Moments of Zeta and the Vertical Distribution of its Zeros

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Intro:

Let $s = \sigma + it$, $\sigma, t \in \mathbb{R}$. The Riemann zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1$$

Properties of $\zeta(s)$

- Meromorphic continuation to $\mathbb{C}$ with a simple pole at $s=1$ with residue 1; i.e.

$$\zeta(s) - \frac{1}{s-1} \quad \text{is entire}$$

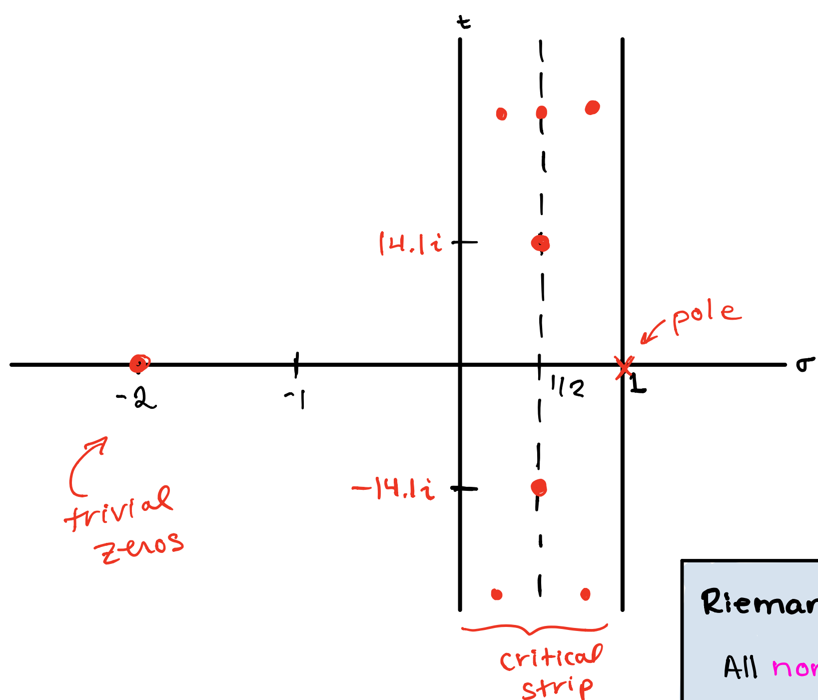
- **Functional equation**: $\zeta(s) = \chi(s)\,\zeta(1-s)$, where

$$\chi(s) = 2(2\pi)^{s-1}\,\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)$$

- Trivial zeros at $s=-2n$, $n=1,2,3,\ldots$

$$\zeta(-2n) = \underbrace{2(2\pi)^{-2n-1}}_{\neq 0}\,\underbrace{\Gamma(2n+1)}_{(2n)!}\,\underbrace{\sin(-\pi n)}_{0}\,\underbrace{\zeta(2n+1)}_{\neq 0}$$

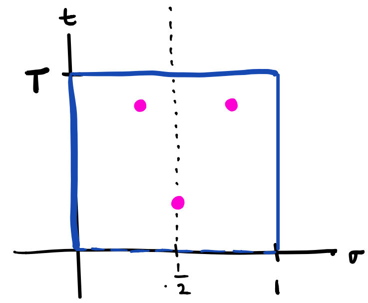
- The zeros are symmetric about the real axis, because $\zeta(s) = \overline{\zeta(\bar{s})}$.



Riemann Hypothesis:

All nontrivial zeros of $\zeta(s)$ have real part equal to $1/2$.

· **Counting zeros :**



$N(T) := \#\{\text{zeros } \rho = \beta + i\gamma : 0 \le \beta \le 1, 0 < \gamma \le T\}$

Argument Principle yields

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1})$$

where $S(T) = \pi^{-1}\arg\zeta(1/2 + iT) \ll \log T$

The Average Gap Between γ_n and γ_{n+1}

$$\sim \frac{2\pi}{\log(\gamma_n)}$$

Consider the ordinates of zeros in the upper half plane:

$$0 < \gamma_1 \le \gamma_2 \le \cdots \le \gamma_n \le \gamma_{n+1} \le \cdots$$

Then $\dfrac{\gamma_{n+1} - \gamma_n}{2\pi/\log\gamma_n}$ is 1 on averge.

