 588
9 Multiplicity one results 590

1 Introduction

Let p be a prime number and let f be a modular form of level N with $p \nmid N$, weight k and character ε, with coefficients in $\bar{\mathbb{F}}_p$. Suppose that f is an eigenform for all Hecke operators T_l^*, l prime, say with eigenvalues $a_l \in \bar{\mathbb{F}}_p$. Then there exists a unique 2-dimensional semi-simple continuous representation $\rho_f : G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$, which is unramified outside pN and has the property that trace $(\rho_f(\mathrm{Frob}_l)) = a_l$ and $\det(\rho_f(\mathrm{Frob}_l)) = \varepsilon(l) l^{k-1}$ for all $l \nmid pN$. For a historical account of this result see [24, §6], see also [3]. It follows from the identities $\det(\rho_f(\mathrm{Frob}_l)) = \varepsilon(l) l^{k-1}$ that ρ_f is odd, i.e., $\det(\rho_f(c)) = -1$ for $c \in G_{\mathbb{Q}}$ a complex conjugation.

Serre conjectured in 1973 that every continuous semi-simple odd representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ arises in this way. In his article [26] he stated a more precise version of this conjecture for irreducible ρ: ρ should arise from a form of level $N(\rho)$, weight k_ρ and character $\varepsilon(\rho)$, where $N(\rho)$, k_ρ and $\varepsilon(\rho)$ are described in terms of ρ;

1

Aim of the paper

weight k_ρ and character $\varepsilon(\rho)$, where $N(\rho)$, k_ρ and $\varepsilon(\rho)$ are described in terms of ρ; $N(\rho)$ and k_ρ are meant to be as small as possible. The aim of this paper is to show that if ρ comes from a modular form at all, say of some type (N, k, ε), then ρ also comes from a modular form of type (N, k_ρ, ε) and k_ρ is (almost) minimal.

Serre's conjecture (Khare-Wintenberger, building on work of many)

Let p be a prime, and let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$. Suppose

$$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

is a continuous irreducible representation and that $\det(\rho(c)) = -1$, where c is a complex conjugation.

Theorem (Serre's conjecture)
There are positive integers k and N such that ρ arises from an eigenform in $S_k(\Gamma_1(N))$ (weak form). In fact, ρ arises from an eigenform in $S_{k(\rho)}(\Gamma_1(N(\rho)))$ where $k(\rho) \geq 2$ and $N(\rho)$ is relatively prime to p (strong form).

Serre predicted these are *minimal* for ρ, as follows: if ρ is isomorphic to some $\rho_{f'}$ with prime-to-p level N' and weight $k' \geq 2$, then $N(\rho) \mid N'$ and $k(\rho) \leq k'$.Carayol has shown that $N(\rho) \mid N'$. Note we have $k \geq 2$!

The weight recipe - the irreducible case

Theorem (Fontaine)
Let f be a cuspidal Hecke eigenform of type (N, k, ε) with $2 \leq k \leq p+1$ with eigenvalues a_l. Suppose that $a_p = 0$ (supersingular case). Then $\rho_{f,p}$ is irreducible and

$$\rho_f|_{I_p} = \begin{pmatrix} \omega_2^{k-1} & 0 \\ 0 & \omega_2'^{k-1} \end{pmatrix}$$

with $\omega_2, \omega_2' : I_{p,t} \to \overline{\mathbb{F}}_p^{\times}$ the two level two fundamental characters.

In this case we have

$$\rho_{I_p} \cong \begin{pmatrix} \omega_2^{a+bp} & 0 \\ 0 & \omega_2'^{a+bp} \end{pmatrix}$$

with $0 \leq a < b \leq p-1$. We set $k(\rho) = 1 + pa + b$.

Galois representations of low weight forms

Write ω for the mod p cyclotomic character.

Theorem (Deligne)
Let f be a cuspidal Hecke eigenform of type (N, k, ε) with $2 \leq k \leq p+1$ with eigenvalues a_l. Suppose that $a_p \neq 0$ (the ordinary case). Then $\rho_{f,p}$ is reducible and

$$\rho_{f,p} = \begin{pmatrix} \omega^{k-1}\lambda(\varepsilon(p)/a_p) & * \\ 0 & \lambda(a_p) \end{pmatrix}.$$

with $\lambda(a) : G_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^{\times}$ the unramified character sending Frob_p to $a \in \overline{\mathbb{F}}_p^{\times}$.

