# The Brauer group and the Brauer-Manin obstruction on K3 surfaces

I acknowledge that I live and work on the traditional territories of the Duwamish and Coast Salish people

http://native-land.ca/

https://mathematicallygiftedandblack.com/

Black History Month

February 23 2021

# Honoree of the Day

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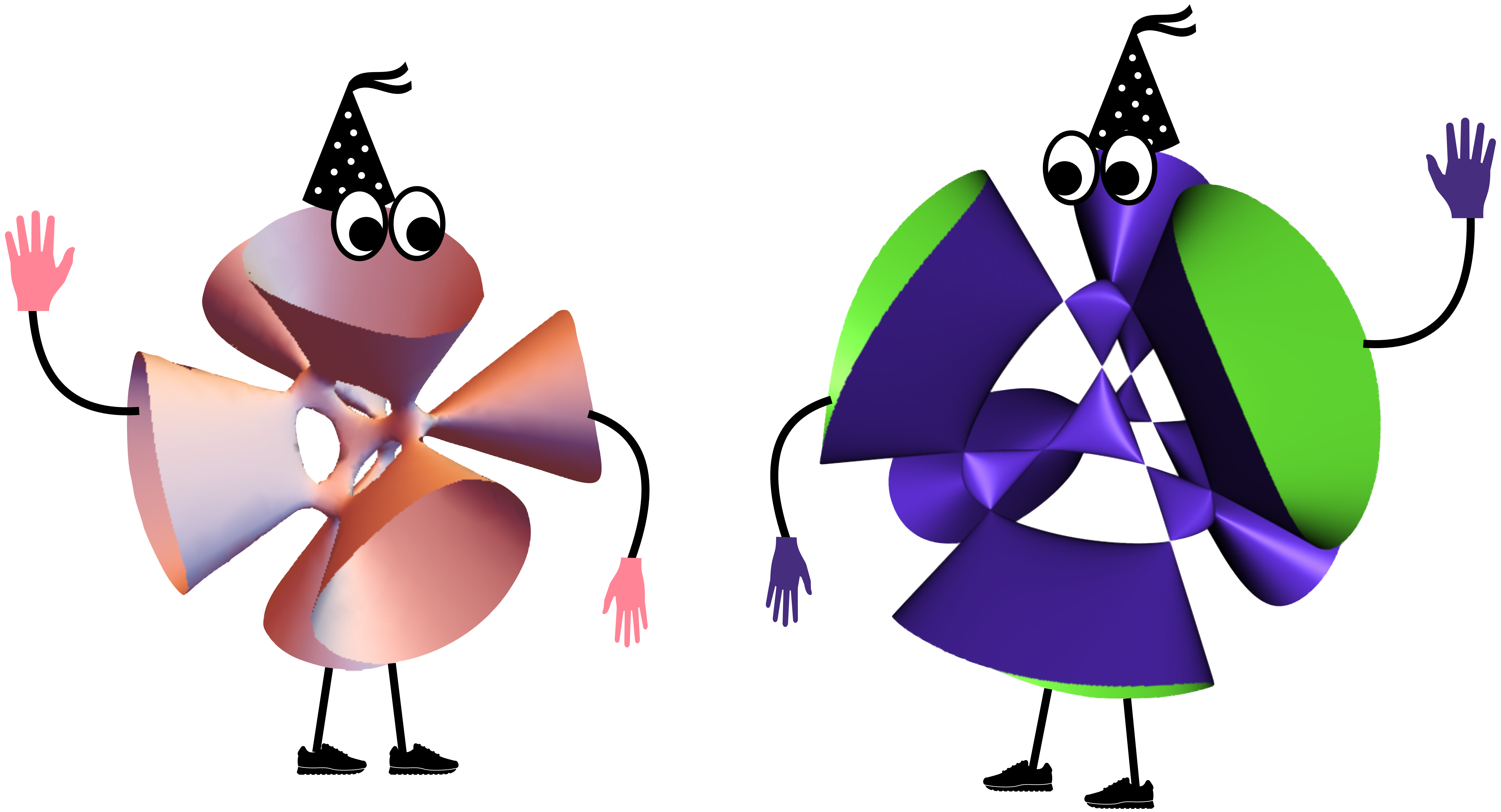
Where do we go from here?

