Descent on K3 surfaces: Brauer group computations and challenges

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Kodaira dimension of a curve

$C/\mathbb{Q}$: a nice curve (sm; proj; geom int).

The Kodaira dimension of C is

$$\kappa(C) := \begin{cases} -1 & g = 0, \\ 0 & g = 1, \\ 1 & g \geq 2. \end{cases}$$

It indicates the Riemannian curvature of $C(\mathbb{C})$.

Kodaira dimension and Rational Points

$\kappa(C) = -1:$ rational curves, i.e., $C \simeq_{\overline{\mathbb{Q}}} \mathbb{P}^1$.

$C(\mathbb{Q}) \neq \emptyset \implies C \simeq_{\mathbb{Q}} \mathbb{P}^1$.

(stereographic projection)

$\kappa(C) = 0:$ $C(\mathbb{Q}) \neq \emptyset \implies C$ is an elliptic curve.

Euler: $C(\mathbb{Q})$ is a group (abelian).

Mordell (1922): $C(\mathbb{Q})$ is finitely generated.

$\kappa(C) = 1:$ Curves of general type.

Faltings (1983): $C(\mathbb{Q})$ is finite.

Kodaira dimension of a surface

For a nice surface $X/\mathbb{C}$:

$$\kappa(X) = \begin{cases} -1 & \text{rational or ruled;} \\ 0 & \text{Abelian, K3, Enriques or bi-elliptic;} \\ 1 & \text{properly elliptic;} \\ 2 & \text{general type.} \end{cases}$$

Intermediate type

K3 surfaces

$X/\mathbb{Q}$ nice surface with $\omega_X \simeq \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$.

Examples:

$w^2 = x^6 + y^6 + z^6$ in $\mathbb{P}(1,1,1,3)$ — degree 2

$x^4 + 2y^4 = z^4 + 4w^4$ in $\mathbb{P}^3$ — degree 4

$Q \cap C$ in $\mathbb{P}^4$ — degree 6

$Q_1 \cap Q_2 \cap Q_3$ in $\mathbb{P}^5$ — degree 8

Polarizations and coarse moduli

Polarized K3 surface of degree 2d: (X, L) with

$X/\mathbb{Q}$ a K3 surface

$L \in \mathrm{Pic}(X)$ primitive in $\mathrm{Pic}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})$ with $L^2 = 2d$.

$M_{2d} =$ (coarse) moduli space of polarized K3s of degree $2d$.

19-dimensional quasi-projective variety $/\mathbb{Q}$

Theorem (Gritsenko, Hulek, Sankaran ’07):
M_{2d} is of general type for $d > 61$.

Picard number 1

Recall that Pic$(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}) \simeq \mathbb{Z}^\rho$, and $1 \leq \rho \leq 20$.

For a 'very general' $(X, L) \in M_{2d}(\mathbb{C})$, we have $\rho = 1$.

Theorem (Ellenberg '04): Fix $2d$. There exists a number field K and a polarized K3 surface $(X, L) \in M_{2d}(K)$ with $\rho = 1$.

Question: Fix $2d$. Does there exist a polarized K3 surface $(X, L) \in M_{2d}(\mathbb{Q})$ with $\rho = 1$?

Question: What is the largest d for which we can show there exists $(X, L) \in M_{2d}(\mathbb{Q})$ with $\rho = 1$?

K3 surfaces: hopes and dreams

$X/\mathbb{Q}$: a K3 surface, given as a system of homogeneous polynomial equations.

Question: Is there an effective procedure to determine whether $X(\mathbb{Q}) \neq \varnothing$?

