Unikely intersections and the André-Oort conjecture

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Lang's Conjecture: Polynomial Relations in Roots of Unity

Let $C \subset (\mathbb{C}^\times)^2$ be an algebraic curve, defined by (irreducible) $F(X, Y) = 0, F(X, Y) \in \mathbb{C}[X, Y, X^{-1}, Y^{-1}]$.

Theorem (Lang 1965)

If C contains infinitely many points (ζ, η) with ζ, η roots of unity, then C is of the form

$$x^m y^n = \zeta,$$

with $n, m \in \mathbb{Z}, \zeta$ a root of unity.

$\therefore$ C - coset of a subgroup by a torsion point; we call C a **Torsion Coset**. For $V \subset (\mathbb{C}^\times)^n, \dim V > 1$ it may be that V contains positive dimensional torsion cosets.

Theorem (Laurent, 1983)

For $V \subset (\mathbb{C}^\times)^n$, then V contains finitely many ***Maximal*** *torsion cosets.*

Manin-Mumford: Abelian Varieties

We replace $(\mathbb{C}^{\times})^n$ by an abelian variety $A(\mathbb{C}) = \mathbb{C}^n/\Lambda$.

- $B \subset A$ - Abelian subvariety
- $\zeta \in A$ - Torsion point
- $\zeta + B$ - Torsion coset

Theorem (Manin-Mumford Conjecture; Raynaud, 1983)

For $V \subset (\mathbb{C}^{\times})^n$, then V contains finitely many ***Maximal*** *torsion cosets.*

André-Oort: Shimura Varieties

- $\mathcal{H}$-Hermitian Symmetric space.
- Γ- discrete, arithmetic group acting on $\mathcal{H}$.
- S - Shimura variety. $S(\mathbb{C}) = \Gamma\backslash\mathcal{H}$
- S contains a discrete, dense set of **Special (CM) Points**
- S contains a countable set of **Special Subvarieties** T, which are themselves "Shimura Subvarieties"

Conjecture (André)

*For $V \subset S$, then V contains finitely many **Maximal** Special Subvarieties.*

- André (1998): $S = Y(1)^2$
- Edixhoven, Klingler, Ullmo, Yafaev (2014) : True conditional on GRH.
- Pila, Pila-Zannier Strategy: $S = Y(1)^n$
- T (Pila-T): $S = \mathcal{A}_g$
- Pila-Shankar-T: General case

Examples of Shimura Varieties: $Y(1)^2$

- $Y(1)$ - (coarse) moduli space of elliptic curves
- $Y(1)(\mathbb{C}) \cong \mathrm{Sl}_2(\mathbb{Z})\backslash\mathbb{H}, \pi : \mathbb{H} \to Y(1)$

Special Points in $Y(1)$

- The CM points are classes $[E]$ of complex elliptic curves with $\mathbb{Z} \subsetneq \mathrm{End}(E)$
- $\pi^{-1}(Y(1)_{\mathrm{CM}}) = \{\tau \in \mathbb{H} \mid [\mathbb{Q}(\tau) : \mathbb{Q}] = 2\}$

Special curves in $Y(1)^2$:

- $\{x\} \times Y(1)$, where x is a special point
- $Y(1) \times \{x\}$, where x is a special point
- For $N > 0$, $T_N = \{([E], [E']), \exists\phi : E \to E', \ker(\phi) \cong \mathbb{Z}/N\mathbb{Z}\}$

Theorem (André, 1998)

A curve $C \in Y(1)^2$ containing infinitely many CM points is special.

Proved effectively by Kühne, Bilu-Masser Zannier, 2012/13.

Examples of Shimura Varieties: $\mathcal{A}_g$

- $\mathcal{A}_g$ - (coarse) moduli space of g-dimensional, principally polarized abelian varieties
- $\mathbb{H}_g = \{Z = X + iY, Y > 0, Z \in \mathrm{M}_g(\mathbb{C})^{\mathrm{sym}}\}$,
  $\mathcal{A}_g(\mathbb{C}) \cong \mathsf{Sp}_{2g}(\mathbb{Z})\backslash\mathbb{H}_g$.