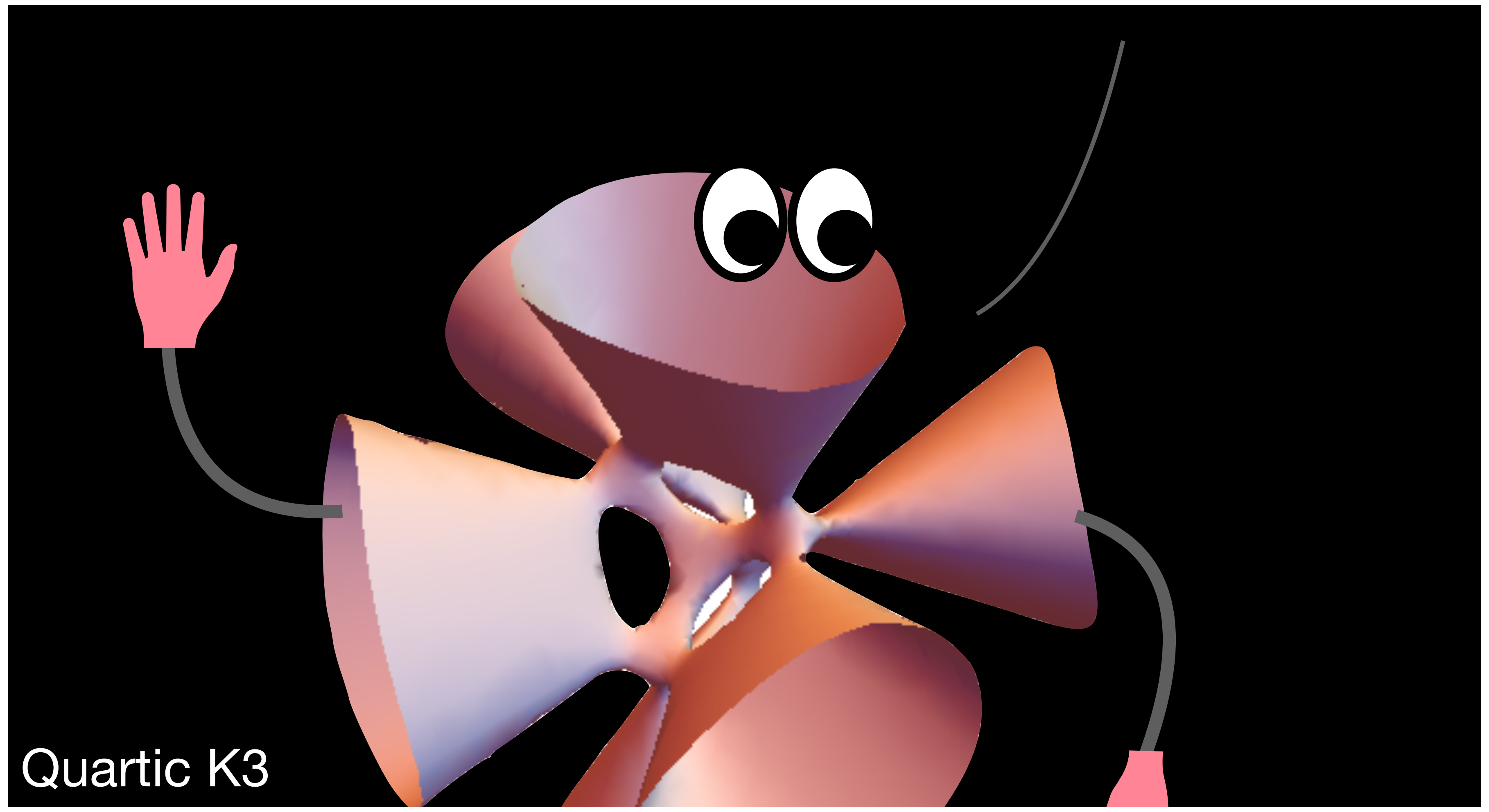
# Two K3 surfaces meet at a party…



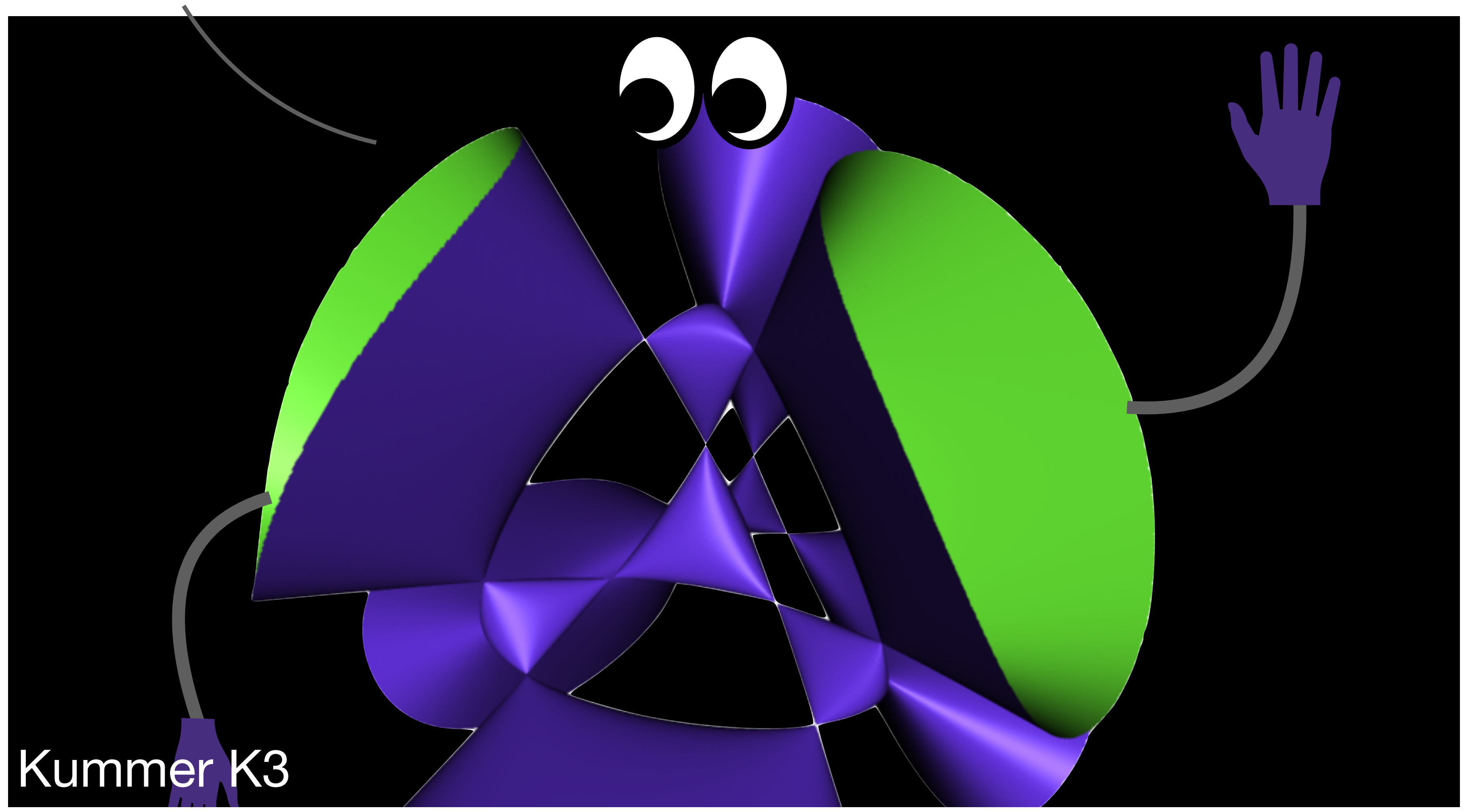
Surfaces created by Burt Totaro (L) and Claudio Rocchini (R)

# Two K3 surfaces meet on zoom

Nice to meet you, Kummer, I'm Quartic.




Hey! I'm Kummer.

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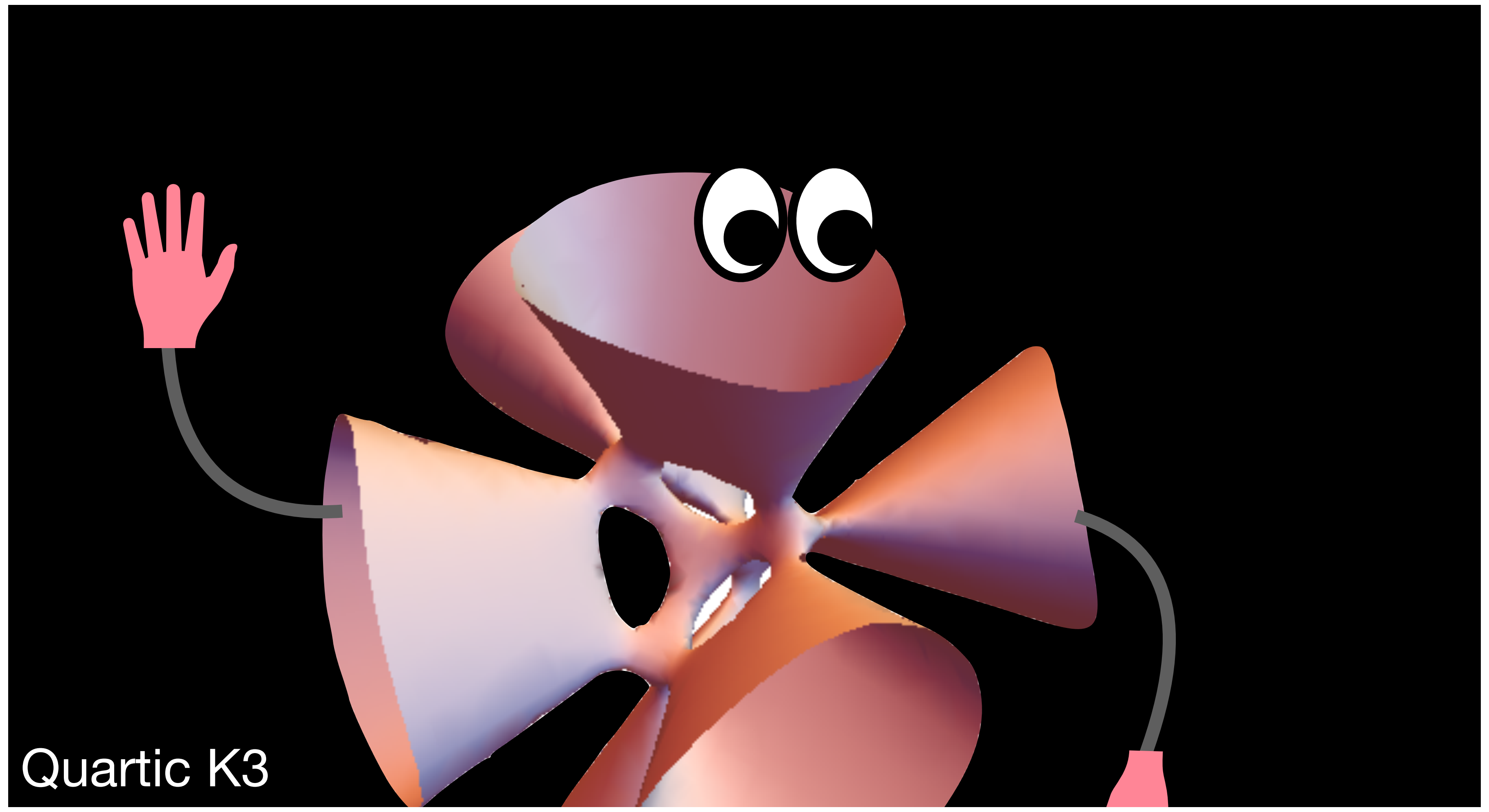
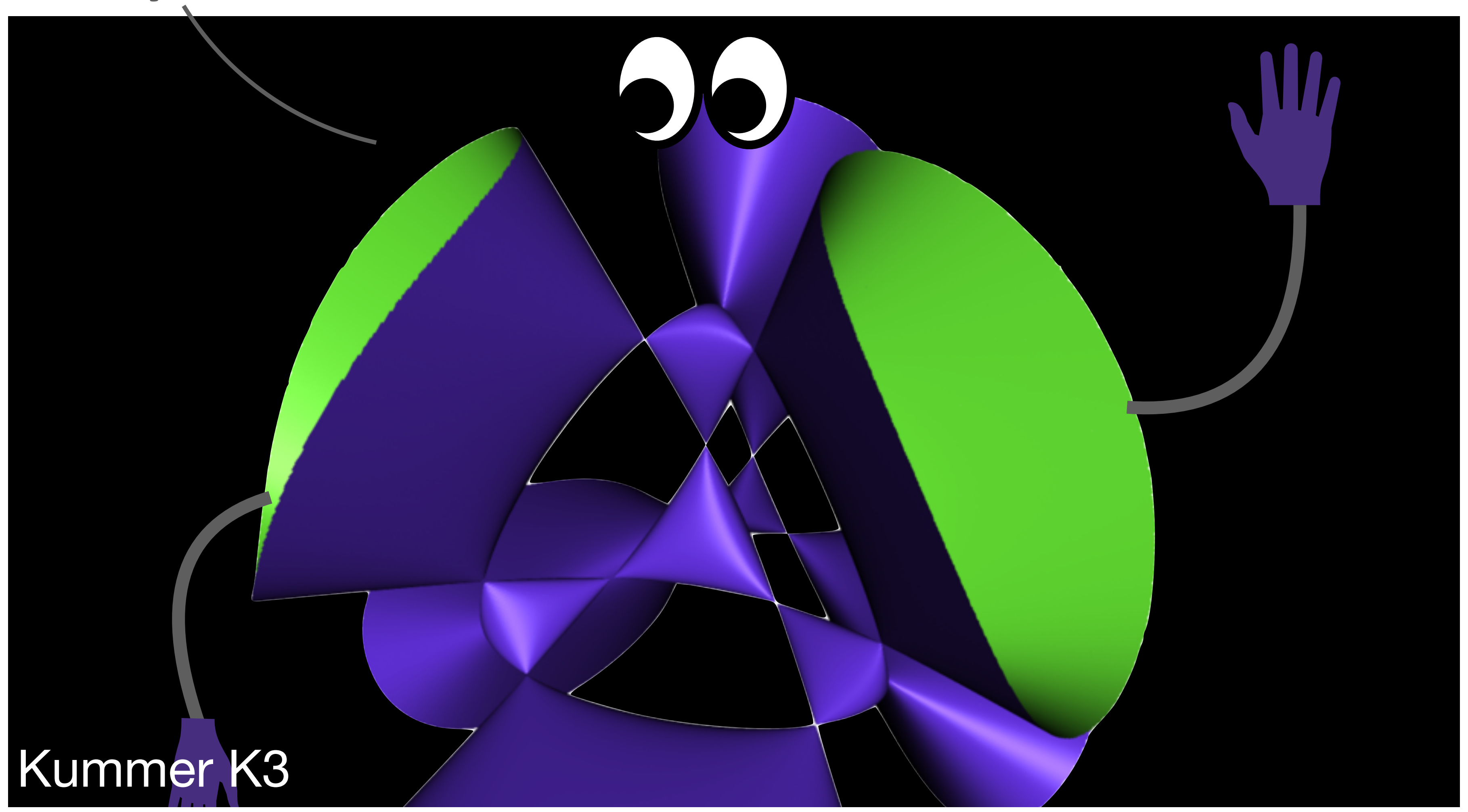
# Two K3 surfaces meet on zoom

