Exploring angle rank using the LMFDB

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February 15, 2022

Overview of the talk

This talk has two aims:

1. Give an intro to the LMFDB
2. Give some angle rank results

References

The main references for the material I will present today are

- Database of isogeny classes of abelian varieties `https://www.lmfdb.org/Variety/Abelian/Fq/`
- Isogeny Classes of Abelian Varieties over Finite Fields in the LMFDB by Dupuy, Kedlaya, Roe, V.
- Counterexamples to a Conjecture of Ahmadi and Shparlinski by Dupuy, Kedlaya, Roe, V.
- Angle ranks of abelian varieties by Dupuy, Kedlaya, Zureick-Brown

Quick background

Recall that an **abelian variety** is a projective variety that is also an algebraic group.

An **isogeny** is a morphism between two abelian varieties of the same dimension, sending the identity to the identity, and which has finite kernel.

The relation of isogeny is an equivalence relation, so we can study isogeny classes of abelian varieties.

Isogeny classes over finite fields

Over $\mathbb{F}_q$, to an isogeny class we can associate its **Weil polynomial**.

This polynomial is the characteristic polynomial of the action of Frobenius on the first cohomology group.

Honda-Tate theory

Fix g the dimension of the isogeny class, and $\mathbb{F}_q$ the base field.

Then the Weil polynomial is of the form

$$P(T) = T^g Q(T + \frac{q}{T})$$

for a

- monic polynomial $Q(T) \in \mathbb{Z}[T]$ of degree g,
- with all real roots in $[-2\sqrt{q}, 2\sqrt{q}]$.

Honda-Tate theory

Even better, (basically) every polynomial like this is a Weil poly!

More precisely, if isog class is **simple**, then Weil poly is of the form

$$P(T) = h(T)^e,$$

for e prescribed by h and h irreducible.

Every irreducible poly $T^g Q(T + \frac{q}{T})$ with all real roots in $[-2\sqrt{q}, 2\sqrt{q}]$ is the h of some simple isogeny class.

Upshot

Isogeny classes are parametrized by polynomials!

But also

- These polys have special properties that allow us to list them; and
- basically every polynomial with those special properties corresponds to an isogeny class.

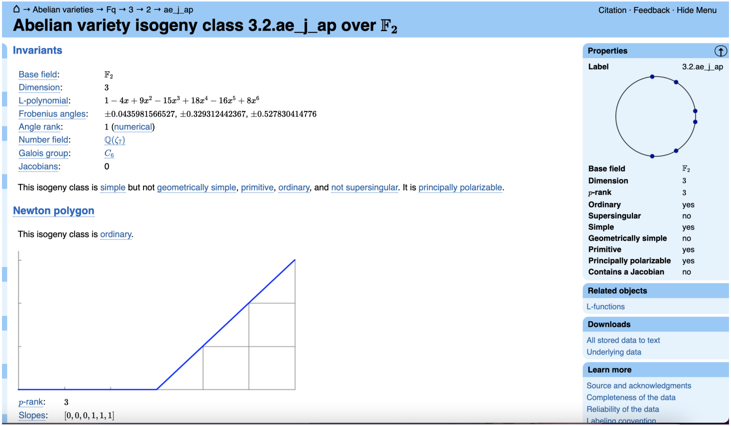
Let's goooo

In 2016, we went ahead and started listing them all.

We also started computing the properties of each isogeny class, to assemble into a database for research.

The LMFDB

"LMFDB" stands for "L-functions and modular forms database."



⌂ → Abelian varieties → Fq → 3 → 2 → ae_j_ap

Citation · Feedback · Hide Menu

Abelian variety isogeny class 3.2.ae_j_ap over $\mathbb{F}_2$

Invariants

Base field:	$\mathbb{F}_2$
Dimension:	3
L-polynomial:	$1 - 4x + 9x^2 - 15x^3 + 18x^4 - 16x^5 + 8x^6$
Frobenius angles:	$\pm0.0435981566527, \pm0.329312442367, \pm0.527830414776$
Angle rank:	1 (numerical)
Number field:	$\mathbb{Q}(\zeta_7)$
Galois group:	C_6
Jacobians:	0

This isogeny class is simple but not geometrically simple, primitive, ordinary, and not supersingular. It is principally polarizable.

