Height zeta functions

Plan

1. Background
2. Analytic tools
3. Equivariant compactifications of additive groups

Background

Basic problem: Let X be a smooth projective variety over a number field k and $\mathcal{L}$ an adelically metrized (very) ample line bundle on X.

$$H_{\mathcal{L}} : X(k) \to \mathbb{R}_{>0}$$

be the associated height function. What is the asymptotic of

$$N(B) := \#\{x \in U(k) \mid H_{\mathcal{L}}(x) \leq B\}, \quad B \to \infty,$$

for suitable Zariski open $U \subset X$?

Background

This is the subject of conjectures of Manin, Batyrev, Peyre, ...

Main idea: Interplay between arithmetic and geometry, connecting recent (at that time) developments in the Minimal Model Program to classical questions in analytic number theory.

Marta Pieropan and Will Sawin gave very nice overviews, touching on

- torsors
- special subvarieties, thin sets and freeness
- geometric analogs, with k replaced by a function field of a curve over a finite field

Background

There are many interesting ideas and new directions in this area. To add a few to those already mentioned in previous talks – a very(!) incomplete list:

- extensions to integral points, Campana points
- mixing, ergodic theory (Duke–Rudnick–Sarnak, Eskin–McMullen, Oh, Mozes, Shah, Gorodnik, Maucourant, ...)
- geometric consistency (Lehmann–Tanimoto, Sengupta 2018)
- motivic framework (Chambert-Loir–Loeser 2013)
- homotopy theory ideas (Manin-Marcolli 2021)
- harmonic analysis on spherical varieties (Sakellaridis–Venkatesh 2018)

Plan

Most of my work in this area was joint with Franke, Manin, Batyrev, Strauch, Chambert-Loir, Shalika, Takloo-Bighash, Tanimoto, Gorodnik.

Today, I will focus on some basic methods that are common to many of our papers.

1. Review of tools
2. Simplest examples

Analytic tools

- Tauberian theorem(s)
- Convexity
- Harmonic analysis (Poisson summation formula)
- p-adic integration
- Convergence, integration by parts
- (Iterated residues)

Tauberian theorem

Consider

$$f(s) = \sum_{n \geq 1} \frac{a_n}{n^s}, \quad a_n > 0.$$

Assume that f

- is holomorphic for $\Re(s) > a > 0$,
- has an isolated pole at $s = a$, of order $b \in \mathbb{N}$, with leading coefficient $c \in \mathbb{R}$, $c \neq 0$.

Then

$$N(B) := \sum_{n \leq B} a_n \sim \frac{c}{a\Gamma(b)} \cdot B^a \log(B)^{b-1}, \quad B \to \infty.$$

Better versions assuming bounds in vertical strips.

Tauberian theorem – application

Consider the height zeta function

$$Z(s) := \sum_{x \in U(k)} H_{\mathcal{L}}(x)^{-s}, \quad U \subseteq X.$$

Its analytic properties translate into asymptotic properties of

$$N(B) = N(U, \mathcal{L}, B).$$

Convexity I

Let $V \subset \mathbb{R}^n$ be a connected open subset and $\bar{V}$ its convex envelope. $V + i\mathbb{R}^n$. Then $Z(\mathbf{s})$ is holomorphic in $\bar{V} + i\mathbb{R}^n$.

Convexity I – Application

To show that a function admits meromorphic continuation to a neighborhood of a cone it suffices to study its behavior near the vertex of the cone.

Convexity II

Theorem (Phragmen-Lindelöf)

Let ϕ *be a regular function for* $\Re(s) \in [\sigma_1, \sigma_2]$. *Assume that*

- $\phi(s) = \mathcal{O}(e^{\epsilon|t|})$, *for all* $\epsilon > 0$,
- $\phi(\sigma_1 + it) = \mathcal{O}(|t|^{k_1})$
- $\phi(\sigma_2 + it) = \mathcal{O}(|t|^{k_2})$

Then (uniformly):

$$\phi(\sigma + it) = \mathcal{O}(|t|^k),$$

with

$$\frac{k - k_1}{\sigma - \sigma_1} = \frac{k_1 - k_2}{\sigma_2 - \sigma_1}.$$

Convexity II – Application

For all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|\frac{(s-1)}{s} \cdot \zeta(\sigma + it)| = \mathcal{O}(|t|^{\epsilon}), \qquad \text{for } \sigma > 1 - \delta.$$

Proof: functional equation + bounds for the Gamma-function.

Similar estimates for Eisenstein series, etc.