Tell me a little bit about yourself.

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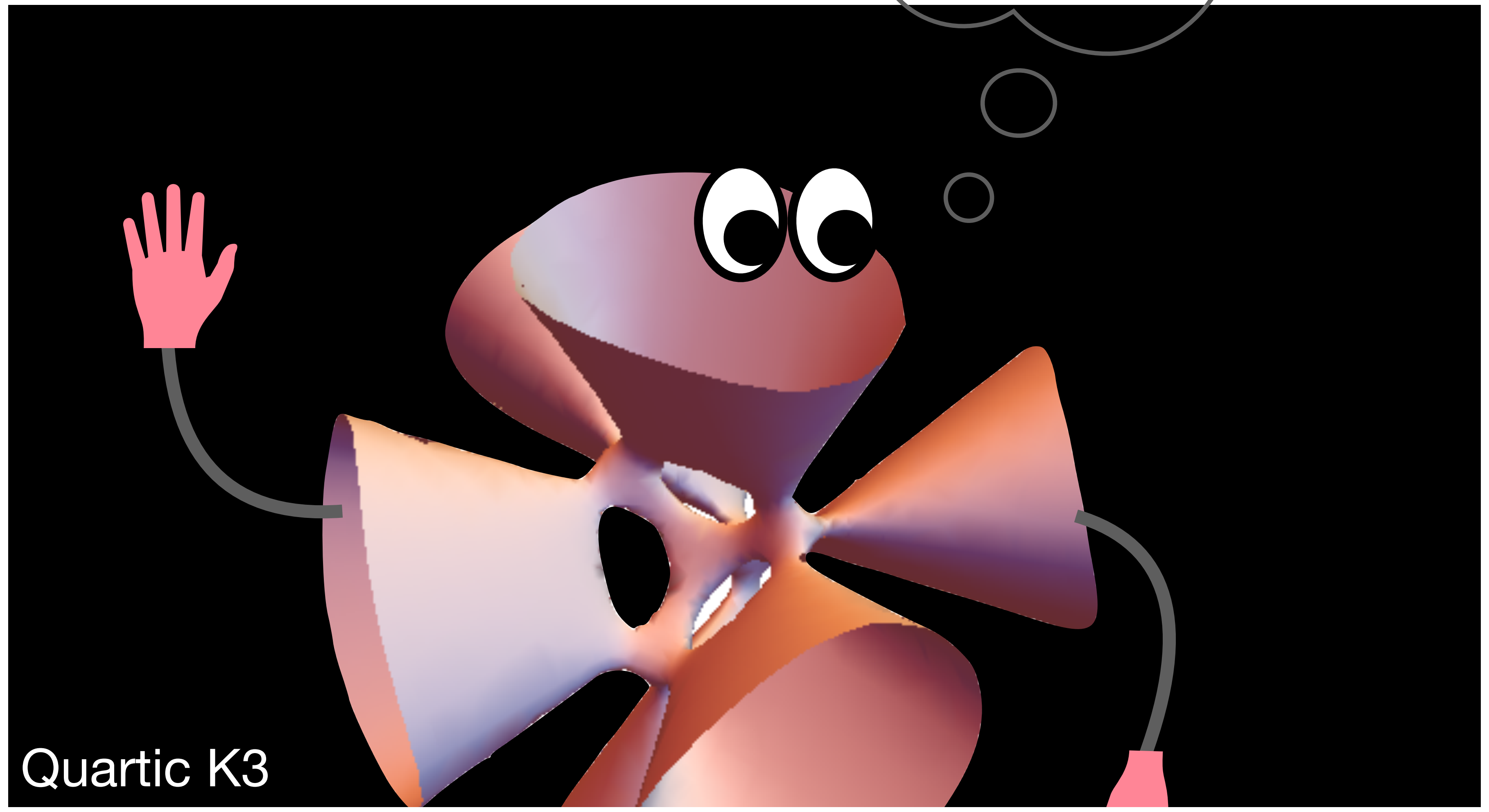
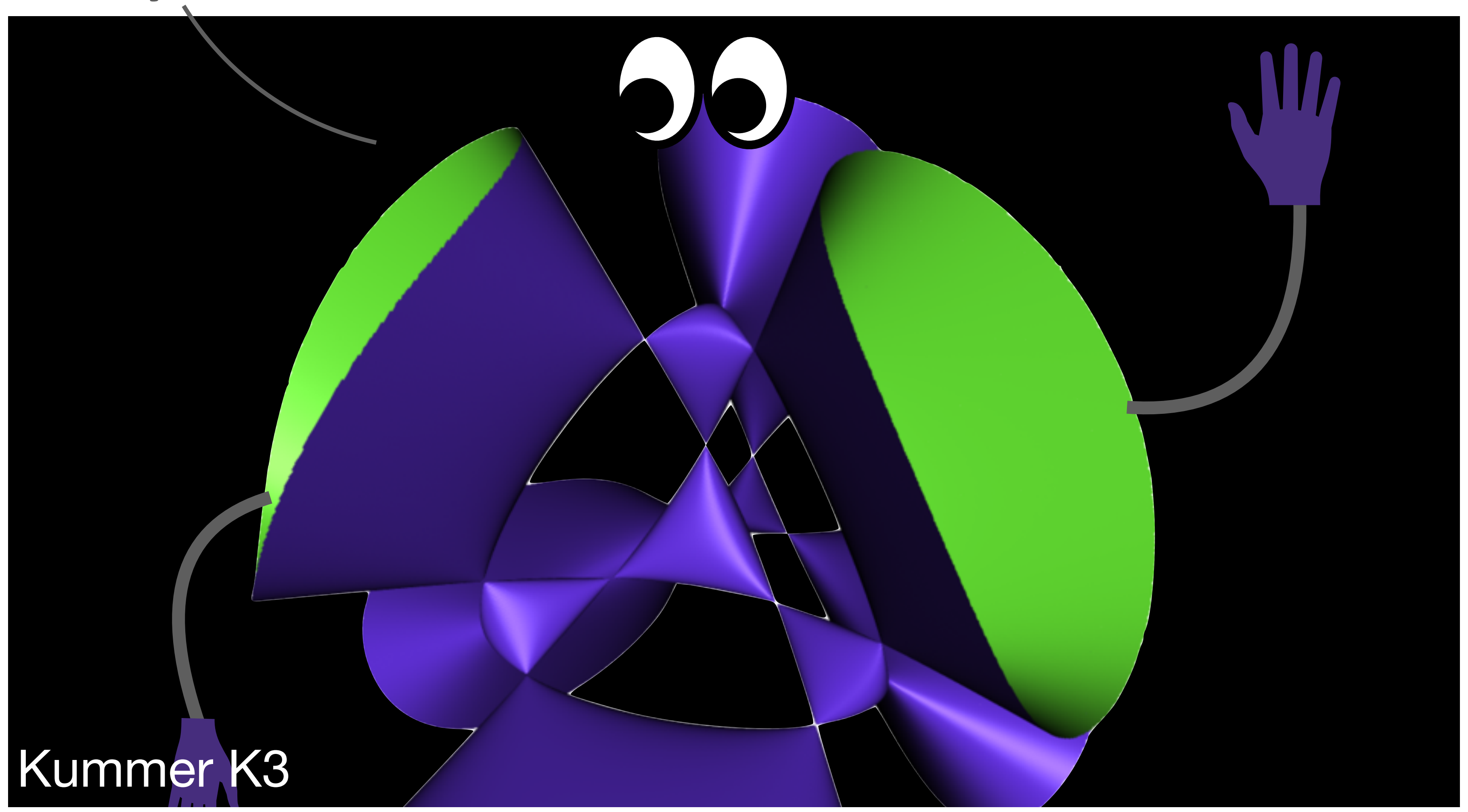
# Two K3 surfaces meet on zoom