Special Points in $\mathcal{A}_g$

- The CM points are classes $[A]$ of complex Abelian varieties, containing a commutative $R \subset \mathrm{End}_{\mathbb{Q}}(A), [R : \mathbb{Q}] = 2g$.
- $\pi^{-1}(Y(1)_{\mathrm{CM}}) \subset \{\tau \in \mathbb{H}_g \mid [\mathbb{Q}(\tau) : \mathbb{Q}] \leq 2g\}$

Special subvarieties in $\mathcal{A}_g$:

- $\mathrm{sym}^g Y(1) \subset \mathcal{A}_g$
- F- totally real field, $[F : \mathbb{Q}] = g$, $Y_F \cong \mathsf{Sl}_2(\mathcal{O}_F)\backslash\mathbb{H}^g \subset \mathcal{A}_g$.
- Other Endomorphism structures, etc…

Pila-Zannier Strategy: Lang's conjecture for $n = 2$

Proof.

Assume $n = 2$.

- $Z \subset (\mathbb{C}^\times)^2$- algebraic curve, $\forall i \in \mathbb{N}, w_i \in Z$ are torsion points.
- w_i defined over $\overline{\mathbb{Q}} \Rightarrow Z$ defined over $\overline{\mathbb{Q}}$
- $\pi : \mathbb{C}^2 \to (\mathbb{C}^\times)^2, (a, b) \to (e^{2\pi ia}, e^{2\pi ib})$.

$$\pi^{-1}(\mathbb{C}^\times_{tor}) = \mathbb{Q}$$

- **Lower Bound:** $\pi(\frac{a}{n}, \frac{b}{n}) \in Z \Rightarrow \forall (c, n) = 1, \pi(\frac{ca}{n}, \frac{cb}{n}) \in Z$. Z contains $\mathbf{1}$ point of order $n \Rightarrow Z \cap [0, 1)$ contain $\phi(n)$ such points.
- **Upper bound:** Z not torus coset $\Leftrightarrow \pi^{-1}(Z)$ not algebraic . Z contains $O_\epsilon(n^\epsilon)$ points in $(\frac{1}{n}\mathbb{Z}/\mathbb{Z})^2$
- $n^{1-\epsilon} \ll_\epsilon \phi(n)$, which is a contradiction. □

Pila-Zannier Strategy:General case

- Consider $\pi : \mathcal{H} \to X$.
- Suppose $V \subset X$ has a Zariski-dense set of special points.
- $\pi^{-1}(V)$ contains **many** points in $\pi^{1}(X_{\mathrm{sp}})$, which are "rational".
- $\pi^{-1}(V)$ cannot contain many rational points, unless $\pi^{-1}(V)$ **contains** algebraic subvarieties W.
- Functional transcendence: This only happens if V is a (weakly) special variety (Ax-Lindemann Theorem).

Pila-Zannier Strategy

3 fundamental Ingredients

- **Rational Points on Transcendental sets** : If a nice (definable) real analytic variety contains many rational points, it is algebraic.
- **Functional Transcendence**: There is no interaction between the algebraic structures on $\mathcal{H}$ and X, except that which is mandated by (weakly) special varieties.
- **Large Galois Orbits**: If $x \in X$ is a special point, then $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x$ is big compared to the complexity of x.

Pila-Wilkie: Counting Rational Points

- $\gcd(a, b) = 1, H(\frac{a}{b}) = \mathrm{Max}(|a|, |b|)$
- $H(z_1, .., z_m) = Max_i H(z_i)$
- $N(S, X) := \#\{z \in \mathbb{Q}^m \mid H(z) \leq X, z \in S\}$

Theorem (Bombieri-Pila, '89)

Let C be an irreducible, compact, real analytic, transcendental curve. Then $N(C, X) = O_\epsilon(X^\epsilon)$

Idea: (Jarnik) Let $C \subset \mathbb{R}^2$ be a circle of radius L. Then $\#C(\mathbb{Z}) \ll L^{2/3}$.

Proof: Let P, Q, R be integer points on an arc of length $L\theta$. Then

$$1/2 \leq \mathrm{Area}(\Delta(PQR)) \ll (L\theta)^2 \sin(\theta) \approx L^2\theta^3.$$

If $\theta = L^{\frac{-1}{3}}$ there are at most 2 points per arc.