The weight recipe - the reducible case

1. Suppose first $\rho_{I_{p,w}}$ is trivial, then

$$\rho_{I_p} \cong \begin{pmatrix} \omega^a & 0 \\ 0 & \omega^b \end{pmatrix}$$

   with $0 \leq a \leq b \leq p-2$. We set $k(\rho) = 1 + pa + b$, unless $(a, b) = (0, 0)$, then we set $k(\rho) = p$.

2. Suppose next $\rho_{I_{p,w}}$ is non-trivial, then

$$\rho_{I_p} \cong \begin{pmatrix} \omega^\beta & * \\ 0 & \omega^\alpha \end{pmatrix}$$

   for unique α and β with $0 \leq \alpha \leq p-2$ and $1 \leq \beta \leq p-1$. Let $a = \min(\alpha, \beta), b = \max(\alpha, \beta)$. If $\omega^{\beta-\alpha} = \omega$ and $\rho_{G_p} \otimes \omega^{-\alpha}$ is not finite at p, then $k(\rho) = 1 + pa + b + p - 1$, otherwise $k(\rho) = 1 + pa + b$.

Mod p modular forms

In the case where ρ_{I_p} is trivial, Serre sets $k(\rho) = p$. Serre originally avoids weight 1 modular forms as he considers mod p modular forms as reductions of forms in characteristic 0 (and these cannot necessarily be lifted).

However, using Katz' geometric definition of mod p modular forms one can allow the weight 1 modular forms.

As such, one can refine Serre's weight prediction. In the case where ρ_{I_p} is trivial, Edixhoven sets $k(\rho) = 1$.

Difference k_ρ and $k(\rho)$

Next write k_ρ for Serre's original weight recipe. We almost always have $k(\rho) = k_\rho$, and we always have $k(\rho) \leq k_\rho$.

It differs only in the reducible case:

1. If $\rho_{I_{p,w}}$ is trivial, $a = 0 = b$, then $k(\rho) = 1$ and $k_\rho = p$.
2. If $p = 2$, if $\rho_{I_{p,w}}$ is non-trivial, $\alpha = 0, \beta = 1$ and ρ_{G_p} is not finite at p, then $k(\rho) = 3$ and $k_\rho = 4$.

The weight part of Serre's conjecture

Theorem (Edixhoven)
Let $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ be continuous, irreducible and odd. Suppose we have a cuspidal eigenform g of type (N, k, ε) such that $\rho \cong \rho_g$. Then

1. *(existence): there exists a cuspidal eigenform f of type (N, k_ρ, ε) (Serre's recipe) with the same eigenvalues for all $l \neq p$ such that $\rho \cong \rho_f$*
2. *(refinement): there exists an eigenform f of type $(N, k(\rho), \varepsilon)$ (Edixhoven's adjusted weight) with the same eigenvalues for all $l \neq p$ such that $\rho \cong \rho_f$.*
3. *(minimality): there is no eigenform of level prime to p and of weight less than $k(\rho)$ whose associated Galois representation is isomorphic to ρ.*

How does one prove this: reduction to lower weight forms

Any form of some weight k 'comes from' a form of a lower weight:

Theorem
Let f be an eigenform of some type (N, k, ε), then there exist integers i and k' with $0 \leq i \leq p-1, k' \leq p+1$ and an eigenform g of type (N, k', ε) such that f and $\theta^i g$ have the same eigenvalues for all Hecke operators $T_l (l \neq p)$.

If $a_p(g) \neq 0$, g might have a companion form: a form g' of weight $p+1-k'$ such that $l^{p+1-k'} a_l(g) = l a_l(g')$.
This happens if and only if $\rho_{g,p}$ is tamely ramified (Gross, Coleman-Voloch).

Theta operator

Recall the θ operator:

$$\theta : \sum a_n q^n \to \sum n a_n q^n$$

with $a_n \in \overline{\mathbb{F}}_p$.

If f is an eigenform of type (N, k, ε) with eigenvalues a_l, then θf is an eigenform of type $(N, k+p+1, \varepsilon)$ with eigenvalues $l a_l$.

We can use this to translate twists of a modular form by the θ operator to twists of its representation by the cyclotomic character:

$$\rho_{\theta f} = \rho_f \otimes \omega.$$

We need to study more carefully what the θ operator does to the weight of a mod p modular form.

Weight filtration

Let f be a mod p modular form of level N and weight k.

Definition (Filtration)
Then we define the filtration $w(f)$ of f to be the smallest integer k' for which there exists a form of level N and weight k' which (at some cusp) has the same q-expansion.