Where should I start?

Tell me a little bit about yourself.

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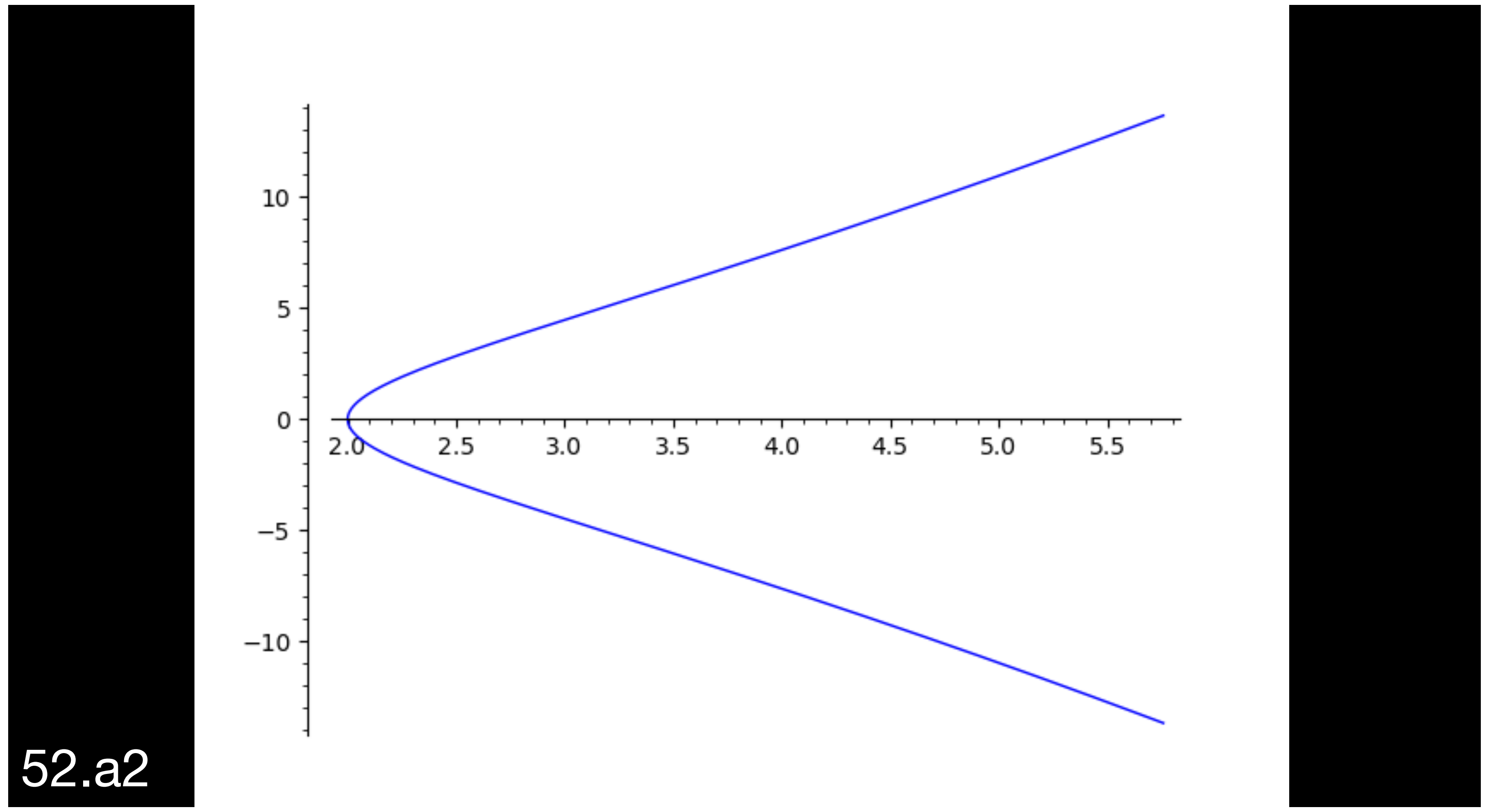
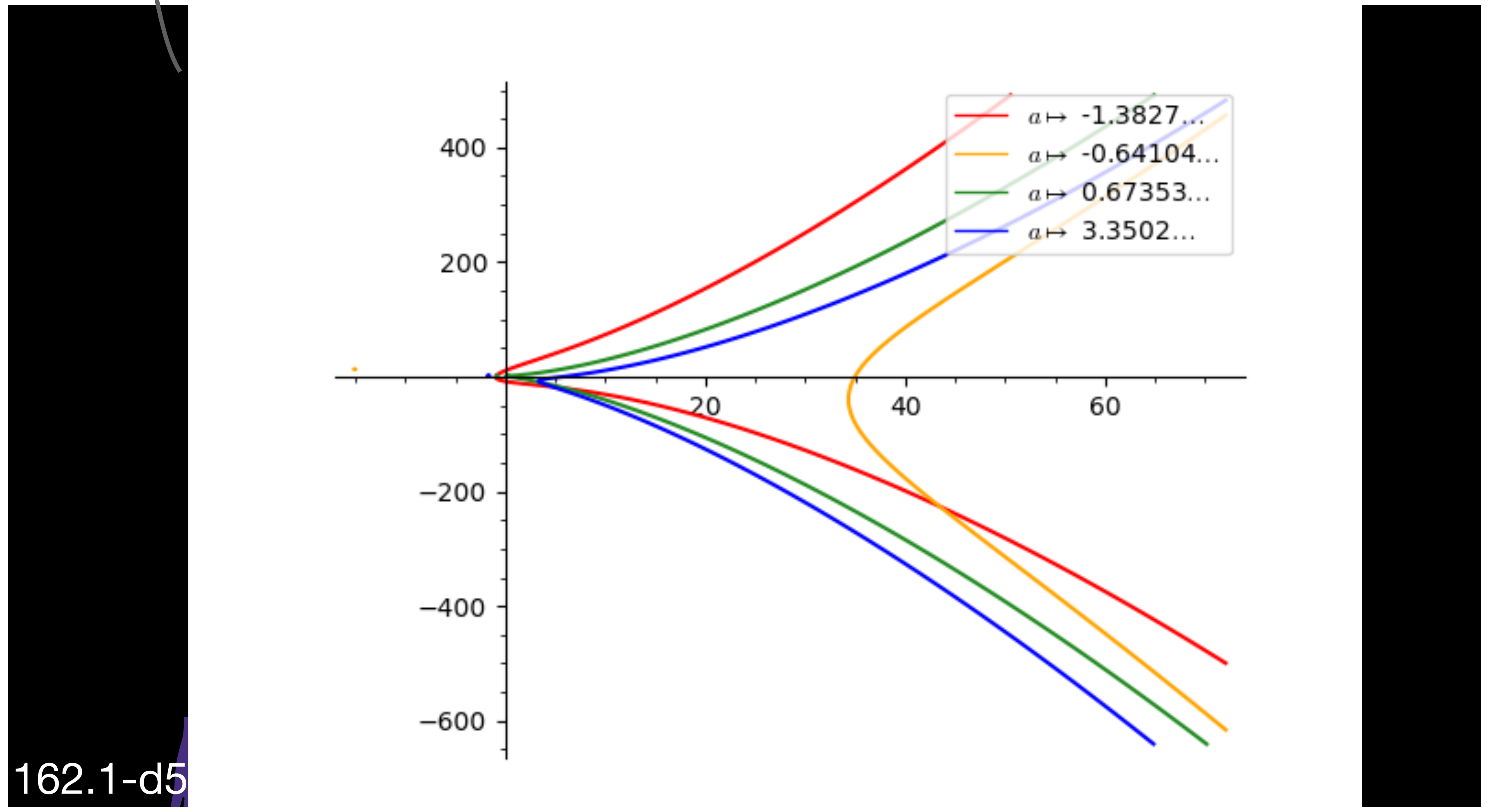
# Two elliptic curves meet on zoom