Newton polygon

This isogeny class is ordinary.

p-rank:	3
Slopes:	$[0, 0, 0, 1, 1, 1]$

Properties

Label	3.2.ae_j_ap
Base field	$\mathbb{F}_2$
Dimension	3
p-rank	3
Ordinary	yes
Supersingular	no
Simple	yes
Geometrically simple	no
Primitive	yes
Principally polarizable	yes
Contains a Jacobian	no

Related objects

L-functions

Downloads

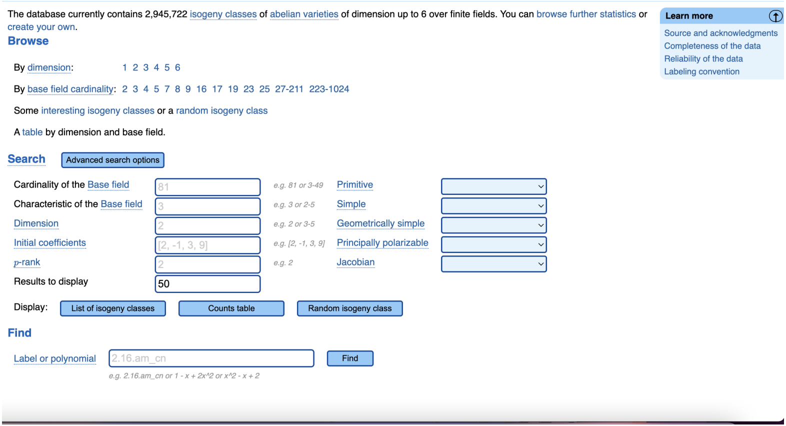
All stored data to text
Underlying data

Learn more

Source and acknowledgments
Completeness of the data
Reliability of the data
Labeling convention

The LMFDB

This is what is at `lmfdb.org/Variety/Abelian/Fq/`.


The database currently contains 2,945,722 isogeny classes of abelian varieties of dimension up to 6 over finite fields. You can browse further statistics or create your own.

Learn more

Source and acknowledgments
Completeness of the data
Reliability of the data
Labeling convention

Browse

By dimension: 1 2 3 4 5 6

By base field cardinality: 2 3 4 5 7 8 9 16 17 19 23 25 27-211 223-1024

Some interesting isogeny classes or a random isogeny class

A table by dimension and base field.

Search Advanced search options

Cardinality of the Base field	81	e.g. 81 or 3-49	Primitive	
Characteristic of the Base field	3	e.g. 3 or 2-5	Simple	
Dimension	2	e.g. 2 or 3-5	Geometrically simple	
Initial coefficients	[2, -1, 3, 9]	e.g. [2, -1, 3, 9]	Principally polarizable	
p-rank	2	e.g. 2	Jacobian	
Results to display	50			

Display: List of isogeny classes | Counts table | Random isogeny class

Find

Label or polynomial 2.16.am_cn Find

e.g. 2.16.am_cn or 1 - x + 2x^2 or x^2 - x + 2

The reason for all this

One of the reasons to build this database was to study **angle ranks** of abelian varieties over finite fields.

Frobenius angles

Recall: the Weil polynomial is the char poly of Frob acting on H^1.

As a result, the absolute value of each of the roots is $\sqrt{q}$.

Abelian variety isogeny class 3.2.ae_j_ap over $\mathbb{F}_2$

Invariants

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Properties

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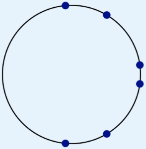


Base field	$\mathbb{F}_2$
Dimension	3
p-rank	3
Ordinary	yes

Angle rank

Roughly, angle rank is the dim of the $\mathbb{Q}$-vs spanned by Frob angles.