Question : What can we say about

$\mu := \liminf\limits_{n\to\infty} \dfrac{\gamma_{n+1} - \gamma_n}{2\pi/\log\gamma_n}$ and $\lambda := \limsup\limits_{n\to\infty} \dfrac{\gamma_{n+1} - \gamma_n}{2\pi/\log\gamma_n}$?

- By definition, we trivially have $\mu \leq 1 \leq \lambda$

- Selberg (1940s) / Fujii (1970s) : observed $\underline{\mu < 1 < \lambda}$

Heath-Brown's notes in Titchmarsh

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Conjecture: $\mu = 0$ and $\lambda = \infty$

(there are ∞ly many pairs of zeros of $\zeta(s)$ that are arbitrarily close (or far apart) relative to the average spacing.)

Q: Where does this conjecture come from?

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<u>Montgomery's Pair Correlation Conjecture</u> (1972)

For any fixed $c > 0$,

$$N(T,c) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum 1 \sim \int^{c} \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du$$

$$N(T,c) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma,\gamma'\le T \\ 0<\gamma-\gamma'\le \frac{2\pi c}{\log T}}} 1 \sim \int_0^c \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du$$

Note: $$0 < \int_0^c \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du < c$$

Thus we can make c very small, and still get $N(T;c) > 0$.

$$\text{PCC} \Rightarrow \mu = 0$$

What about large gaps?

Montgomery's Pair Correlation Conjecture (1972)

For any fixed $c > 0$,

$$N(T,c) := \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{\substack{0<\gamma,\gamma'\le T \\ 0<\gamma-\gamma'\le \frac{2\pi c}{\log T}}} 1 \sim \int_0^c \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) du$$

$$1 - \left(\frac{\sin \pi u}{\pi u}\right)^2$$ "Pair Correlation Function" of zeros of $\zeta(s)$.

- Dyson noted eigenvalues of random Hermitian matrices have the same pair correlation function.

  - This connection has been supported by extensive numerical estimates by Odlyzko.

- Further observations in this realm $\Rightarrow$ $\lambda = \infty$

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Questions so far?

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Small Gaps - Progress:

Conjecture: $\mu = 0$

$\mu < 1$

Author(s), year	upper bound on μ (under RH)	
Montgomery / Goldston '72	0.6072	positive proportion of zeros
Carneiro, Chandee, Littmann, Milinovich '17	0.606894	
Chirre, Goncalves, DeLatte '19	0.6039	
Montgomery - Odlyzko '81	0.5179	∞ly many zeros
Conrey - Ghosh - Gonek '84	0.5172	
Bui - Milinovich - Ng '10	0.5155	
Feng - Wu '12	0.515398	
Preobrazinskii '16	0.515396	

The Class number problem & Exceptional Zeros

Let $d < 0$ be a fundamental discriminant.

$K = \mathbb{Q}(\sqrt{|d|})$

$|$

$\mathbb{Q}$

The ideal class group of K:

$$Cl(K) := \left(\text{group of fractional ideals of } K\right) / \left(\text{subgroup of principal ideals of } K\right)$$

The class number of K:

$$h(K) := |Cl(K)|.$$

$h(K) = 1 \iff Cl(K) = \{id\} \iff \mathcal{O}_K$ is a PID $\iff \mathcal{O}_K$ is a UFD.

Gauss: Conjectured $h(K) \to \infty$ as d runs through negative discriminants.

- Proved by Heilbronn in 1934.

- This implies that there are only finitely many imaginary quadratic fields K with $h(K)=n$, where $n \geq 1$ is fixed.

Class Number Problem for Imaginary Quadratic Fields:

Give a complete list of fundamental discriminants so that $h(K)=n$, n fixed.

$n=1$: Solved by Stark / Heegner / Baker

$d = -3, -4, -7, -8, -11, -19, -43, -67, -163$

How about $n \geq 2$?

<u>Class Number Formula:</u> (assuming $d<-4$)

$$h(K) = \pi^{-1} \sqrt{|d|}\, L(1, \chi_d)$$

- $L(1,\chi_d) = \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s}$, where $\chi_d(n) = \left(\frac{d}{n}\right)$

Want: Lower bound on $h(K)$.

$$h(K) \geq 1 \qquad \left(\text{so } L(1,\chi_d) \gg \frac{1}{\sqrt{|d|}}\right)$$

—

Under RH:

$$L(1,\chi_d) \gg \frac{1}{\log\log|d|} \quad \Longrightarrow \quad h(K) \gg \frac{\sqrt{|d|}}{\log\log|d|}$$

Question: Can we get a stronger, unconditional lower bound on $L(1,\chi_d)$?