Recall: $$X(\mathbb{Q}) \subseteq X(\mathbb{A})^{\text{Br}} \subseteq X(\mathbb{A}) = \prod_{p \leq \infty} X(\mathbb{Q}_p)$$

Defined using $\text{Br}(X)/\underbrace{\text{Br}_0(X)}_{\text{im}\left(\text{Br}(\mathbb{Q}) \to \text{Br}(X)\right)}$

K3 surfaces: hopes and dreams

Conjecture (Skorobogatov '09):
For $X/\mathbb{Q}$ a K3 surface we have

$$X(\mathbb{A})^{\text{Br}} \neq \varnothing \implies X(\mathbb{Q}) \neq \varnothing .$$

Theorem (Kresch, Tschinkel '11):
Let $X/\mathbb{Q}$ be a K3 surface, given as a system of homogeneous polynomial equations. Assume we have equations for generators of Pic$(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})$, and a bound for #Br$(X)/Br_0(X)$. Then $X(\mathbb{A})^{\text{Br}}$ is effectively computable.

K3 surfaces: hopes and dreams

Theorem (Charles '14):
Let $X/\mathbb{Q}$ be a K3 surface, given as a system of homogeneous polynomial equations. Then equations for generators of Pic$(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})$ are effectively computable.

Conjecture (V.-A. '15 + Shafarevich '94):
Fix $n \in \mathbb{Z}_{>0}$. There is a constant $C = C(n)$ such that

$$\#\mathrm{Br}(X)/\mathrm{Br}_0(X) < C$$

for all K3 surfaces X/K with $[K : \mathbb{Q}] = n$.

K3 surfaces: hopes and dreams

Theorems | Conjectures

Kresch—Tschinkel + Charles + V.A. + Shafarevich

$\downarrow$

Expectation: There an effective procedure to determine whether $X(\mathbb{Q}) \neq \varnothing$.

Effective $\longrightarrow$ Practical?

Brauer Groups

$X/\mathbb{Q}$ a nice variety. $\qquad \mathrm{Br}(X) := \mathrm{H}^2_{\mathrm{et}}(\mathrm{X}, \mathbb{G}_{\mathrm{m}})_{\mathrm{tors}}$

$$\underbrace{\mathrm{Br}_0(X)}_{\substack{\mathrm{im}\left(\mathrm{Br}(\mathbb{Q}) \to \mathrm{Br}(X)\right) \\ \text{constant classes}}} \subseteq \underbrace{\mathrm{Br}_1(X)}_{\substack{\ker\left(\mathrm{Br}(X) \to \mathrm{Br}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})\right) \\ \text{algebraic classes}}} \subseteq \mathrm{Br}(X)$$

$$0 \to \frac{\mathrm{Br}_1(X)}{\mathrm{Br}_0(X)} \to \frac{\mathrm{Br}(X)}{\mathrm{Br}_0(X)} \to \frac{\mathrm{Br}(X)}{\mathrm{Br}_1(X)} \to 0.$$

$$\frac{\mathrm{Br}_1(X)}{\mathrm{Br}_0(X)} \cong \mathrm{H}^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}))$$

$$\frac{\mathrm{Br}(X)}{\mathrm{Br}_1(X)} \hookrightarrow \mathrm{Br}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}$$

Gvirtz
Orr
Skorobogatov
Valloni

Nontrivial extensions!

Theorem (Gvirtz, Skorobogatov '19):

For $X/\mathbb{Q} : x^4 + y^4 = 2(z^4 - w^4)$ we have

$$\begin{array}{ccccccccc} 0 & \to & \dfrac{\mathrm{Br}_1(X)}{\mathrm{Br}_0(X)} & \to & \dfrac{\mathrm{Br}(X)}{\mathrm{Br}_0(X)} & \to & \dfrac{\mathrm{Br}(X)}{\mathrm{Br}_1(X)} & \to & 0 \\ & & \wr\wr & & \wr\wr & & \wr\wr & & \\ 0 & \to & \mathbb{Z}/4\mathbb{Z} & \to & \mathbb{Z}/8\mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} & \to & 0. \end{array}$$

Algebraic Brauer Groups

Lemma:

Fix $1 \leq \rho \leq 20.$
$X/\mathbb{Q}$: a K3 surface with Pic$(X \times_{\mathbb{Q}} \overline{\mathbb{Q}}) \simeq \mathbb{Z}^{\rho}$.
There is an $M = M(\rho) \in \mathbb{Z}$, indep. of X, such that

$$\#\mathrm{Br}_1(X)/\mathrm{Br}_0(X) < M\,.$$

Algebraic Brauer Groups

Idea of the proof:

Pass to a finite Galois extension $K/\mathbb{Q}$ such that $\mathrm{Pic}(X_K) \simeq \mathrm{Pic}(X \times_{\mathbb{Q}} \overline{\mathbb{Q}})$. Let $G = \mathrm{Gal}(K/\mathbb{Q})$.

LES in Galois cohomology for

$$0 \to \mathbb{Z}^\rho \xrightarrow{\cdot|G|} \mathbb{Z}^\rho \to \mathbb{Z}^\rho/|G| \to 0$$

gives

$$\mathrm{H}^1(G, \mathbb{Z}^\rho) \simeq \frac{\left(\mathbb{Z}^\rho/|G|\right)^G}{(\mathbb{Z}^\rho)^G/|G|}$$

$\Longrightarrow$ $\#\mathrm{H}^1(G, \mathbb{Z}^\rho)$ divides $|G|^\rho$, regardless of action

Algebraic Brauer Groups

Idea of the proof:

There are finitely many possibilities for G:

G acts on $\mathbb{Z}^\rho$ through a finite subgroup of $\mathrm{GL}_\rho(\mathbb{Z})$.

Minkowski: for $m \geq 3$ the kernel of

$$\mathrm{GL}_\rho(\mathbb{Z}) \to \mathrm{GL}_\rho(\mathbb{Z}/m\mathbb{Z})$$

is torsion free.

Algebraic Brauer Groups

Question:
Can we give sharp bounds for $M(\rho)$, $1 \leq \rho \leq 20$?

$M(1) = 1$ (Galois cohomology)

$M(2) \leq 2$ (Wolff, 2019). Expect that $M(2) = 1$.

Transcendental Brauer Groups

$X/\mathbb{C}$ a K3 surface.

$$T(X) := \big(\mathrm{Pic}(X)\big)^{\perp} \subseteq \underbrace{\mathrm{H}^2(X, \mathbb{Z})}_{U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}}$$

Transcendental lattice

Important observation: $\mathrm{Br}(X)[n] \simeq \mathrm{Hom}(T(X), \mathbb{Z}/n\mathbb{Z})$

If $n = p$, a prime, this implies:

$$\langle \alpha \rangle \longleftrightarrow T_{\langle \alpha \rangle} \subseteq T(X)$$

Order p elt of $\mathrm{Br}(X)$

Sublattice of index p

Sample classification theorem

McKinnie, Sawon, Tanimoto, V.-A. '17:

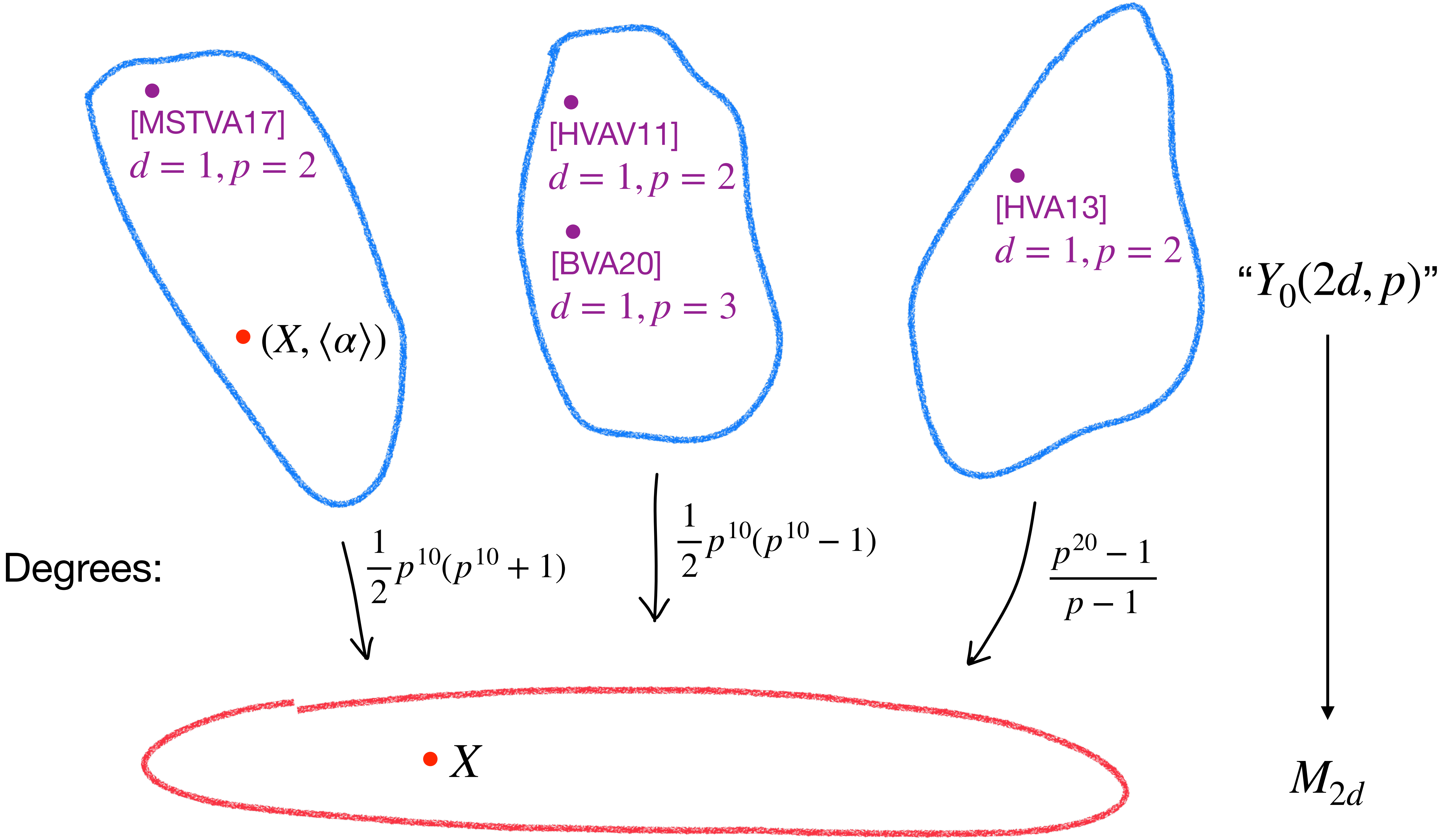
$X/\mathbb{C}$ a K3 surface; assume Pic$(X) \simeq \mathbb{Z}L$ with $L^2 = 2d$.

$p \nmid 2d$ prime; $\alpha \in$ Br$(X)[p]$ nontrivial.

There are three isomorphism classes for $T_{\langle\alpha\rangle}$:

$d(T_{\langle\alpha\rangle})$	Distinguishing feature	# of lattices
$\mathbb{Z}/2dp^2\mathbb{Z} = \langle v \rangle$	$-2dp^2q(v) \equiv \square \bmod p$	$\frac{1}{2}p^{10}(p^{10}+1)$
$\mathbb{Z}/2dp^2\mathbb{Z} = \langle v \rangle$	$-2dp^2q(v) \not\equiv \square \bmod p$	$\frac{1}{2}p^{10}(p^{10}-1)$
$\mathbb{Z}/2dp^2\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$		$\frac{p^{20}-1}{p-1}$

Mental Picture



$Y_1(2d, p)$ does not exist

$$\mathscr{M}_{2d}\colon (\mathrm{Sch}/\mathbb{Q})^{\mathrm{op}} \to (\mathrm{Sets})$$

$$T \to \left\{(f\colon X \to T, L \in \mathrm{H}^0(T, R^1 f_* \mathbb{G}_m))\right\} / \simeq$$

For $\mathrm{Spec}(\Omega) \to T$, the pair (X_Ω, L_Ω) is polarized K3 with $L_\Omega^2 = 2d$

Write $\Phi\colon \mathscr{M}_{2d} \to M_{2d}$ for the coarse moduli map.