Equivalently, it is the smallest integer $k - i(p-1)$ such that f is divisible by the i-th power of the Hasse invariant.

Theorem (Serre)
Let f, f' be two mod p modular forms of level N and weight k and k' respectively. If they have the same q-expansion, then $k \equiv k' \mod p-1$.

Theta cycles

Recall

Theorem
Let f be an eigenform of some type (N, k, ε), then there exist integers i and k' with $0 \leq i \leq p-1, k' \leq p+1$ and an eigenform g of type (N, k', ε) such that f and $\theta^i g$ have the same eigenvalues for all Hecke operators $T_l (l \neq p)$.

Suppose $\theta(f) \neq 0$. Then the θ cycle are the p integers

$$(w(f), w(\theta f), \ldots, w(\theta^{p-1} f))$$

Lemma (Serre)
If f has filtration k and $p \nmid k$, then θf has filtration $k+p+1$. If $p \mid k$, then $w(\theta f) < w(f) + p + 1$.

Theta cycles – a picture I

The cycles are classified by Edixhoven for weight $\leq p+1$.

Example
Let f be a cuspidal eigenform of type (N, k, ε) with $3 \leq k \leq p-1$ with eigenvalues a_l and $w(f) = k$. Suppose $a_p = 0$, then the θ-cycle of f is

$$(k, k+p+1, \dots, k+(p-k)(p+1), k_1, \dots, k_1+(k-3)(p+1), k)$$

where $k_1 = p+3-k$.

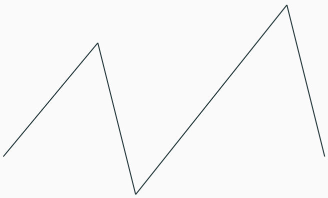


Figure 1: Schematic view of θ-cycles for f as above (where $k_1 < k$)

Theta cycles – a picture II

Figure 2: Schematic view of θ-cycles for ordinary f with $2 \leq w(f) \leq p-1$ on the left and $w(f) = 1, p, p+1$ on the right

Idea of the proof - existence

Suppose ρ is modular of weight k and level N, we want to show it also arises from a form of weight $k(\rho)$ and level N.

- There exists a form f_1 of weight $k_1 \leq p+1$, and $0 \leq i \leq p-1$ such that $\rho \otimes \omega^{-i} \cong \rho_{f_1}$.
- We know a lot about $\rho_{f_1,p}$ since f_1 is of low weight.
- Find k_1 and i in terms of ρ_p.
- Untwist f_1: set $f = \theta^i f_1$ and compute the theta cycle of f_1.
- Show $w(f) = k(\rho)$.

This k_1 is not necessarily unique: we need the work of Gross and Coleman-Voloch concerning companion forms.
For deciding between 2 and $p+1$ we use work of Mazur about finiteness at p.

Other characterisations of weights of Galois representations

Reformulation

We write $\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p^2$ for the $(k-2)$-th symmetric power of the standard representation of $\mathrm{GL}_2(\mathbb{F}_p)$ on $\overline{\mathbb{F}}_p^2$.

Proposition (Ash and Stevens)
Let $k \geq 2$. Then ρ is modular of level N and weight k if and only if the corresponding system of Hecke eigenvalues appears in $H^1(\Gamma_1(N), \mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p{}^2)$.

Remark
In fact, this is equivalent to ρ appearing in $H^1(\Gamma_1(N), V)$, with V a Jordan-Hölder constituent of $\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p{}^2$.

These *weights* are easy to list!

Serre weights

Definition (Serre weights)
The V are irreducible representations of $\mathrm{GL}_2(\mathbb{F}_p)$ over $\overline{\mathbb{F}}_p$,

$$V_{t,s} = \det^s \otimes \mathrm{Sym}^{t-1}\, \overline{\mathbb{F}}_p^2, \quad 0 \leq s < p-1, 1 \leq t \leq p$$

We call these *Serre weights*.

Buzzard, Diamond and Jarvis defined *algebraic modularity*. Given ρ, for what $V_{t,s}$ is ρ algebraically modular?

Buzzard, Diamond and Jarvis define a set of Serre weights $W(\rho)$.

Theorem (The BDJ conjecture in the classical case)
If $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{F}}_p)$ is modular of some weight, then

$$W(\rho) = \{V | \rho \textit{ is modular of weight } V\}.$$

The weight set $W(\rho)$ - examples

Remark
The recipe for $W(\rho)$ depends purely on the local representation.