Tell me a little bit about yourself.



# Elliptic Curve Facebook™

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⌂ → Elliptic curves → $\mathbb{Q}$ → 52 → a → 2

## Elliptic curve with LMFDB label 52.a2 (Cremona label 52a1)

**Introduction**
Overview Random
Universe Knowledge

**L-functions**
Rational All

**Modular forms**
Classical Maass
Hilbert Bianchi

**Varieties**
Elliptic curves over $\mathbb{Q}$
Elliptic curves over $\mathbb{Q}(\alpha)$
Genus 2 curves over $\mathbb{Q}$
Higher genus families
Abelian varieties over $\mathbb{F}_q$

**Fields**
Number fields
$p$-adic fields

**Representations**
Dirichlet characters
Artin representations

**Groups**
Galois groups
Sato-Tate groups

Show commands for: Magma / Pari/GP / SageMath

### Minimal Weierstrass equation

$y^2 = x^3 + x - 10$

### Mordell-Weil group structure

$\mathbb{Z}/2\mathbb{Z}$

### Torsion generators

$(2, 0)$

### Integral points

$(2, 0)$

### Invariants

| | | | | |
|---|---|---|---|---|
| Conductor: | $52$ | = | $2^2 \cdot 13$ | |
| Discriminant: | $-43264$ | = | $-1 \cdot 2^8 \cdot 13^2$ | |
| j-invariant: | $\frac{432}{169}$ | = | $2^4 \cdot 3^3 \cdot 13^{-2}$ | |
| Endomorphism ring: | $\mathbb{Z}$ | | | |
| Geometric endomorphism ring: | $\mathbb{Z}$ | (no potential complex multiplication) | | |
| Sato-Tate group: | $\mathrm{SU}(2)$ | | | |

### BSD invariants

| | |
|---|---|
| Analytic rank: | $0$ |
| Regulator: | $1$ |
| Real period: | $1.6909664172912925709611680072$ |
| Tamagawa product: | $2 = 1 \cdot 2$ |
| Torsion order: | $2$ |
| Analytic order of Ш: | 1 (exact) |

### Modular invariants

Modular form 52.2.a.a

$q + 2q^5 - 2q^7 - 3q^9 - 2q^{11} - q^{13} + 6q^{17} - 6q^{19} + O(q^{20})$

⌂ → Elliptic curves → 4.4.10273.1 → 162.1 → d → 5

## Elliptic curve 162.1-d5 over number field 4.4.10273.1

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### Base field 4.4.10273.1

Generator $a$, with minimal polynomial $x^4 - 2x^3 - 5x^2 + x + 2$; class number $1$.

### Weierstrass equation

$y^2 + (a^3 - 3a^2 - a + 3)\,xy + (2a^3 - 5a^2 - 6a + 4)\,y = x^3 + (-2a^3 + 5a^2 + 6a - 4)\,x^2 + (-223a^3 + 597a^2 + 699a - 671)\,x - 2029a^3 + 5367a^2 + 6672a - 6309$

This is a global minimal model.

### Invariants

| | | | |
|---|---|---|---|
| Conductor: | $(3a)$ | = | $(a) \cdot (-a^2 + 1) \cdot (-a^3 + 4a^2 - 7)$ |
| Conductor norm: | $162$ | = | $2 \cdot 3 \cdot 27$ |
| Discriminant: | $(-351a^3 + 756a^2 + 1134a + 1998)$ | = | $(a) \cdot (-a^2 + 1)^3 \cdot (-a^3 + 4a^2 - 7)^8$ |
| Discriminant norm: | $-15251194969974$ | = | $2 \cdot 3^3 \cdot 27^8$ |
| j-invariant: | $\frac{19109112025}{13122}a^3 - \frac{26566531225}{6561}a^2 - \frac{57838815025}{13122}a + \frac{72137636141}{13122}$ | | |
| Endomorphism ring: | $\mathbb{Z}$ | | |
| Geometric endomorphism ring: | $\mathbb{Z}$ | | (no potential complex multiplication) |
| Sato-Tate group: | $\mathrm{SU}(2)$ | | |

### Mordell-Weil group

| | |
|---|---|
| Rank: | $1$ |
| Generator | $\left(\frac{35}{4}a^3 - 23a^2 - \frac{119}{4}a + \frac{109}{4} : -\frac{119}{8}a^3 + 38a^2 + \frac{435}{8}a - \frac{385}{8} : 1\right)$ |
| Height | $1.31652931384029$ |
| Torsion structure: | $\mathbb{Z}/2\mathbb{Z}$ |
| Torsion generator: | $\left(-\frac{15}{4}a^3 + \frac{39}{4}a^2 + \frac{25}{2}a - \frac{49}{4} : \frac{7}{2}a^3 - \frac{75}{8}a^2 - \frac{43}{4}a + \frac{93}{8} : 1\right)$ |

### BSD invariants

| | |
|---|---|
| Analytic rank: | $1$ |
| Mordell-Weil rank: | $1$ |
| Regulator: | $1.31652931384029$ |
| Period: | $61.7430800534858$ |
| Tamagawa product: | $2 = 1 \cdot 1 \cdot 2$ |
| Torsion order: | $2$ |
| Leading coefficient: | $6.41593812456013$ |

$$E(k)_{\mathrm{tors}}$$

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- Fundamental attribute of an elliptic curve.

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Is there an analog for K3 surfaces?

# Let's get creative!

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$E$ elliptic curve

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$$E(k)_{\mathrm{tors}} \quad \rightsquigarrow^{\mathrm{H}_{\mathrm{et}}^{\dim Y}\left(Y, \mathbf{G}_m\right)_{\mathrm{tors}}}$$

$E$ elliptic curve

$X$ K3 surface

$$E(k)_{\mathrm{tors}} \quad \overset{\mathrm{H}^{\dim Y}_{\mathrm{et}}\left(Y, \mathbf{G}_m\right)_{\mathrm{tors}}}{\rightsquigarrow}$$

$E$ elliptic curve

$X$ K3 surface

$$E(k)_{\mathrm{tors}} \xrightarrow{\ \mathrm{H}^{\dim Y}_{\mathrm{et}}\left(Y, \mathbf{G}_m\right)_{\mathrm{tors}}\ } \operatorname{Br} X$$

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$$\operatorname{Br} X \subset \operatorname{Br} \mathbf{k}(X) = \frac{\left\{Y : Y_{\overline{\mathbf{k}(X)}} \simeq \mathbf{P}^n_{\overline{\mathbf{k}(X)}}\right\}}{\simeq}, \quad (X \text{ smooth variety})$$

Subgroup of everywhere unramified elements

$$\operatorname{Br} X \subset \operatorname{Br} \mathbf{k}(X) = \frac{\left\{ Y : Y_{\overline{\mathbf{k}(X)}} \simeq \mathbf{P}^n_{\overline{\mathbf{k}(X)}} \right\}}{\simeq}, \quad (X \text{ smooth variety})$$

