Inverse Galois Problem: Does every finite group appear as some Galois group of some number field K? (over $\mathbb{Q}$)

$$K \cong \mathbb{Q}[x]/f(x)$$
$$\cong \mathbb{Q}[x]/g(x)$$

$\mathrm{Gal}(K/\mathbb{Q})$ = galois group (finite)
"group of symmetries" acting on the roots of $f(x)$

Q: How many number fields w/ Galois group G are there?

Hope: infinitely many (if there is one)

Let $d \geq 2$ be an integer (degree of numberfield)

Let $G \hookrightarrow S_d$ be a transitive subgroup

$$\mathcal{F}_{d,G} = \{ K \text{ - number field w/ deg } d \mid \mathrm{Gal}(\tilde{K}/\mathbb{Q}) = G \}$$

$$N_{d,G}(X) = \#\{ K \in \mathcal{F}_{d,G} \mid |\mathrm{Disc}(K)| \leq X \}$$

Precise Q: How does $N_{d,G}(X)$ grow as $X \to \infty$?

simplest case: $d = 2$ $G = S_2 = \mathbb{Z}/2\mathbb{Z} = C_2$

Quadratic field: $\mathbb{Q}(\sqrt{d})$ - d square free integer
$d \neq 0, 1$

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}$$

$$\mathcal{F}_{2,S_2} \overset{1-1}{\longleftrightarrow} \{\text{square free integers that are not } 0, 1\}$$

$N_{2,S_2}(X) \longleftrightarrow$ How many s.f. integers are there? upto X (as $X \to \infty$)

$$\lim_{X \to \infty} N_{2,S_2}(X) = \frac{6}{\pi^2} X + O(\sqrt{X})$$

Idea in general: $\lim_{X \to \infty} N_{d,G}(X) \sim c_{d,G} X^{a_{d,G}} (\log X)^{b_{d,G}}$

Q: Are there general formulas for $a_{d,G}, b_{d,G}$ and $c_{d,G}$?

$c_{2,S_2} = \frac{6}{\pi^2}$ $b_{2,S_2} = 0$ $a_{2,S_2} = 1$

(if $a_{d,G} = 1$, then $b_{d,G} = 0$) expectations of the asymptotics.

Cohn (1954): c_{3,C_3} $b_{3,C_3} = 0$ $a_{3,C_3} = \frac{1}{2}$

$$c_{3,C_3} = \frac{11\sqrt{3}}{36\pi} \prod_{p \equiv 1 \bmod 6} \frac{(p+2)(p-1)}{p(p+1)}$$

class field theory $\boxed{\text{CFT}}$

Davenport-Heilbronn (1971): $c_{3,S_3} = \frac{1}{3\zeta(3)}$

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geometry-of-numbers

$\boxed{\text{GON}}$

$b_{3,S_3} = 0$

$a_{3,S_3} = 1$

Wright, Maki : G abelian, they prove:

$$a_{d,G} = \frac{1}{d}\left(1 - \frac{1}{p}\right) \qquad b_{d,G} = \frac{n_p}{p-1} - 1$$

p = smallest prime dividing $|G|$

n_p = # of elements of G of order p.

Malle, Türkelli: G non-abelian, they predict:

$a_{d,G}$ and $b_{d,G}$.

What about $c_{d,G}$?

Bhargava: $G = S_d$, wrote really beautiful formulas.

$G \neq S_d$ ($d=4$, $G = D_4$

↳ constants c_{4,D_4} does not satisfy the analogous prediction).