Example
Let

$$\rho_{I_p} \sim \begin{pmatrix} \omega^a & 0 \\ 0 & 1 \end{pmatrix}$$

with $2 \leq a \leq p-3$, then

$$W(\rho) = \{V_{a,0}, V_{p-1-a,a}\}.$$

Example
Let

$$\rho_{I_p} \sim \begin{pmatrix} \omega_2^b & 0 \\ 0 & \omega_2^{pb} \end{pmatrix}$$

with $1 \leq b \leq p-1$ then

$$W(\rho) = \{V_{b,0}, V_{p+1-b,b-1}\}.$$

The minimal algebraic weight

Question: How to compare this with $k(\rho)$?

Definition (Minimal weight)

$$k_{\min}(V_{t,s}) = \min_k \{k \geq 2 \mid V_{t,s} \in \mathrm{JH}(\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p^2)\}$$

e.g. for $V_{t,0} = \mathrm{Sym}^{t-1}\,\overline{\mathbb{F}}_p^{\,2}$, we see $k_{\min}(V_{t,0}) = t+1$.

Formula for $k_{\min}(V_{t,s})$

Definition (Minimal weight)

$$k_{\min}(V_{t,s}) = \min_k \{k \geq 2 \mid V_{t,s} \in \mathrm{JH}(\mathrm{Sym}^{k-2}\,\overline{\mathbb{F}}_p^2)\}$$

Proposition (W.)
Let $0 \leq s < p-1, 1 \leq t \leq p$ and let $V_{t,s}$ be a Serre weight. Then

$$k_{\min}(V_{t,s}) = \begin{cases} s(p+1)+t+1, & s+t<p, \\ (s+1)(p+1)+tp-p^2, & s+t \geq p. \end{cases}$$

This reflects the behaviour of θ-cycles!

We saw earlier that $\rho_{\theta f} \cong \rho_f \otimes \omega$.If ρ is modular of some weight $V_{t,s}$, $\omega \otimes \rho$ will be modular of weight $V_{t,s+1}$.

Behaviour of Theta cycles

Let $p = 5$. Consider the Serre weight $V_{3,0}$. We have $k_{\min}(V_{3,0}) = 4$. We consider a 'theta' cycle by twisting by the determinant: $k_{\min}(V_{3,0}), k_{\min}(V_{3,1}), k_{\min}(V_{3,2}), \ldots$.

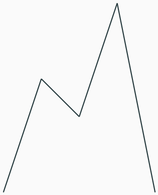


Figure 3: Algebraic theta cycle: (4,10,8,14,4)

By Edixhoven's classification, if $a_p = 0$, this is exactly the theta cycle of a mod p modular form with filtration 4.

Minimal algebraic weight

Definition: We set $k_{\min}(W(\rho)) = \min_k\{k_{\min}(V_{t,s}) \mid V_{t,s} \in W(\rho)\}$.

Theorem (Equality of two weight invariants)

$$k(\rho) = k_{\min}(W(\rho))$$

Example
Let $p = 5$ and suppose

$$\rho_{I_5} \sim \begin{pmatrix} \omega_2^3 & 0 \\ 0 & \omega_2'^3 \end{pmatrix}$$

here $k(\rho) = 4$, we have $W(\rho) = \{V_{3,0}, V_{3,2}\}$. We find $k_{\min}(V_{3,0}) = 4$ and $k_{\min}(V_{3,2}) = 8$, so

$$k(\rho) = 4 = k_{\min}(W(\rho))$$

Algebraic θ cycles

Example
Let ρ be as in the previous example. Recall the Serre weights were $V_{3,0}$ and $V_{3,2}$. We find cycles:

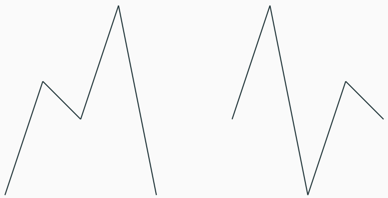
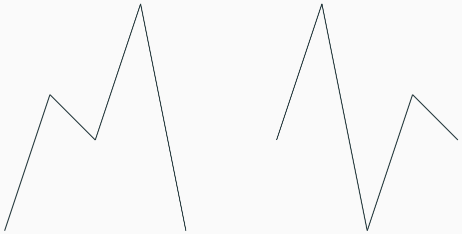


Figure 4: Algebraic theta cycles $(4, 10, 8, 14, 4)$ and $(8, 14, 4, 10, 8)$

Algebraic θ cycles

What about the cycle
$k_{\min}(W(\rho)), k_{\min}(W(\omega \otimes \rho)), \ldots, k_{\min}(W(\omega^{p-1} \otimes \rho))$?