Pila-Wilkie: Counting Rational Points

- $Y \subset \mathbb{R}^m$
- $Y^{alg}: = \bigcup_{I \subset Y} Y$, union over irreducible (semi)-algebraic curves I.
- $Y^{tran} := Y \backslash Y^{alg}$
- $N(S, X) := \#\{z \in \mathbb{Q}^m \mid H(z) \leq X, z \in S\}$

Theorem (Pila-Wilkie,'04; Bombieri-Pila)

Asssume Y is compact, real analytic. Then

$$N(Y^{tran}; X) = O_\epsilon(X^\epsilon)$$

Generalizations:

$\mathbb{Q} \Rightarrow$ Points of bounded degree

compact, real analytic $\Rightarrow$ Definable in an o-minimal structure $(\mathbb{R}_{an,exp})$.

Classical Transcendence: exponential

Theme: e^x is as transcendental as possible, subject to $e^{x+y} = e^x \cdot e^y$.

Conjecture (Schanuel)

Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ be $\mathbb{Q}$-linearly independent. Then

$$Tr.deg_{\mathbb{Q}} \mathbb{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) \geq n.$$

- $n = 2, \vec{\alpha} = (1, \pi) \Leftrightarrow e, \pi$ algebraically independent.
- (Lindemann-Weierstrass) true if α_i are all algebraic.
- $\Gamma \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ - graph of co-ordinate wise exponential.
- Schanuel conjecture $\Leftrightarrow \forall x \in \Gamma(\mathbb{C}), Tr.deg(x) \geq n$.

Functional Transcendence: Conjectures

Setup: $\pi : \mathcal{H} \to X$ transcendental map between algebraic varieties. $V_X \subset X, V_{\mathcal{H}} \subset \mathcal{H}$ algebraic subvarieties. Γ_π-graph of π.
$V \subset \mathcal{H}$ is called bi-algebraic if $\pi(V)$ is algebraic. *For e^x these are pre-images of torus cosets, affine linear subspaces with $\mathbb{Q}$-slopes*

Conjecture (Ax-Lindmenann)

If $\pi(V_{\mathcal{H}}) \subset V_X, \exists$ bi-algebraic S such that $W \subset S, \pi(S) \subset V$.

Conjecture (Ax-Schanuel)

Let $V \subset \mathcal{H} \times X$ be a subvariety, $U \subset V \cap \Gamma_\pi$ be an analytic component. If

$$dim U > dim V - dim X$$

then $\exists$ bi-algebraic S such that $U \subset S$ and $dim U = dim W \cap S + dim V \cap S - dim S$.

Functional Transcendence: Results

- AS for $X = \mathbb{C}^n$ or $A(\mathbb{C})$, (Ax, 1971)
- AL for $X = Y(1)^n$, (Pila, 2011)
- AL for all Shimura Varieties (Klingler-Ullmo-Yafaev, 2015)
- AL for Mixed Shimura Varieties (Gao, 2017)
- AL for non-arithmetic rank 1 quotients (Mok, 2018)
- AS for $X = Y(1)^n$, (Pila-T, 2016)
- AS for all Shimura varieties, (Mok-Pila-T, 2018)
- AS for Variations of Hodge Structures, (Bakker-T, 2018)
- AS for variations of Mixed Hodge structures (Chiu, Gao-Klingler)

Functional Trancendence: Proof ingredients

- Pila-Wilkie (Again!)
- o-minimality, Definable Chow Theorem (Peterzil-Starchenko)
- Hyperbolic Geometry (Hwang-To)

Largeness of Galois Orbits

- **The $(\mathbb{C}^\times)^n$ case**
  - $x \in (\mathbb{C}^\times)^n_{\mathrm{tor}}$
  - Need $|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \gg \mathrm{ord}(x)^\delta$
  - Reduces to $\phi(n) = n^{1-o(1)}$.
- **The Abelian variety case**
  - $A/\mathbb{Q}$ - abelian variety
  - $x \in A_{\mathrm{tor}}$
  - Need $|\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \gg \mathrm{ord}(x)^\delta$
  - Follows from Masser, Wintenberger

Largeness of Galois Orbits

- $x \in \mathcal{A}_g$ Corresponds to PPAV A_x
- $\mathrm{End}(A_x) = R \subset K, [K : \mathbb{Q}] = 2g$

Theorem (T,2015)

$\exists \delta_g > 0$ *with*

$$|Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \gg |Disc(R)|^{\delta_g}$$

Corollary

For any $g \geq 1$, there are finitely many CM points in $\mathcal{A}_g$ over $\mathbb{Q}$.