Abstract Fourier analysis

- G - locally compact abelian group
- $\chi : G \to \mathbb{S}^1 \subset \mathbb{C}^\times$ a character, i.e., a continuous homomorphism; these form an abelian group, $\chi_0 = id$.
- $H \subseteq G$ a closed subgroup

Abstract Fourier analysis

Theorem (Poisson summation formula)

$$\int_H f\, dh = \int_{(G/H)^\perp} \hat{f}\, d\hat{g}$$

where

$$\hat{f}(\chi) := \int_G f(g)\chi(g) dg$$

and

$$(G/H)^\perp := \{\chi \quad | \quad \chi \text{ is trivial on } H\}$$

and $dh, d\hat{g}$ are suitably normalized Haar measures

Issues:

- measure
- integrability

Fourier analysis – applications

Let $G = \mathbb{G}_a^d$, or $\mathbb{G}_m^d$, and X an equivariant compactification of G. There is a way to extend

$$H_{\mathcal{L}} : G(\mathbb{A}_k) \to \mathbb{C},$$

such that the restriction to $G(k)$ is a standard height, as before. Then, for $\Re(s) \gg 0$, the height zeta function

$$Z(s, g) = \sum_{x \in G(k)} H_{\mathcal{L}}(xg)^{-s} \in \mathrm{L}^2(G(k) \backslash G(\mathbb{A}_k)),$$

and one can apply the Poisson summation formula.

Fourier analysis – applications

More generally, let G be a linear algebraic group, X its smooth projective equivariant compactification, $D := X \setminus G$ the boundary, assumed to be normal crossings:

$$D := \cup_{\alpha \in \mathcal{A}} D_\alpha .$$

There is a height pairing

$$H : G(\mathbb{A}_k) \times \prod_\alpha \mathbb{C} D_\alpha$$

(distance to the boundary).

Fourier analysis – applications

One can consider the height zeta function, for $\Re(\mathbf{s}) \gg 0$,

$$Z(\mathbf{s}, g) = \sum_{x \in G(k)} H(xg, \mathbf{s})^{-1} \in \mathrm{L}^2(G(k) \backslash G(\mathbb{A}_k))$$

and apply to it a spectral expansion

$$Z = \sum_{\pi} Z_{\pi},$$

"sum" over irreducible unitary representations.

Fourier analysis – applications

Sometimes, the main pole comes from the trivial representation

$$\int_{G(\mathbb{A}_k)} H(g, \mathbf{s})^{-1} \, dg.$$

This is analogous to

"number of lattice points is equal to the volume"

used in the torsor method.

Integrals of this type can be computed in closed form, in terms of the geometry of the boundary (Denef-Loeser).

p-adic integration / Denef-Loeser, ...

Igusa integrals and volume asymptotics (Chambert-Loir–T. 2010):

- Metrics, local heights, Tamagawa measures
- Clemens complexes: (X, D) smooth projective variety, with an snc divisor $D = \cup_{\alpha \in \mathcal{A}} D_\alpha$,

  $$D_A := \cap_{\alpha \in A} D_\alpha, \quad D_A^\circ = D_A \setminus \text{ intersections with other } D_\alpha,$$

  the corresponding geometric structure (Galois orbits) and analytic structure (points)
- Geometric Igusa integrals

What is left

- matching analysis with geometry
- convergence of the spectral extension

Example $G = \mathbb{G}_a$

An additive (p-adic) character is a continuous homomorphism

$$\psi_p : \mathbb{Q}_p \to \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}.$$

Concretely, put

$$\psi_a(x) = e^{2\pi i \{ax\}},$$

where

$$\{ax\} = ax - \underbrace{[ax]}_{\in \mathbb{Z}_p}.$$

Note that the left side is in $\mathbb{Q}$.

Measures

Fix Haar measures μ_p, for all primes p, normalized by

$$\mu_p(\mathbb{Z}_p) = 1,$$

we have

$$\mu_p(p^n\mathbb{Z}_p) = \frac{1}{p^n}, \quad \mu_p(\mathbb{Z}_p^\times) = 1 - \frac{1}{p}.$$

We also put

$$\mu_\infty = dx,$$

the standard Lebesgue measure on $\mathbb{R}$. This gives a Haar measure on $\mathbb{A}_\mathbb{Q}$.

Dualities

This gives a well-defined measure on the adeles:

$$\prod_p \mu_p \times \mu_\infty.$$

We also have a character

$$\psi_a : \mathbb{A}_\mathbb{Q} \to \mathbb{S}^1,$$

where $a \in \mathbb{A}_\mathbb{Q}$, defined as

$$\psi_a = \prod_p \psi_{a_p} \times \psi_{a_\infty},$$

a product of local characters. This is indeed a continuous homomorphism.