Difficulty: $L(s,\chi_d)$ could have an exceptional zero: a real, simple zero β lying very close to 1, making it difficult to produce a lower bound on $L(1,\chi_d)$ and hence on $h(K)$.

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Two goals: Obtain...

- an unconditional lower bound on $h(K)$ to solve the class number problem.
- a strong enough unconditional lower bound on $L(1,\chi_d)$ to eliminate the possibility of the exceptional zero.

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Siegel (1935) : For every $\varepsilon>0$, $h(K) \gg d^{\frac{1}{2}-\varepsilon}$, but the implied constant is not computable.

· Goldfeld-Gross-Zagier : For every $\varepsilon>0$, $h(K) \gg (\log|d|)^{1-\varepsilon}$.

· With this lowerbound, Watkins has computed complete lists of fundamental discriminants with $h(K)=2,3,4,\dots,100$.

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Conrey-Iwaniec (2002) : If for all large T there are $\gg T(\log T)^{4/5}$ nontrivial zeros of $\zeta(s)$ such that $0<\gamma\leq T$ and

$$\frac{\gamma_{n+1}-\gamma_n}{2\pi/\log\gamma_n} \leq \frac{1}{2}\left(1-\frac{1}{\sqrt{\log\gamma}}\right)$$

then

$$L(1,\chi_d) \gg \frac{1}{(\log|d|)^{90}}$$

and the implied constant is computable.

Small Gaps - Progress:

Conjecture: $\mu = 0$

Author(s), Year		Upper bound on μ (under RH)
Montgomery / Goldston	'72	0.6072
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—

State of the art approach to small gaps (under RH)

- Developed by Julia Mueller (1982) in the context of large gaps.

- Determined to also apply to small gaps by Conrey-Ghosh-Gonek (1984).

- Equivalent to a method due to Montgomery-Odlyzko around the same time.

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Set up:

$$A(t) = \sum_{k \leq X} \frac{a_k}{k^{it}} \quad , \quad X = T^{1-\delta} \quad , \quad \delta \text{ small}$$

Define

$$M_1 := \int_{T/2}^{2T} |A(t)|^2 \, dt \qquad \text{"global" average}$$

and

$$M_2(c) = \int_{-\pi c/\log T}^{\pi c/\log T} \sum_{\frac{T}{2} \leq \gamma \leq 2T} |A(\gamma+\alpha)|^2 \, d\alpha \qquad \text{"local average" near zeros of } \zeta(s)$$

—

Remarks:

① $M_2(c)$ is monotonically increasing:

$$M_2(c) = \int_{-\pi c/\log T}^{\pi c/\log T} \sum_{\frac{T}{2} \le \gamma \le 2T} |A(\gamma+\alpha)|^2 \, d\alpha$$

② Claim: $M_2(\mu) \le M_1 \le M_2(\lambda)$:

μ: small gaps
λ: large gaps

Recall, by Selberg/Fujii $\mu < 1 < \lambda$.

For zeros $\gamma, \gamma' \in [T/2, 2T]$, the average spacing is $2\pi/\log T$.

$$M_1 := \int_{T/2}^{2T} |A(t)|^2 \, dt$$

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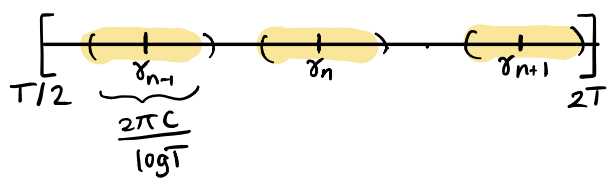
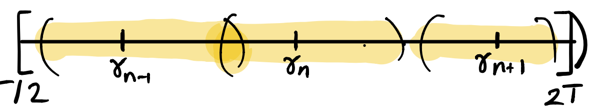
In the range of integration, if $c<1$:



$$\int_{-\pi c/\log T}^{\pi c/\log T} \sum_{\frac{T}{2} \le \gamma \le 2T} |A(\gamma+\alpha)|^2 \, d\alpha \le \int_{T/2}^{2T} |A(t)|^2 \, dt$$