$E$ elliptic curve

$X$ K3 surface

$$E(k)_{\text{tors}} \xrightarrow{\mathrm{H}^{\dim Y}_{\text{et}}\left(Y, \mathbf{G}_m\right)_{\text{tors}}} \operatorname{Br} X$$

# One caveat

$E$ elliptic curve

$$E(k)_{\text{tors}}$$

$$\overset{\mathrm{H}^{\dim Y}_{\text{et}}\left(Y, \mathbf{G}_m\right)_{\text{tors}}}{\rightsquigarrow}$$

$X$ K3 surface

$$\operatorname{Br} X$$

# One caveat

$E$ elliptic curve

$X$ K3 surface

$$E(k)_{\mathrm{tors}} = E(\bar{k})^{G_k}_{\mathrm{tors}}$$

$$\overset{\mathrm{H}^{\dim Y}_{\mathrm{et}}(Y, \mathbf{G}_m)_{\mathrm{tors}}}{\rightsquigarrow}$$

$$\mathrm{Br}\, X$$

# One caveat

$E$ elliptic curve

$X$ K3 surface

$$E(k)_{\text{tors}} \xrightarrow{\;\mathrm{H}^{\dim Y}_{\text{et}}(Y, \mathbf{G}_m)_{\text{tors}}\;} \rightsquigarrow \quad \operatorname{Br} X$$

$$\|$$

$$E(\bar{k})^{G_k}_{\text{tors}} \rightsquigarrow$$

# One caveat

$E$ elliptic curve

$X$ K3 surface

$$E(k)_{\mathrm{tors}} \xrightarrow{\ \mathrm{H}^{\dim Y}_{\mathrm{et}}(Y, \mathbf{G}_m)_{\mathrm{tors}}\ } \operatorname{Br} X$$

$$\|$$

$$E(\bar{k})^{G_k}_{\mathrm{tors}} \rightsquigarrow (\operatorname{Br} \overline{X})^{G_k}$$

# One caveat

$E$ elliptic curve $\qquad\qquad$ $X$ K3 surface

$$
\begin{array}{ccc}
E(k)_{\mathrm{tors}} & \xrightarrow[\quad]{\mathrm{H}^{\dim Y}_{\mathrm{et}}(Y,\mathbf{G}_m)_{\mathrm{tors}}} \!\!\!\rightsquigarrow & \operatorname{Br} X \\
\| & & \nparallel \\
E(\bar{k})^{G_k}_{\mathrm{tors}} & \rightsquigarrow & (\operatorname{Br} \bar{X})^{G_k}
\end{array}
$$

# Which one is important?

$E$ elliptic curve

$X$ K3 surface

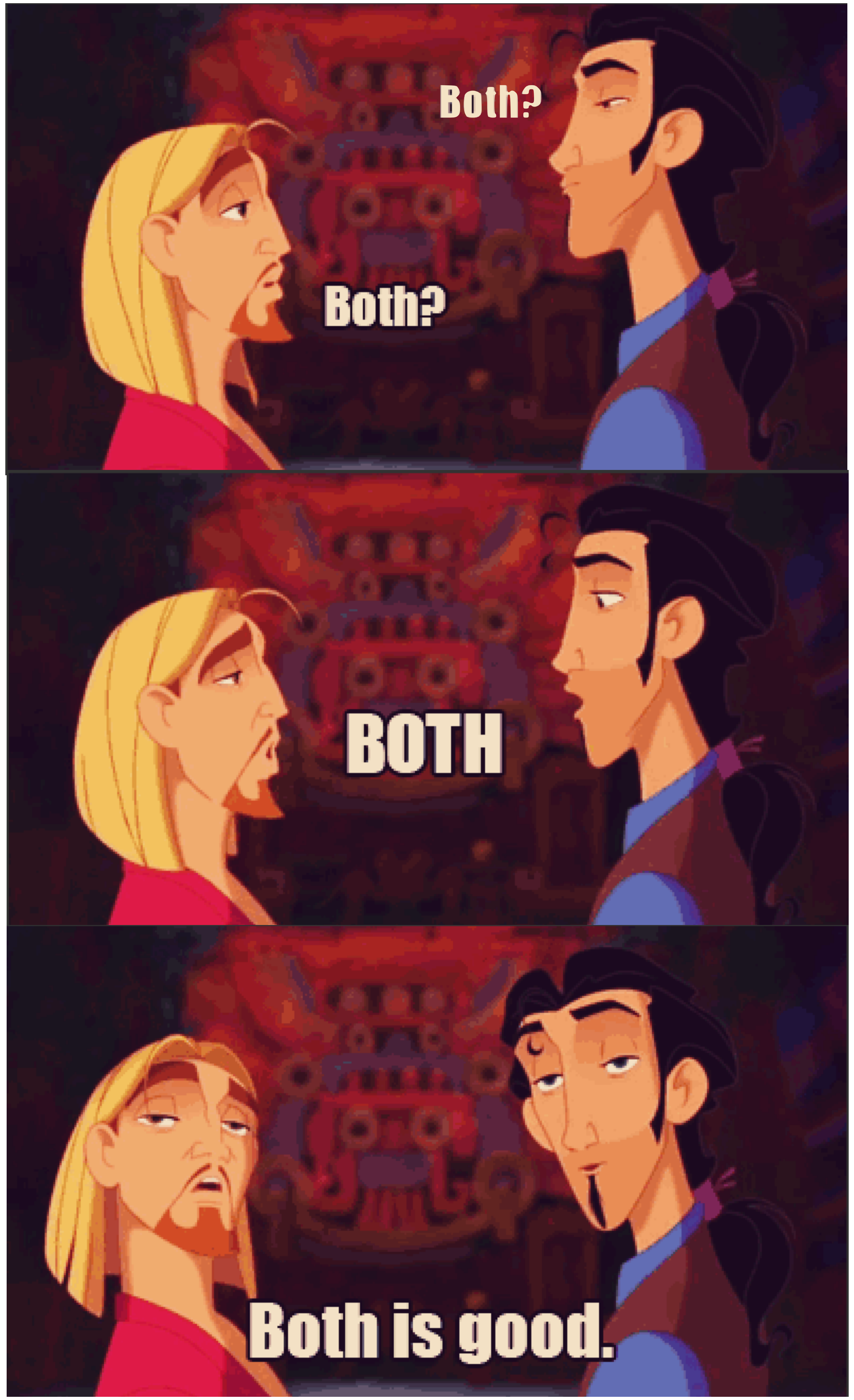
$$E(k)_{\mathrm{tors}} \quad \overset{\mathrm{H}^{\dim Y}_{\mathrm{et}}(Y, \mathbf{G}_m)_{\mathrm{tors}}}{\rightsquigarrow} \quad \mathrm{Br}\, X$$

$$\| \qquad\qquad\qquad \nparallel$$

$$E(\bar{k})^{G_k}_{\mathrm{tors}} \quad \rightsquigarrow \quad (\mathrm{Br}\, \bar{X})^{G_k}$$

$E$ elliptic curve

$$E(k)_{\mathrm{tors}} = E(\bar{k})^{G_k}_{\mathrm{tors}}$$




$X$ K3 surface

$$\mathrm{Br}\,X \neq (\mathrm{Br}\,\bar{X})^{G_k}$$

# How reasonable is this analogy?

$$E(k)_{\text{tors}} = E(\bar{k})^{G_k}_{\text{tors}} \rightsquigarrow \operatorname{Br} X$$

$$E(k)_{\text{tors}} = E(\bar{k})^{G_k}_{\text{tors}} \rightsquigarrow (\operatorname{Br} \overline{X})^{G_k}$$

# $E(k)_{\text{tors}}$

- Fundamental attribute of an elliptic curve.
- Rigidifies the moduli problem.
- Helpful or even necessary for computing other properties/attributes.

# What can we say structurally?

$$E(k)_{\text{tors}} = E(\overline{k})^{G_k}_{\text{tors}} \rightsquigarrow \operatorname{Br} X$$

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# Over $k = \overline{k}$?

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$$(\mathbf{Q}/\mathbf{Z})^2$$

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$\mathrm{Br}\, X$, $X$ K3 surface

# Over $k = \bar{k}$?

$$(\mathbf{Q}/\mathbf{Z})^2$$

$E(k)_{\text{tors}}$, $E$ elliptic curve

$$(\mathbf{Q}/\mathbf{Z})^{22-\rho}$$

$$1 \leq \rho \leq 22$$

$\operatorname{Br} X$, $X$ K3 surface

# Over finite fields?

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Finite!