For $g = 1$ this follows immediately from Complex Multiplicaiton theory and the Bruaer-Siegel Theorem. In general, the size is the image of one class group in another $\mathrm{CL}(K) \to \mathrm{CL}(L)$. The issue is torsion in class groups.

Largeness of Galois Orbits: Proof of $\mathcal{A}_g$ case

- Suppose x is defined over $\mathbb{Q}$, $R = \mathcal{O}_K$.
- $S_K = \{y \in \mathcal{A}_g, \mathrm{End}(A_y) = \mathcal{O}_K\}$.
- $|S_K| \gg |D_K|^{1/4}$, all points in S_K Isogenous, same Faltings height, field of definition.
- Faltings height of A_x is small $\left(|D_K|^{o(1)}\right)$, by **Average Colmez conjecture** (Andreatta-Goren-Howard-Madapusi-Pera,Yuan-Zhang).
- **Masser-Wustholz** $\Rightarrow$ there exist low degree isogenies between points of S_K.
- Not enough low degree isogenies exist.

General Shimura Varieties: Reduction to Height bound

- In general, no moduli interpertation, so no Masser-Wustholz. But recently Binyamini-Schmidt-Yafaev found another reduction.
- On the **transcendental** graph $\Gamma \subset \mathcal{H} \times S$, the CM points give many algebraic points.
- Using breakthrough results of Binyamini, greatly improving Pila-Wilkie estimates for special transcendental varieties (defined by foliations over number fields), one can conclude. Improvements include:
  - Poly-log bounds instead of sub-exponential bounds in the height. i.e. $N(X^{tran}, T) \ll \log(T)^{O(1)}$.
  - Polynomial bounds in the *degree* of the algebraic points, allowing one to count not just rational but algebraic points of varying degree.

Abelian Type vs. Non-Abelian Type

- Special subvarieties of $\mathcal{A}_g$ are *abelian-type*
- These have a universal family of Abelian Varieties $f : A \to S$.
- This gives models of S over $\mathbb{Z}$, not just $\mathbb{Q}$.
- Local systems $R^1 f_* \mathbb{Z}_\ell$, coherent sheaves $R^1 f_* \mathcal{O}$, all on $S_{\mathbb{Z}}$,

Abelian Type vs. Non-Abelian Type II

- Non-abelian type subvarieties: $G = E_6, E_7$ and some Orthogonal groups.
- No universal family, but on the other hand they all satisfy Margulis super-rigidity.
- Can still build Local systems $R^1 f_* \mathbb{Z}_\ell$, coherent sheaves $R^1 f_* \mathcal{O}$, but now only on $S_{\mathbb{Q}}$ (and with more work and less reward)
- Conjecturally, S still admits a universal family of Motives $M \to S$, but this is not known in a **single case**.
- CM points - $x \in S$ fixed by a maximal torus $T \subset G$. Can understand Galois action on these explicitly by embedding Abelian type subvarieties.

Idea: Canonical heights

- $\mathcal{L}$ - Automorphic line bundle on $S = S_K(G, X)$.
- Define a canonical height $h_{\mathcal{L}}$ on $S(\overline{\mathbb{Q}})$ with good functoriality properties (generalization of Faltings height).
- For a CM point $(T, X_T) \subset (G, X)$, bound $h_{\mathcal{L}}(T, X_T)$ by relating it to the Faltings height bounds coming from $\mathcal{A}_g$.
- It is crucial that the height function we choose is canonical beyond the usual $O(1)$ error term, since we are using different comparisons for different CM points.

Height functions: Shimura Varieties

- For abelian type Shimura varieties, we can use the **Faltings Height**, coming from the Neron model of Abelian varieties. Note this is still subtle, $h(j(di))$ is extremely hard to get a grip on as j is transcendental!
- For exceptional type Shimura Varieties, work of Diao-Lan-Liu-Zhu (Scholze) and Esnault-Groechenig establishes a p-adic Riemann-Hilbert correspondence, allows one to get out p-adic coherent sheaves from the p-adic local systems, and carry along an integral structure.
- This gives us a canonical height on arbitrary Shimura varieties!