$$\therefore \; M_2(\mu) \le M_1 .$$

On the other hand, if $c > 1$, then



$$\int_{-\pi c/\log T}^{\pi c/\log T} \sum_{\frac{T}{2} \le \gamma \le 2T} |A(\gamma+\alpha)|^2 \, d\alpha \geqslant \int_{T/2}^{2T} |A(t)|^2 \, dt$$

$$\therefore \; M_2(\lambda) \geqslant M_1 .$$

Point: since $M_2(\mu) \le M_1 \le M_2(\lambda)$,

- If $M_2(c) < M_1$, then $\lambda > c$.
- If $M_2(c) > M_1$, then $\mu < c$.

Thus for small gaps, we must choose $A(t)$ and c such that

$$\frac{M_2(c)}{M_1} > 1 ,$$

and for large gaps, we must choose $A(t)$ & c such that

$$\frac{M_2(c)}{M_1} < 1.$$

- If we multiple out the numerator and denominator, we can show

$$\frac{M_2(c)}{M_1} = c - \frac{\operatorname{Re}\left(\sum_{nk \le X} a_k \overline{a_{nk}}\, g_c(n) \Lambda(n) n^{-1/2}\right)}{\sum_{k \le X} |a_k|^2} + o(1)$$

where
$$g_c(n) = \frac{2 \sin\left(\pi c \frac{\log n}{\log X}\right)}{\pi \log n} \quad \text{and} \quad \Lambda(n) = \begin{cases} \log p & n = p^k \\ 0 & \text{else} \end{cases}.$$

- For small gaps, we want $M_2(c)/M_1 > 1$. This suggests we pick $A(t)$ to be big around zeros of $\zeta(s)$.

Note:
$$\frac{\zeta(2s)}{\zeta(s)} = \sum_{k=1}^{\infty} \frac{\lambda(k)}{k^s} \qquad \lambda(k) = -1^{\text{TOTAL \# of PRIME DIVISORS of } k}$$

is large around zeros of $\zeta(s)$, so we could take

$$A(t) = \sum_{k \le X} \frac{\lambda(k)}{k^{1/2+it}}. \qquad a_k = \frac{\lambda(k)}{k^{1/2}}$$

This choice of $A(t)$ gives:

$$\frac{M_2(c)}{M_1} = c + \frac{2}{\pi}\int_0^1 \frac{\sin(\pi c v(1-\delta))}{v}(1-v)\,dv.$$

—

Using Mathematica, we can find the smallest $c>0$ for which

$$c + \frac{2}{\pi}\int_0^1 \frac{\sin(\pi c v)}{v}(1-v)\,dv > 1.$$

$c = 0.5181$ works! $\therefore\ \mu \le 0.5181$

- Conrey-Ghosh-Gonek choose:

$$a_k = \frac{\lambda(k)}{k^{1/2}}\, d_r(k)$$

$d_r(k)$ is a multiplicative function with $d_r(p^k) = \frac{\Gamma(k+r)}{\Gamma(r)\,k!}$.

Produces $\mu \le 0.5172$, $r = 1.1$ (under RH)

Subsequent Refinements:

Smooth

$$a(k) = \underbrace{\frac{\lambda(k)}{k^{1/2}}\, d_r(k)}\, f\left(\frac{\log X/k}{\log X}\right)$$

Limitation: Conrey-Ghosh-Gonek show that this method cannot attain $\mu < \frac{1}{2}$ for any choice of a_k.

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Questions?

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Large gaps and moments of zeta

Method: due to R.R. Hall (1999), unconditional

Consider the ordinates of nontrivial zeros on the critical line:

$$0 < t_1 \leq t_2 \leq t_3 \leq \dots$$

and let

$$\Lambda := \limsup_{n\to\infty} \frac{t_{n+1} - t_n}{2\pi/\log t_n}$$

RH $\Rightarrow$ $\lambda = \Lambda$. Unconditionally, $\Lambda \geq \lambda > 1$.

$$\Lambda := \limsup_{n\to\infty} \frac{t_{n+1} - t_n}{2\pi/\log t_n}$$

Conjecture
$\Lambda = \infty$

Results on Λ:

Author / Year	Lower bound on Λ
Hall ('99)	2.26
Hall ('02)	2.34520
Hall ('05)	2.630637
Bredberg ('11)	2.76
Bui-Milinovich ('17)	3.18

Best results on λ using Mueller method:

$\lambda > 2.9$ (under RH) by Bui 2011

$\lambda > 3.072$ (under GRH for Dirichlet L-functions), Feng & Wu 2013

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Hall's Method

<u>Wirtinger's Inequality</u> : Suppose that $f(t)$ is a real, continuously differentiable function which satisfies $f(0) = f(\pi) = 0$. Then

$$\int_0^\pi f(t)^2\, dt \le \int_0^\pi f'(t)^2\, dt .$$

Extend to f complex-valued & continuously differentiable, if $f(a) = f(b) = 0$:

$$\int^b |f(t)|^2\, dt \le \frac{(b-a)^2}{\;} \int^b |f'(t)|^2\, dt .$$

$$\int_a \cdots \qquad \pi^2 \int_a \cdots$$

Toy example of Hall's Method:

- based on a comment in Hall's 1999 article

- We will show that $\Lambda \geqslant \sqrt{3}$.

- Thus, under RH, $\lambda \geqslant \sqrt{3}$.

Choice of function: Take $f(t)$ to be the Hardy Z-function:

$$Z(t) := e^{i\theta(t)} \zeta(\tfrac{1}{2}+it) = \underbrace{\left\{\pi^{-it} \frac{\Gamma(\frac{1}{4}+\frac{1}{2}it)}{\Gamma(\frac{1}{4}-\frac{1}{2}it)}\right\}^{1/2}}_{:= (\chi(\frac{1}{2}+it))^{-1/2}} \zeta(1/2+it).$$

Proof. Suppose, towards contradiction, that $\Lambda \leq \kappa$, for κ some real number. Denote all of the zeros of $Z(t)$ in the interval $[T, 2T]$ by

$$t_1 \leq t_2 \leq \cdots \leq t_n \leq t_{n+1} \leq t_N.$$

By our assumption we have

$$t_{n+1} - t_n \leq (1+o(1)) \frac{2\pi\kappa}{\log T} \quad \text{as } T \to \infty.$$

—

By Wirtinger's inequality

$$\int_{t_n}^{t_{n+1}} Z^2(t)\,dt \le \left(\frac{t_{n+1}-t_n}{\pi}\right)^2 \int_{t_n}^{t_{n+1}} Z'(t)^2\,dt .$$

Summing for all zeros in the range $[T, 2T]$, we have

$$\int_{t_1}^{t_N} Z^2(t)\,dt \le \frac{(1+o(1))\,4\kappa^2}{\log^2 T} \int_{t_1}^{t_N} Z'(t)^2\,dt$$

—

Since $|Z(t)| = |\zeta(1/2+it)| \ll t^{1/6+\varepsilon}$ (Weyl's bound)

and our assumption, we have

$$\int_{T}^{2T} Z^2(t)\,dt \le (1+o(1))\,\frac{4\kappa^2}{\log^2 T} \int_{T}^{2T} Z'(t)^2\,dt .$$

Therefore, if

$$\limsup_{T\to\infty} \frac{\log^2 T}{4\kappa^2}\,\frac{\int_T^{2T} Z^2(t)\,dt}{\int_T^{2T} Z'(t)^2\,dt} > 1 .$$

we will have contradicted our assumption and may conclude

$$\Lambda > \kappa.$$

Want:

$$\limsup_{T\to\infty} \frac{\log^2 T}{4\kappa^2} \frac{\int_T^{2T} Z^2(t)\,dt}{\int_T^{2T} Z'(t)^2\,dt} > 1.$$

Moments:

- $\displaystyle\int_T^{2T} Z^2(t)\,dt = \int_T^{2T} |\zeta(1/2+it)|^2\,dt \overset{\text{Hardy \& Littlewood}}{\sim} T\log T$

- $\displaystyle\int_T^{2T} Z'(t)^2\,dt \overset{\text{Hall}}{\sim} \frac{T}{12}(\log T)^3$