So obtain

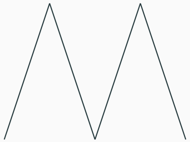


Figure 5: Algebraic theta cycles for ρ

What about more general Galois representations?

There has been a lot of interest of generalising Serre's conjecture due to its close relations with the Langlands program.

We focus on $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, where F is a totally real field.

Conjecture
Let $\rho : G_F \to GL_2(\overline{\mathbb{F}}_p)$ be continuous, irreducible and totally odd. Then there exists a Hilbert modular form f such that $\rho \cong \overline{\rho}_f$.

What are the weights of Hilbert modular forms that ρ shows up with? Is there a notion of a minimal weight?

First breakthrough on this is due to Buzzard, Diamond and Jarvis.

Algebraic modularity

Recall earlier Serre weights were irreducible $\overline{\mathbb{F}}_p$-representations of $\mathrm{GL}_2(\mathbb{F}_p)$. Now we get $\overline{\mathbb{F}}_p$-representations of

$$\mathrm{GL}_2(\mathcal{O}_F/p) = \prod_{\mathfrak{p}|p} \mathrm{GL}_2(\mathcal{O}_F/\mathfrak{p})$$

Definition (More general Serre weights)
Let $\mathfrak{p} \mid p$, we have

$$V_{\vec{t},\vec{s}} = \bigotimes_{\tau\in\Sigma} \left(\det^{s_\tau} \otimes_{k_\mathfrak{p}} \mathrm{Sym}^{t_\tau - 1} k_\mathfrak{p}^2\right) \otimes_{k_\mathfrak{p},\tau} \overline{\mathbb{F}}_p.$$

with $\Sigma = \{\tau : k_\mathfrak{p} \to \overline{\mathbb{F}}_p\}$ and $t_\tau \leq p$ for all $\tau \in \Sigma$.

Then a Serre weight is $V = \bigotimes_{\mathfrak{p}|p} V_\mathfrak{p}$.

More general BDJ conjecture [Gee et al.]

Theorem (The BDJ conjecture)
If $\rho : G_F \to GL_2(\overline{\mathbb{F}}_p)$ is modular, then

$$W(\rho) = \{V | \rho \text{ is modular of weight } V\}$$

Question
How do we define analogues of $k_{\min}(V_{t,s})$ and $k_{\min}(W(\rho))$?

What do we mean by minimal?

For simplicity suppose p is inert in F.

What does minimal mean?

We need a partial ordering for $F \neq \mathbb{Q}$. We write e_τ for the basis element of $\mathbb{Z}^\Sigma$ associated to τ. We set

$$\Xi_{\mathrm{Ha}}^{\mathbb{Z}} = \Big\{ \sum_{\tau \in \Sigma} y_\tau h_\tau \in \mathbb{Z}^\Sigma \mid y_\tau \geq 0 \text{ for all } \tau \in \Sigma \Big\},$$

where $h_\tau = p e_{\mathrm{Fr}^{-1} \circ \tau} - e_\tau$ is the weight of a partial Hasse invariant, e.g. $(0, \dots, 0, -1, p, 0, \dots, 0)$.

Definition (Partial ordering)
We say $\vec{k} \leq_{\mathrm{Ha}} \vec{k}' \iff \vec{k}' - \vec{k} \in \Xi_{\mathrm{Ha}}^{\mathbb{Z}}$.

Motivation: If $\vec{k} \leq_{\mathrm{Ha}} \vec{k}'$, then if ρ is modular of weight $\vec{k}$, it is also modular of weight $\vec{k}'$.

Some small cases

For $F = \mathbb{Q}$, the Hasse cone is $\Xi^{\mathbb{Z}}_{\mathrm{Ha}} = \{y_\tau(p-1) \in \mathbb{Z} \mid y_\tau \geq 0\}$ where $p-1$ is the weight of the Hasse invariant.

For F quadratic, we obtain the following picture:

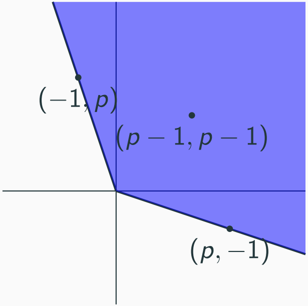


Figure 6: The Hasse cone in the quadratic case

Now we can define analogues $k_{\min}(V_{t,s})$ and $k_{\min}(W(\rho))$.