Dualities

$$\mathbb{Q} \subset \mathbb{A}_{\mathbb{Q}}$$

is cocompact, and self-dual, $(\mathbb{A}/\mathbb{Q})^{\perp} = \mathbb{Q}$. The Poisson summation formula takes the form

$$\sum_{x \in \mathbb{Q}} f(x) = \sum_{a \in \mathbb{Q}} \hat{f}(a),$$

where

$$\hat{f}(a) = \int_{\mathbb{A}_{\mathbb{Q}}} f(x)\psi_a(x)\mu(x),$$

provided we have convergence.

Application

Recall that

$$\mathbb{P}^1(\mathbb{Q}) := \{(x_0, x_1) \in (\mathbb{Z}^2_{prim} \setminus 0)/\pm\}.$$

Consider a height function:

$$H : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{R}$$

given by

$$H(x_0, x_1) = \sqrt{x_0^2 + x_1^2}$$

Let

$$N(B) := \#\{(x_0, x_1) \mid H(x_0, x_1) \leq B\}$$

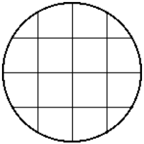
be the counting function.

Application

We are interested in the asymptotic of

$$N(B), \quad \text{for } B \to \infty.$$

This is nothing but the Gauss circle problem except that we are looking at coprime coordinates.



Application

We now translate this into the adelic language. Recall the product formula:

$$\prod |x|_p \cdot |x|_\infty = 1, \quad x \in \mathbb{Q}^\times.$$

The coprimality condition is nothing but

$$\max(|x_0|_p, |x_1|_p) = 1, \quad \forall p.$$

This allows to rewrite the problem as follows.

Application

Consider the following height function

$$H = \prod_p H_p \times H_\infty : \mathbb{A}_\mathbb{Q} \to \mathbb{R},$$

with local factors given by

$$H_p(x) := \max(1, |x|_p), \quad H_\infty(x) := (1 + x^2)^{1/2}.$$

Note that, for all p, the local height H_p is invariant under translation by $\mathbb{Z}_p$. We are interested in the asymptotic of

$$N(B) := \{x \in \mathbb{Q} \mid H(x) \leq B\}, \quad B \to \infty.$$

Height zeta function

Therefore, we introduce and study the function

$$Z(s) := \sum_{x \in \mathbb{Q}} H(x)^{-s}.$$

The series converges absolutely to a holomorphic function for $\Re(s) > 2$. By the Poisson summation formula, we have

$$Z(s) = \sum_{a \in \mathbb{Q}} \hat{H}(s, \psi_a)$$

where

$$\hat{H}(s, \psi_a) = \int_{\mathbb{A}_{\mathbb{Q}}} H(x)^{-s} \cdot \psi_a(x) \mu(x).$$

Height zeta function

Why is this any better? We started with a sum over $\mathbb{Q}$, and we again have a sum over $\mathbb{Q}$.

However, because H_p is invariant under $\mathbb{Z}_p$, only characters which are trivial on $\mathbb{Z}_p$, for all p contribute. So a must be in $\mathbb{Z}$.

Height zeta function

We write

$$Z(s) = \underbrace{\int_{\mathbb{A}_\mathbb{Q}} H(x)^{-s}\mu(x)}_{\text{trivial character}} + \underbrace{\sum_{a\neq 0} \cdots}_{\text{nontrivial characters}}$$

and analyze the terms.

- put $U(0) := \{x \mid |x|_p \leq 1\}$ and $U(j) := \{x \mid |x|_p = p^j\}$, note that

$$\mathrm{vol}(U(j)) = p^j(1 - \frac{1}{p}).$$

Local integrals

We have

$$\begin{aligned}\int_{\mathbb{Q}_p} H_p(x_p)^{-s}\mu_p &= \int_{U(0)} H_p(x_p)^{-s}\mu_p + \sum_{j\geq 1}\int_{U(j)} H_p(x_p)^{-s}\mu_p \\ &= 1+\sum_{j\geq 1} p^{-js}\mathrm{vol}(U(j)) \\ &= \frac{1-p^{-s}}{1-p^{-(s-1)}}\end{aligned}$$

$$\int_{\mathbb{R}}(1+x^2)^{-s/2}dx = \cdots$$

The Euler product gives

$$\frac{\zeta(s-1)}{\zeta(s)}\cdot\Gamma()$$

which has a simple pole at $s=2$ with residue $\frac{1}{\zeta(2)}\cdot\Gamma()$.