More precisely, let

- $P(T)$ be the Weil polynomial of an isogeny class and
- α_i be its roots,

then the **angle rank** of this isogeny class is

$$\delta = \dim_{\mathbb{Q}}\left(\mathrm{Span}_{\mathbb{Q}}\left\{\frac{\mathrm{Arg}(\alpha_i)}{\pi}\right\}\cup\{1\}\right)-1.$$

Quick angle rank facts

$$\delta = \mathsf{dim}_{\mathbb{Q}}\left(\mathrm{Span}_{\mathbb{Q}}\left\{\frac{\mathrm{Arg}(\alpha_i)}{\pi}\right\}\cup\{1\}\right) - 1.$$

- Angle rank is between 0 and g, inclusively.
- Angle rank is 0 if and only if supersingular.

From now on, assume that the isogeny class is geometrically simple, so that angle rank 0 only happens when $g = 1$.

Quick angle rank history

- First defined in 1979 by Zarhin, to study the Lie algebra of the image of the ℓ-adic Galois representation attached to an abelian variety over a number field.
- Showed that for each ℓ, the rank of some part of the Lie algebra is equal to maximal angle rank of the reductions of the abelian variety.

Tate conjecture

- If angle rank is g, then all Tate classes are gen in codim 1. (Zarhin 1994)

- Tate proved that the Tate conjecture holds in codimension 1.

Conversely, if angle rank is *not* maximal, then the varieties have an **exotic** Tate class; a class that is not generated in codimension 1. (Zarhin 1993, Lenstra-Zarhin 1993)

What you can do with data

Disprove a conjecture is one thing.

Conjecture (Ahmadi–Shparlinski, 2010)

Every ordinary geometrically simple Jacobian over a finite field has maximal angle rank.

- When $g = 2$, every geom simple Jacobian has max angle rank (AS10)
- When $g = 3$, the conjecture is true (Zarhin 2015)
- When $g = 4$, the conjecture is false

But also

You can make new conjectures (and prove them if you're lucky).

One natural line of inquiry is to clarify the relationship of angle rank to other isogeny class invariants.

Shoutout

In 1992, Chi showed results similar to those in DKZB, but in a different, more complicated language.

The Galois group

If the isogeny class is simple, then the Weil polynomial has the form

$$P(T) = h(T)^e,$$

with e determined by h and h irreducible.

The Galois group of the isogeny class is $G = \mathrm{Gal}(\mathbb{Q}(\{\alpha_i\})/\mathbb{Q})$.

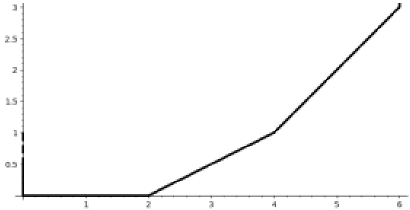
The Newton polygon

Let v be a place of $\mathbb{Q}(\{\alpha_i\})$ above p, normalized so $v(q) = 1$.

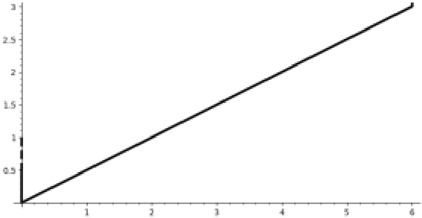
Then the Newton polygon of the isogeny class is the

- continuous piecewise linear function with nondecreasing derivative
- containing a segment of length 1 and slope $v(\alpha_i)$ for each $i = 1, \ldots, 2g$.

Example of Newton polygons



Newton Polygon with Slopes $[0, 0, 1/2, 1/2, 1, 1]$



Newton Polygon with Slopes $[1/2, 1/2, 1/2, 1/2, 1/2, 1/2]$

Important insight

If no relations among the angles, the angle rank is maximal.