$E(k)_{\text{tors}}$, $E$ elliptic curve

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$E(k)_{\mathrm{tors}}$, $E$ elliptic curve

Br $X$,  $X$ K3 surface

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Finite!

$E(k)_{\text{tors}}$, $E$ elliptic curve

$\operatorname{Br} X$, $X$ K3 surface

# Over number fields?

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Finite!

$E(k)_{\text{tors}}$, $E$ elliptic curve

# Over number fields?

## Finite!

$E(k)_{\text{tors}}$, $E$ elliptic curve

$\operatorname{Br} X$, $X$ K3 surface

# Over number fields?

Finite!

Finite!

$E(k)_{\mathrm{tors}}$, $E$ elliptic curve

$\mathrm{Br}\, X$, $X$ K3 surface

# Over number fields?

Finite!

Finite!*

$E(k)_{\mathrm{tors}}$, $E$ elliptic curve

$\mathrm{Br}\, X$, $X$ K3 surface

$$\operatorname{Br} X \text{ vs. } (\operatorname{Br} \overline{X})^{G_k}$$

# Over number fields?

Finite!

$E(k)_{\text{tors}}$, $E$ elliptic curve

Finite!

[Skorobogatov, Zarhin 2008]

$\frac{\operatorname{Br} X}{\operatorname{Br}_0 X}$, $X$ K3 surface

**Theorem** (Merel, after Mazur, Kamienny)

Let $d \in \mathbf{N}$. Then there exists a $C \in \mathbf{N}$ such that for all $k/\mathbf{Q}$ with $[k : \mathbf{Q}] \leq d$, and all elliptic curves $E/k$,

$$\#E(k)_{\text{tors}} \leq C.$$

Let $X/k$ be a K3 surface over a number field.

Is $\#\left(\operatorname{Br} X/\operatorname{Br}_0 X\right)$ uniformly bounded?

## Conjecture (Várilly-Alvarado 2015)

Let $d \in \mathbf{N}$ and let $\Lambda \subset U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ be primitive.

Then there exists a $C \in \mathbf{N}$ such that for all $k/\mathbf{Q}$ with

$[k : \mathbf{Q}] \leq d$, and all K3 surfaces $X/k$ with $\Lambda \simeq \operatorname{NS} \overline{X}$,

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# Moduli of K3 surfaces

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- Infinite countable union of 19-dimensional varieties.

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# Moduli of K3 surfaces

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More complicated than the $j$-line!

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$[k : \mathbf{Q}] \leq d$, and all K3 surfaces $X/k$ with $\Lambda \simeq \operatorname{NS} \overline{X}$,

$$\#\left(\operatorname{Br} X / \operatorname{Br}_0 X\right) \leq C.$$

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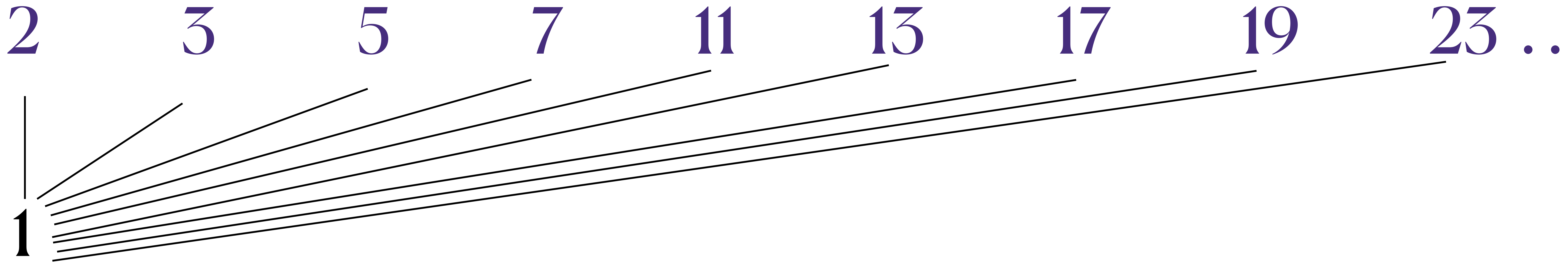
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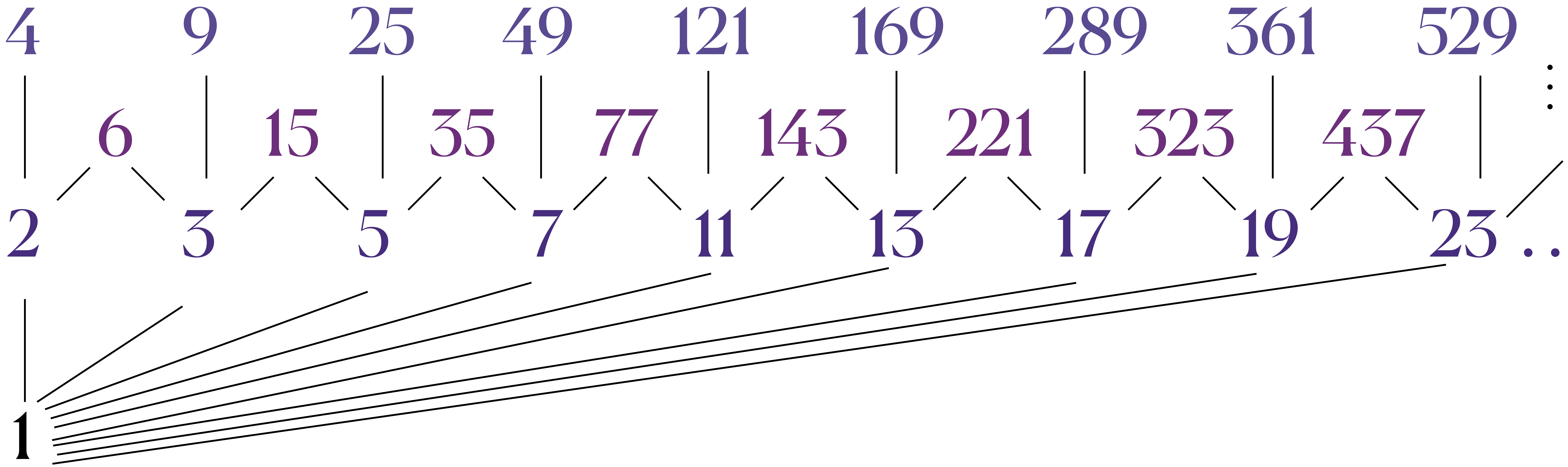
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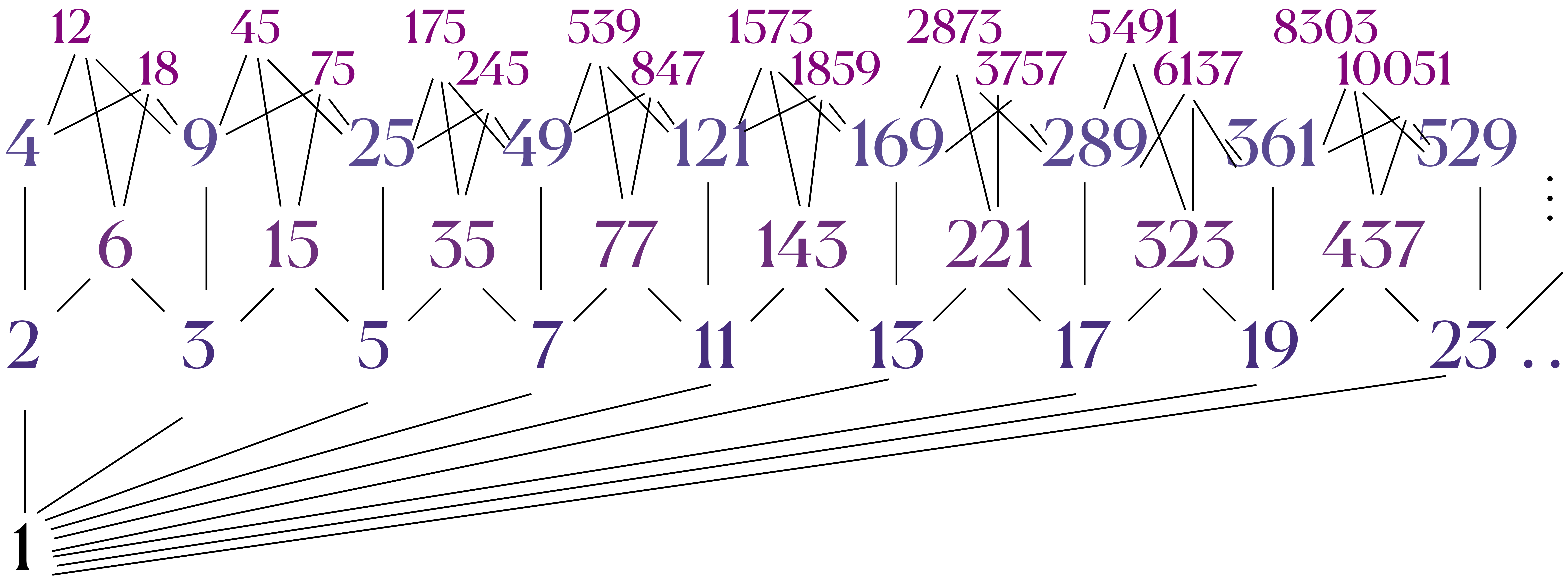