Characters, once again

- $\mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}/\mathbb{Z}$
- $\psi_p : x_p \mapsto e^{2\pi i a_p \cdot x_p}$, with $a_p \in \mathbb{Q}_p$
- unramified: trivial on $\mathbb{Z}_p$, i.e., $a_p \in \mathbb{Z}_p$
- $\psi_\infty : x \mapsto e^{2\pi i a \cdot x}$, $a \in \mathbb{R}$
- $\psi = \prod_p \psi_p \cdot \psi_\infty = \psi_a$, with $a \in \mathbb{A}_\mathbb{Q}$
- duality $\hat{\mathbb{Q}}_p = \mathbb{Q}_p$ and $\hat{\mathbb{R}} = \mathbb{R}$
- $(\mathbb{A}_\mathbb{Q}/\mathbb{Q})^\vee$ (characters trivial on $\mathbb{Q}$) $= \mathbb{Q}$
- $\psi = \psi_a$ unramified for all $p \Rightarrow a \in \mathbb{Z}$.

Characters, once again

For $a \in \mathbb{Z} \setminus 0$ and $p \nmid a$, we compute

$$\hat{H}_p(s, \psi_a) = 1 + \sum_{j \geq 1} p^{-sj} \int_{|x|_p = p^j} \psi_a(x_p)\mu_p = 1 - p^{-s}.$$

Characters, once again

Proof: let $V(i)$ be the set of $x_p \in \mathbb{Q}_p$ with $H_p(x) \leq p^i$. Then

$$\int_{V(i)} \psi_a(x)\mu_p = p^{in} \int_{\mathbb{Z}_p} \psi_{a/p^i}(x)\mu_p.$$

For $i \geq 1$ and $p \nmid a$, we integrate a nontrivial character over a compact group, thus get 0. For $i = 0$, we get 1. Since $U(i) = V(i) \setminus V(i-1)$, we have

$$\int_{U(i)} \psi_a(x)\mu_p = \begin{cases} 0 & i \geq 2 \\ -1 & i = 1 \end{cases},$$

which implies the claim.

Characters, once again

For $\Re(s) > 1 + \epsilon$, and $p \mid a$, replace ψ by 1 and estimate

$$|\hat{H}_p(s, \psi_a)| \leq \frac{1}{1 - p^{-\epsilon}}.$$

For any positive integer a we have

$$\prod_{p|a} \frac{1}{p^{\epsilon}} \ll \log(1 + a).$$

Thus

$$\prod_{p|a} |\hat{H}_p(s, \psi_a)| \ll (1 + |a|)^{\delta}$$

for some (small) $\delta > 0$.

continued

$$Z(s) = \frac{\zeta(s-1)}{\zeta(s)} \cdot \Gamma() +$$

$$\sum_{a \in \mathbb{Z}} \prod_{p \nmid a} \frac{1}{\zeta_p(s)} \cdot \prod_{p | a} \hat{H}_p(a_p, s) \cdot \int_{\mathbb{R}} (1 + x^2)^{-s/2} \cdot e^{2\pi i a x} dx$$

For $\Re(s) > 2 - \delta$, one has:

- $| \prod_{p|a} \hat{H}_p(s, a)| \ll | \prod_{p|a} \int_{\mathbb{Q}_p} H_p(x_p)^{-s} \mu_p| \ll (1 + |a|)^{\delta}$
- $| \int_{\mathbb{R}} (1 + x^2)^{-s/2} \cdot e^{2\pi i a x} dx| \ll_N \frac{1}{(1+|a|)^N}$, for any $N \in \mathbb{N}$, (integration by parts)

This gives a meromorphic continuation of $Z(s)$ and its pole at $s = 2$.

Results

All conjectures holds when X is one of the following varieties:

- (Franke) G/P, (Strauch) twisted products of G/P
- (Batyrev-T.) $X \supset$ T
- (Strauch-T.) $X \supset$ G/U
- (Chambert-Loir-T.) $X \supset \mathbb{G}_a^n$
- (Shalika-T.) $X \supset$ U (bi-equivariant)
- (Shalika-Takloo-Bighash-T.) $X \supset$ G
  De Concini-Procesi varieties
- etc

Open problems?

How about the $(ax + b)$-group, i.e.,

$$1 \to \mathbb{G}_a \to G \to \mathbb{G}_m \to 1$$

Partial results in joint work with Tanimoto, general case still open!