Computing Isolated Points on Modular Curves

Nicholas Triantafillou
University of Georgia

VaNTAGe Seminar
November 10, 2020

Kaan Bilgin, Galatasaray University
Atharva Korde, University of British Columbia
Michelle Manes, National Science Foundation
Travis Morrison, Virginia Tech
Matias Moya Giusti, Université de Lille
Soumya Sankar, MSRI / Ohio State
Bianca Viray, University of Washington
David Zureick-Brown, Emory University

Mazur's Torsion Theorem (1978)

Let $E/\mathbb{Q}$ be an elliptic curve. Then, $E(\mathbb{Q})_{\text{tors.}}$ is isomorphic to one of the following groups:

$\mathbb{Z}/N\mathbb{Z}$ for $1 \leq N \leq 10$ or $N=12$, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2N\mathbb{Z}$ for $1 \leq N \leq 4$.

Mazur's Isogeny Theorem (Mazur + Kenku 1978-1982)

Let $E/\mathbb{Q}$ be an elliptic curve. If E has a cyclic subgroup of order N which is defined over $\mathbb{Q}$ then

$$N \in \{1, 2, \ldots, 19, 21, 25, 27, 37, 43, 67, 163\}.$$

Geometric Version (Modular Curves)

$$\left\{ (\underset{\text{elliptic curve}}{E}, \underset{\text{subgroup} \cong \mathbb{Z}/N\mathbb{Z}}{G}) \right\} \Big/ \sim \quad \longleftrightarrow \quad \text{points on modular curve } X_0(N) \text{ except "cusps".}$$

$$X_0(N)(\mathbb{Q}) \subseteq \{\text{cusps}\} \quad \text{unless} \quad N \in \{ \ldots, 163\}.$$

Can explicitly parameterize $(E, G)/\sim$ over $\mathbb{Q}$ by $X_0(N)(\mathbb{Q})$

$\mathbb{P}^1$

$\Gamma_{\mathbb{Q}}$, $\mathbb{Q}$-points of an elliptic curve, or finite set.

–

What happens over number fields?

1996 (Merel) For $[K:\mathbb{Q}] \le d$, finitely many possibilities for $E(K)_{tors.}$

Which $E(K)_{tors.}$ appear...	$[K:\mathbb{Q}] = 2$	$[K:\mathbb{Q}] = 3$
∞'ly often	1980s Kamienny Kenku Momose	2004 Jeon, Kim, Schweizer
At least once		2020 Etropolski, Morrow, Zureick-Brown, Derickx, van Hoeij

one extra possibility. ~ "isolated" point on $X_1(21)$. (Najman 2014)

("Isolated Points" – Bourdon–Ejder–Liu–Odumodu–Viray)

(Isolated Points - Bourdon Ejder Liu Odumodu Viray)

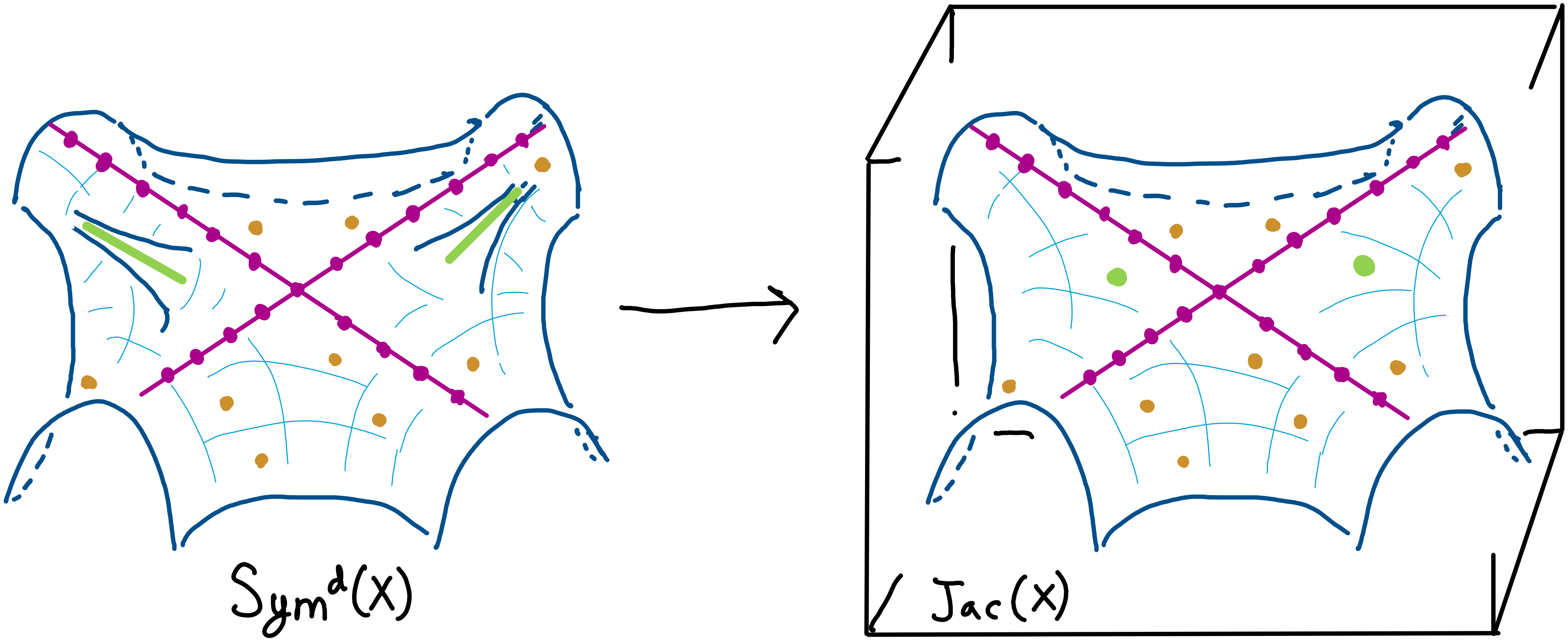
Let $X/\mathbb{Q}$ be a nice curve, $P_0 \in X(\mathbb{Q})$.