Thus $$\limsup_{T\to\infty} \frac{\log^2 T}{4\kappa^2} \frac{\int_T^{2T} Z^2(t)\,dt}{\int_T^{2T} Z'(t)^2\,dt} > 1 \quad \text{for} \quad \kappa=\sqrt{3}$$

so we have contradicted $\Lambda \leq \sqrt{3}$. Thus $\Lambda > \sqrt{3}$. ∎

Other choices of $f(t)$:

$$\boxed{\int_a^b |f(t)|^2 dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b |f'(t)|^2 dt.}$$

- Hall (1999) : $f(t) = Z^2(t) \Rightarrow \Lambda > 2.263509.$

Uses : $\int_T^{2T} Z^4(t)\, dt \sim \frac{T}{2\pi} \log^4 T$ (Ingham)

$$\int_T^{2T} Z'(t)^4 dt \sim \frac{T}{1120\pi^2} \log^8 T \quad \text{(Hall)}$$

—

Modifications:

- Hall (2005) : Modification with shifts : $f(t) = Z(t)Z(t+a)$ where $a \ll \frac{1}{\log T}$.

Reasoning: Consider $[T, T+b]$ (length b)

If $a \leq b$ and $f(t) \neq 0$ for $t \in (T, T+b)$, then you get that $Z(t) \neq 0$ in an interval of length $a+b$.

This idea yields $\Lambda > 2.630637...$

- Current Record: Bui & Milinovich $\Lambda > 3.18$

Their $f(t)$ incorporates shifts and an idea of Bredberg:

$$f(t) = \underbrace{e^{ivt\log\frac{T}{2\pi}}}_{\substack{\text{mimic}\\ \text{real valued}\\ \text{function}}} \underbrace{\zeta(\tfrac{1}{2}+it)\,\zeta(\tfrac{1}{2}+it+i\tfrac{\kappa\pi}{\log T/2\pi})}_{\substack{\text{know mean-}\\ \text{value}\\ \text{theorem}}} \underbrace{M(\tfrac{1}{2}+it)}_{\substack{\text{Dirichlet}\\ \text{polynomial}}}$$

Choice of M:

$$\sum_{n \le T^{\theta}} \frac{d_r(n)}{n^s}(\text{polynomial}), \quad \theta < 1/4 \text{ by Bettin, Bui, Li, Radziwiłł.}$$

- Note: Bredberg chose $M(s) = \sum_{n \le T^{\theta}} \frac{1}{n^s}$, $\theta < 1/11$ by Hughes-Young

Limitations of Method

① The method relies on being able to compute

$$\int_T^{2T} |f(t)|^2 dt \quad \text{and} \quad \int_T^{2T} |f'(t)|^2 dt.$$

- For $\zeta(s)$, we only have this information for the second & fourth moment.

  - This is why using amplifiers is being explored:

    The longer the length, the better we do.

Conjecture: As $T \to \infty$,

$$\int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^{2k} dt \sim c_k \, T(\log T)^{k^2}, \quad k>0$$

$k=1$: Proved by Hardy & Littlewood (1918)

$k=2$: Proved by Ingham (1926)

Values of c_k:

$k=3$: Conjectured by Conrey - Ghosh (1998)

$k=4$: Conjectured by Conrey - Gonek (2001)

$Re(k) \geq -\frac{1}{2}$: Conjectured by Keating - Snaith (2000)

$k \in \mathbb{N}$: - Conjectured by Diaconu, Goldfeld, Hoffstein (2003)

- Conjectured by Conrey, Farmer, Keating, Rubinstein, Snaith (2005)

Suppose you knew all the moments ... How much will Hall's method give you?

Hughes: Predictions from RMT:

$$\int_0^T Z(t)^{2k-2h} Z'(t)^{2h} \, dt \sim \underbrace{a(k)}_{\substack{\text{product}\\ \text{over}\\ \text{primes}}} \underbrace{b(h,k)}_{\in \mathbb{Q}} T(\log T)^{k^2+2h}$$

Hall has shown: Λ

2002

6th moment	$\Rightarrow$	2.8914
8th moment	$\Rightarrow$	3.392
10th moment	$\Rightarrow$	3.858
12th moment	$\Rightarrow$	4.298

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Conrey, Farmer, Keating, Rubinstein, & Snaith (2005)

"Recipe" for moments of L-functions (with shifts).

One might:

- See what the recipe of CFKRS yields (w/ shifts)

- Try to win, somehow harnessing knowledge of 2nd & 4th moment together.

Thank You
for your attention!