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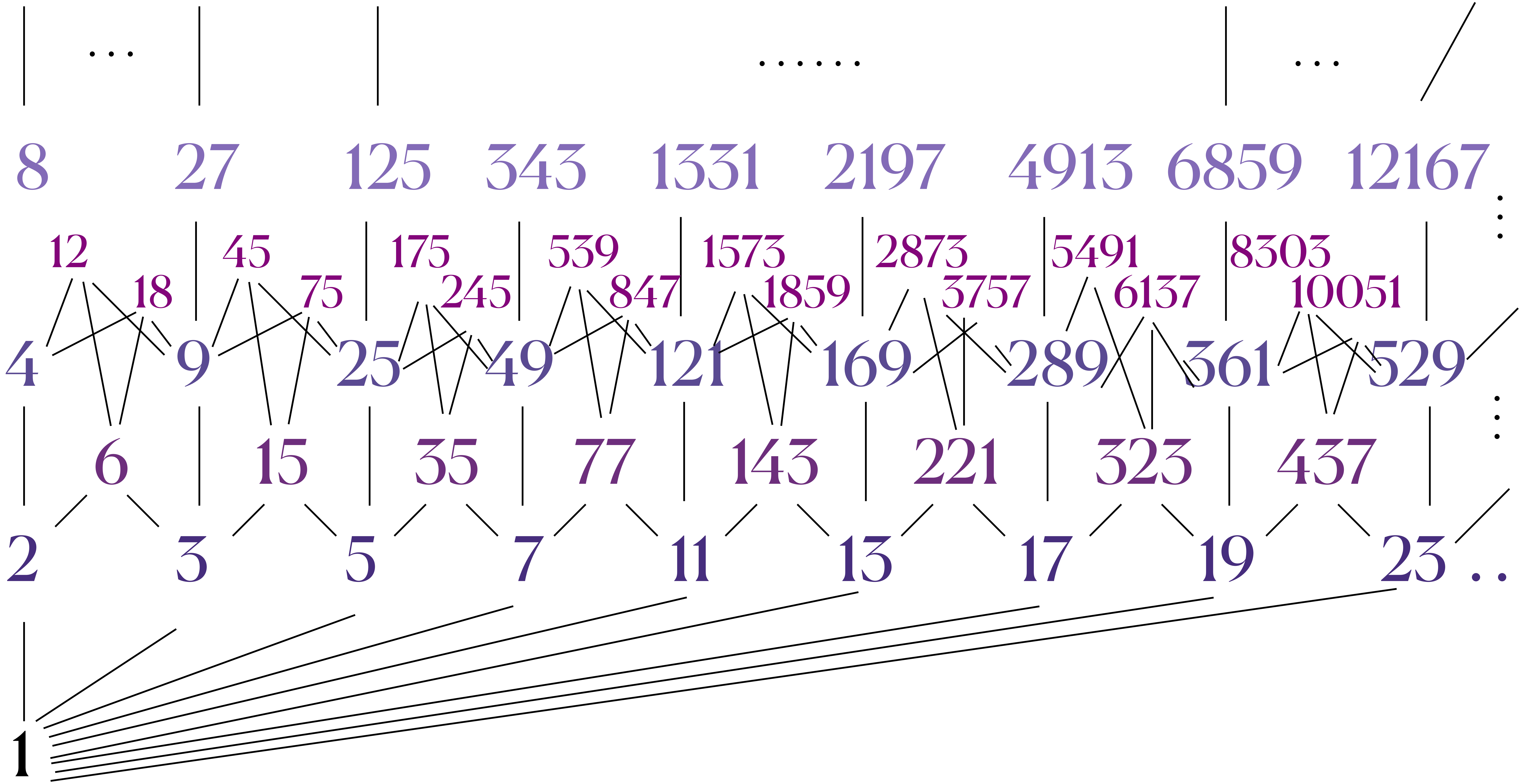
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12 45 175 539 1573 2873 5491 8303
18 75 245 847 1859 3757 6137 10051
4 9 25 49 121 169 289 361 529
6 15 35 77 143 221 323 437
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$\#\left(\operatorname{Br} X / \operatorname{Br}_0 X\right)[\ell^\infty]$ is uniformly bounded when...

# $\#\left(\operatorname{Br} X/\operatorname{Br}_0 X\right)[\ell^\infty]$ is uniformly bounded when...

- $\overline{X} \simeq \operatorname{Kum}(E \times E/C)$, $E$ non-CM (depending on $[k : \mathbf{Q}]$ and $\#C$) [Várilly-Alvarado, Viray 2017]

## $\#\left(\mathrm{Br}\,X/\mathrm{Br}_0\,X\right)[\ell^\infty]$ is uniformly bounded when...

- $\overline{X} \simeq \mathrm{Kum}\,(E \times E/C)$, $E$ non-CM (depending on $[k:\mathbf{Q}]$ and $\#C$)
  [Várilly-Alvarado, Viray 2017]
- $X$ varies along a curve (depending on $[k:\mathbf{Q}]$ and $\mathscr{X}$)
  [Cadoret, Charles 2020]

## $\#\left(\mathrm{Br}\, X/\mathrm{Br}_0\, X\right)[\ell^\infty]$ is uniformly bounded when…

- $\overline{X} \simeq \mathrm{Kum}\,(E \times E/C)$, $E$ non-CM (depending on $[k : \mathbf{Q}]$ and $\#C$)
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- $X$ is a K3 surface of CM type (depending on $[k : \mathbf{Q}]$)
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Is the Brauer group uniformly bounded in geometric 2-dimensional families of K3 surfaces?

# Further arithmetic applications

Let $X$ be a K3 surface over a number field $k$.

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## $\#\left(\mathrm{Br}\, X/\mathrm{Br}_0\, X\right)[\ell^\infty]$ is uniformly bounded when...

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# $\#\left(\operatorname{Br} X/\operatorname{Br}_0 X\right)$ is **effectively** uniformly bounded when...

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- $\overline{X} \simeq \operatorname{Kum}(E \times E/C)$, $E$ CM (depending on $[k : \mathbf{Q}]$, $\#C$, $\operatorname{End}(E)$ ) [Balestrieri, Johnson, Newton (preprint)]

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<u>Theorem</u> (Ieronymou, Skorobogatov 2015):

Let $X/k$ be a diagonal quartic. Then $X(\mathbf{A}_k)^{\mathrm{Br}_{\mathrm{odd}}} \neq \varnothing$.

Is there a $d \in \mathbf{N}$ such that

$$X(\mathbf{A}_k)^{\mathrm{Br}} = \varnothing \Leftrightarrow X(\mathbf{A}_k)^{\mathrm{Br}[d^\infty]} = \varnothing\,?$$

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# For K3's in general?

Is there a $d \in \mathbf{N}$ such that

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Theorem [Berg, Várilly-Alvarado 2020]

$\exists\, X/\mathbf{Q}$ with $\operatorname{rk} \mathrm{NS}\, \overline{X} = 1, X(\mathbf{A}_\mathbf{Q}) \neq \emptyset$, and $X(\mathbf{A}_k)^{\mathrm{Br}[3]} = \emptyset$.

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Theorem [Gvirtz, Loughran, Nakahara (preprint)]

$\exists\, X/\mathbf{Q}$ with $X(\mathbf{A}_\mathbf{Q})^{\mathrm{Br}} = \emptyset$ and $X(\mathbf{A}_k)^{\mathrm{Br}[2^\infty]} \neq \emptyset$.

Fix $n \in \mathbf{N}$. Is there a K3 surface $X$ such that order $n$ Brauer classes are necessary to detect a Brauer-Manin obstruction?

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