A relation among angles corresponds to a relation

$$\alpha_1^{e_1}\alpha_2^{e_2}\cdots\alpha_{2g}^{e_{2g}} = q^d$$

for some integer d.

Therefore the Galois group acts on the relations.

Idea

Recall $G = \mathrm{Gal}(\mathbb{Q}(\{\alpha_i\})/\mathbb{Q})$.

Find a simpler $\mathbb{Q}[G]$-module which carries info of G acting on

$$\Gamma = \langle \alpha_1, \alpha_2, \ldots, \alpha_{2g} \rangle \subset \overline{\mathbb{Q}}^\times.$$

(Note that the angle rank is $\operatorname{rk} \Gamma - 1$.)

Newton hyperplane representation

For each j, let $\beta_j = \dfrac{\alpha_j}{\overline{\alpha}_j}$.

Let V be

$$\mathrm{Span}_{\mathbb{Q}} \left\{ (v(\beta_1), v(\beta_2), \ldots, v(\beta_g)) : v | p \right\}.$$

The right dimension

Theorem (Dupuy, Kedlaya, Zureick-Brown)

The dimension of V is the angle rank.

Proof.

A multiplicative relation among the α_is gives an equation

$$\beta_1^{f_1} \cdots \beta_g^{f_g} = \zeta$$

for ζ a root of unity.
Such an equation holds iff

$$f_1 v(\beta_1) + f_2 v(\beta_2) + \cdots + f_g v(\beta_g) = 0 \quad \text{for all} \quad v|p.$$

$\square$

The G-action

Fix a labeling of the roots of the Weil poly as

$$\alpha_1, \alpha_2, \ldots, \alpha_g, \overline{\alpha}_1, \ldots, \overline{\alpha}_g.$$

Then to describe $\sigma \in G$, we can keep track of

- the fact that the pair $\{\alpha_j, \overline{\alpha}_j\}$ was sent to $\{\alpha_k, \overline{\alpha}_k\}$; and
- a sign keeping track of if α_j went to α_k $(+1)$ or $\overline{\alpha}_k$ (-1).

The G-action

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- a sign keeping track of if α_j went to α_k $(+1)$ or $\overline{\alpha}_k$ (-1).

So $\sigma \in G$ can be represented by a signed $g \times g$ perm matrix, which then acts on the vectors $(v(\beta_1), v(\beta_2), \ldots, v(\beta_g)) \in V$.

More about G

This description gives a extension

$$G \cong C \rtimes \overline{G},$$

where

$$C \leq C_2^g,$$

and

$$\overline{G} \leq S_g.$$

First result

Theorem (Lenstra-Zarhin (1993), DKZB)

Suppose that the Newton slopes of the isogeny class are $1/2$ with multiplicity 2, and all other slopes in $\mathbb{Z}_{(2)}$. Then

- *if g is even, then the isogeny class has maximal angle rank;*
- *if g is odd, then the isogeny class has angle rank $g - 1$ or g.*

Second result

Theorem (DKZB, Generalized Tankeev)

Let m be a certain integer associated to the subgroup $C \leq G$. If $\frac{g}{m}$ is prime, then the isogeny class has angle rank in

$$\{m, g-m, g\}.$$

Third result

If we write $G \cong C \rtimes \overline{G}$, then

Theorem (DKZB)

Suppose that $\overline{G}$ acts primitively on $\{1, \ldots, g\}$ and C is nontrivial, then the isogeny class has maximal angle rank.

Fourth result

Theorem (DKZB, Effective Zarhin)

The vectors $(e_1, e_2, \ldots, e_{2g}) \in \mathbb{Z}^{2g}$ *such that*

$$\alpha_1^{e_1}\alpha_2^{e_2}\cdots\alpha_{2g}^{e_{2g}} = q^d$$

for some d *are generated by vectors of weight less than or equal to*

$$\#G(\#G - \delta)^3(g\delta)^\delta.$$

Thank you!