Analogue of $k_{\min}(V_{t,s})$

We would like to define

$$k_{\min}(V_{\vec{t},\vec{s}}) = \min_{\geq \mathrm{Ha}} \left\{ \vec{k} \in \mathbb{Z}^{\Sigma}_{\geq 2} \cap \Xi^{\mathbb{Q}}_{\min} \mid V_{\vec{t},\vec{s}} \in \mathrm{JH}\left(\bigotimes_{\tau \in \Sigma} \mathrm{Sym}^{\vec{k}_\tau - 2} k_{\mathfrak{p}}^2 \otimes_\tau \overline{\mathbb{F}}_p \right) \right\},$$

where

$$\Xi^{\mathbb{Q}}_{\min} = \Big\{ \sum_{\tau} x_\tau e_\tau \in \mathbb{Q}^{\Sigma} \mid p x_\tau \geq x_{\mathrm{Fr}^{-1} \circ \tau} \text{ for all } \tau \in \Sigma \Big\}.$$

For $V_{(t_0,t_1),(0,0)} = \mathrm{Sym}^{t_0 - 1} k_{\mathfrak{p}}^2 \otimes \mathrm{Sym}^{t_1 - 1} k_{\mathfrak{p}}^2$, we see $k_{\min}(V_{(t_0,t_1),(0,0)}) = (t_0 + 1, t_1 + 1)$.

Formula for $k_{\min}(V_{t,s})$

Let

$$k(\vec{t},\vec{s}) = \sum_{\tau\in\Sigma}(s_\tau(e_\tau + pe_{\mathrm{Fr}^{-1}\circ\tau}) + (t_\tau+1)e_\tau) - \sum_{\substack{\tau\in\Sigma\,|\\ s_\tau+t_\tau\geq p}}(p-t_\tau)(pe_{\mathrm{Fr}^{-1}\circ\tau} - e_\tau).$$

Conjecture

Let p be odd and $k_{\min}(V_{\vec{t},\vec{s}})$ be as above, then we have

$$k_{\min}(V_{\vec{t},\vec{s}}) = k(\vec{t},\vec{s})$$

Again we get behaviour reflecting Θ cycles - for mod p Hilbert modular forms. Here we have partial Θ operators Θ_τ, which add $e_\tau + pe_{\mathrm{Fr}^{-1}\circ\tau}$ to the weight.

An example

Let $p = 3$. Consider the Serre weight $V_{(2,3),(0,0)}$. We can again look at an 'algebraic' theta cycle: We obtain

$$(3,4) \to (5,4) \to (6,7) \to (6,5) \to (8,5) \to (9,8) \to (9,6) \to (11,6) \to (3,4)$$

Minimal algebraic weight

We analogously would like to define

$$k_{\min}(W(\rho)) = \min_{\geq \mathrm{Ha}}\{k(\vec{t},\vec{s}) \mid V_{\vec{t},\vec{s}} \in W(\rho)\}$$

Proposition (W.)
Suppose F is real quadratic and p is inert in F. Then $k_{\min}(W(\rho))$ is well-defined.

Questions about algebraic theta cycles

- Can we show $k_{\min}(W(\rho))$ always exists?
- What are the algebraic theta cycles for $k_{\min}(V_{\vec{t},\vec{s}})$?
- Can we describe

  $$(k_{\min}(W(\rho)), k_{\min}(W(\psi \otimes \rho)), k_{\min}(W(\psi^2 \otimes \rho)), \ldots, k_{\min}(W(\psi^{p^d-1}\rho)))$$

  where ψ is a fundamental character of level d?
- How does this relate to theta cycles of mod p Hilbert modular forms?

Questions?

Minimal weight cones

We set

$$\Xi_{\min}^{\mathbb{Q}} = \Big\{ \sum_{\tau} x_\tau e_\tau \in \mathbb{Q}^{\Sigma} \mid p x_\tau \geq x_{\mathrm{Fr}^{-1} \circ \tau} \text{ for all } \tau \in \Sigma \Big\}.$$

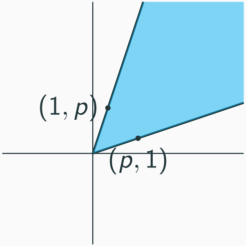


Figure 7: The minimal cone in the quadratic case

Define $\Xi_{\min}^{+} = \Xi_{\min} \cap \mathbb{Z}_{\geq 1}^{\Sigma}$.