CM points: partial CM types

- K-CM field
- $\Phi \subset \mathrm{Hom}(K, \overline{\mathbb{Q}}), \Phi \cap \overline{\Phi} = \emptyset$ - partial CM type

Using this data, get a CM point an an automorphic line bundle:

- $T - \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_m, h_\Phi : \mathbb{S} \to T_\mathbb{R}$
- $\rho_\Phi : T_\mathbb{C} \to \mathbb{G}_m$ gives an automorphic line bundle $\mathcal{L}_\Phi$.
- Need to bound $h_{\mathcal{L}_\Phi}$.
- $\mathcal{A}_g$ - the case of Φ a full CM type, $\Phi \cup \overline{\Phi} = \mathrm{Hom}(K, \overline{\mathbb{Q}})$.

CM points: Delignes Construction

- F- tot. real field, $K_1, K_2, \ldots, K_N$ - CM extensions of F.
- Φ_i - partial CM types on K_i such that $\Phi_i \mid_F$ partition $\mathrm{Hom}(F, \overline{\mathbb{Q}})$.
- $K_{tot} := K_1 K_2 \cdots K_n, \Phi_{tot} := \bigcup_i \pi_i^{-1} \Phi_i$ - Full CM type.
- $\displaystyle\sum_{i=1}^{n} h_{\mathcal{L}_{\phi_i}} = h_{\mathcal{L}_{\Phi tot}}.$
- For given F, there are many CM extensions e.g. $F(\sqrt{-p})$. Combining in all possible ways leads to individual bounds.

Construction of Canonical Height: Relative p-adic Hodge Theory

- Let $S = S_K(G, X)$, $\mathcal{L}$ - an automorphic line bundle. *Starts life over* $\mathbb{C}$*, descends to* $E(G, X)$ *by Milne+*.
- To define a canonical height on $\mathcal{L}$ we need a metric on $\mathcal{L}_v$ for every place v. At Archimedean places use Hodge metric.
- At finite places p need a height on $\mathcal{L}_v$. We begin by finding $\mathcal{L}$ as a hodge piece of a rational representation V of G. Riemann-Hilbert gives a vector bundle ${}_{\mathrm{dR}}V$, which comes with a hodge filtration $F^p_{\mathrm{dR}}V$, and we identify $\mathcal{L} \cong Gr^k_F{}_{\mathrm{dR}}V$.
- Work of Diao-Lan-Liu-Zhu (Scholze) defines a p-adic Riemann-Hilbert correspondence, giving a vector bundle ${}_{p-\mathrm{dR}}V$ on $S_{\mathbb{Q}_p}$ starting from the p-adic local system $V \otimes \mathbb{Q}_p$.
- For Shimura varieties, they show compatibility: ${}_{\mathrm{dR}}V$, when base changed to $\overline{\mathbb{Q}}_p$, becomes naturally isomorphic to ${}_{p-\mathrm{dR}}V$.

Under the hood: Intrinsic height

- K - local field
- L - Hodge-Tate representation of G_K over $\mathbb{Z}_p$.
- $L_n := (L \otimes \mathbb{C}_p(-n))^{G_K}) \otimes \mathbb{C}_p(n) \subset L \otimes \mathbb{C}_p$ has a natural norm on it.
- The *intrinsic norm* on L_n is induced from the quotient norm $\frac{L \otimes \mathbb{C}_p}{L_{<n} \otimes \mathbb{C}_p}$.
- The Liu-Zhu construction interpolates the intrinsic norm and shows that it is an admissible height at p.But hard to glue across primes!
- **Question: Is the Intrinsic norm the same as the 'naive' norm if L is Crystalline and weights are in $[0, p-2]$?**

Crystalline representations: The work of Esnault-Groechenig

- For almost all primes p, S has a good integral model at p.
- Duality: If S is exceptional, then everything is rigid!
- Esnault-Groechenig conclude that V is globally crystalline (Faltings-Fontaine-Lafaille) giving a good integral model, compatible with Liu-Zhu

Thank You!