Ideal moduli functor:

$$\mathscr{Y}_1(2d, p)\colon (\mathrm{Sch}/\mathbb{Q})^{\mathrm{op}} \to (\mathrm{Sets})$$

$$T \to \left\{(f\colon X \to T, L \in \mathrm{H}^0(T, R^1 f_* \mathbb{G}_m)), \alpha \in \mathrm{H}^0(T, R^2 f_* \mathbb{G}_m)\right\} / \simeq$$

$Y_1(2d,p)$ does not exist

Brakkee '20:

$\xi\colon \mathscr{Y}_1(2d,p) \to Y_1(2d,p)$ putative coarse moduli map

$$\begin{array}{ccc} \mathscr{Y}_1(2d,p) & \xrightarrow{\xi} & Y_1(2d,p) \\ \downarrow & & \downarrow \exists \pi \\ \mathscr{M}_{2d} & \xrightarrow{\Phi} & M_{2d} \end{array}$$

$U \subset M_{2d}$: open subset where $\mathrm{Aut}(X,L)$ is trivial $(d > 1)$.

For $y = (X, L, \alpha) \in Y_1(2d,p)$, the fiber above $x = \pi(y)$ is

$$\big(Y_1(2d,p)\big)_x \simeq \mathrm{Br}(X)[p] \simeq (\mathbb{Z}/p\mathbb{Z})^{22-\rho(X)}$$

The fix (Brakkee '20)

Correct moduli functor should parametrize triples

(X, L, α) with (X, L) a polarized K3 of degree $2d$ and

$$\alpha \in \mathrm{Hom}\left(\mathrm{H}^2(X, \mathbb{Z}/r\mathbb{Z})_{\mathrm{pr}}, \mathbb{Z}/r\mathbb{Z}\right)$$

Here $\mathrm{H}^2(X, \mathbb{Z}/r\mathbb{Z})_{\mathrm{pr}} = L^{\perp} \subset \mathrm{H}^2(X, \mathbb{Z}/r\mathbb{Z})$.

When $\rho = 1$, $\mathrm{H}^2(X, \mathbb{Z}/r\mathbb{Z})_{\mathrm{pr}} = L^{\perp} = T(X)$.

Relative version: for a family $(f\colon X \to T, L)$, set

$$R^2_{\mathrm{pr}} f_* \mathbb{Z} := \left(R^2 f_* \mathbb{Z} \xrightarrow{\cdot c_1(L)} R^4 f_* \mathbb{Z}\right)$$

The fix (Brakkee '20)

Let $\mathscr{F}_r = \mathscr{H}om\left(R^2_{\mathrm{pr}} f_* \mathbb{Z}, \underline{\mathbb{Z}/r\mathbb{Z}}\right)$

Define the moduli functor

$$\mathscr{K}_1(2d, r) \colon (\mathrm{Sch}/\mathbb{Q})^{\mathrm{op}} \to (\mathrm{Sets})$$

$$T \to \left\{(f \colon X \to T, L \in \mathrm{H}^0(T, R^1 f_* \mathbb{G}_m)), \alpha \in \mathrm{H}^0(T, \mathscr{F}_r)\right\} / \simeq$$

Theorem (Brakkee '20):
There is a course moduli space

$$\xi : \mathscr{K}_1(2d, r) \to K_1(2d, r)$$

with at most $r \cdot \gcd(r, 2d)$ many components.

Questions/tasks

Construct a lattice-polarized version of Brakkee’s spaces.

Which of these spaces are of general type?

Exact formula for number of components of these spaces?

Geometric interpretation for the $\rho = 1$ locus?