$$\left\{ \begin{array}{c} \text{degree } d \\ \text{points on} \\ X \end{array} \right\} \subseteq \left\{ \begin{array}{c} \mathbb{Q}\text{-points on} \\ \mathrm{Sym}^d(X) \approx X \times \cdots \times X / S_d \end{array} \right\}$$

$$\begin{aligned} j: \mathrm{Sym}^d X &\longrightarrow \mathrm{Jac}(X) \\ (P_1, \ldots, P_d) &\longmapsto \left[\sum_{i=1}^{d} P_i - d \cdot P_0 \right] \end{aligned}$$

1. Fibers $\cong \mathbb{P}^n$, $n \in \mathbb{Z}_{\geq 0}$. (linear systems)
2. (Faltings' Theorem '91)

$$\{\mathbb{Q}\text{-points of image}\} \subseteq \bigcup_{i=1}^{d} (A_i(\overset{\text{rank} > 0}{\mathbb{Q}}) + D_i) \cup \{\text{finitely many points}\}$$



$$\left\{\begin{array}{c}\text{degree } d \\ \text{points on} \\ X\end{array}\right\} = \left\{\begin{array}{c}\mathbb{P}^1\text{-} \\ \text{parameterized} \\ \text{points}\end{array}\right\} \cup \left\{\begin{array}{c}\text{AV-} \\ \text{parameterized} \\ \text{points}\end{array}\right\} \cup \left\{\begin{array}{c}\text{isolated} \\ \text{points} \\ \text{(finite)}\end{array}\right\}$$

Goal: Compute and study {isolated points} in all degrees

Goal: Compute and study {isolated points} in all degrees on $X_0(N)$ when $\operatorname{rank} J_0(N)(\mathbb{Q}) = 0$.

cusps? CM-points? $\mathbb{Q}$-curves?

Strategy: 1. Compute $J_0(N)(\mathbb{Q}) = J_0(N)(\mathbb{Q})_{tors.}$

2. For $D \in J_0(N)(\mathbb{Q})$, compute effective divisors $\sim [D + d \cdot P_i]$ (Riemann-Roch space)

dim $= 0 \Leftrightarrow$ no points   dim $= 1 \Leftrightarrow$ isolated point   dim ≥ 2 - $\mathbb{P}^1$ parameterized

1. To compute $J_0(N)(\mathbb{Q})_{tors.}$, try

Upper bound: $J(\mathbb{Q})_{tors.}$-prime to $p \hookrightarrow J(\mathbb{F}_p)$

Lower bound: Subgroup generated by cusps.

-

Current

Progress

Progress

Before Us:

$X_0(N)$ w/ rank $J_0(N)(\mathbb{Q}) = 0$.

(Bruin - Najman)

Quadratic Points on hyperelliptic $X_0(N)$.

(Ozman - Siksek)

Quadratic Points on non-hyperelliptic $X_0(N)$.

in genus 3, 4, 5

in genus 3, 4, 5.

Rank 0 $J_0(N)(\mathbb{Q})$ by genus

	2	3	4	5	6	...
N=	22 ✓	30 ✓	38 $d \leq 3$	42 tors.	71 ✓	for the future
	23 ✓	33 ✓	44 ✓	51 $d \leq 1$		
	26 ✓	34 ✓	47 ✓	52 ✓		
	28 ✓	35 ✓	54 ✓	55 tors.		17 more
	29 ✓	39 ✓	81 ✓	56 $d \leq 2$		up to N=100
	31 ✓	40 ✓		59 ✓		
	50 ✓	41 ✓		63 tors.		
		45 ✓		72 tors.		
		48 ✓		75 tors.		
		64 ✓				

confirms Bruin-

Najman new content

$j(x)$	$\mathrm{disc}(\mathbf{k}(j(x)))$	$\deg(x)$	#	Notes
∞	1	1	4	
$j+3375$	-7	2	2	CM
$j^2-\frac{18269}{2}j+\frac{41781923}{2}$	-7	2	1	
$j^2-\frac{4421}{2^2}j+\frac{1225043}{2^2}$	-7	2	1	
$j^2-\frac{20446557107}{2^{11}}j+\frac{1398848417083180430}{2^{11}}$	-7	2	1	
$j^2+\frac{15856001239}{4194304}j+\frac{21547612858283}{4194304}$	-7	2	1	
$j^2-\frac{70948241828986}{31381059609}j+\frac{4054697705872832 3}{31381059609}$	-8	2	1	
$j^2+\frac{3878}{3}j+\frac{11697083}{3}$	-8	2	1	
$j^2+\frac{34720274}{3^2}j+\frac{27496736717468 3}{3^2}$	-8	2	1	
$j^2+\frac{31638611846}{3^{11}}j+\frac{619979193949424 3}{3^{11}}$	-8	2	1	
$j^2-\frac{2124233110387369 37}{2^{11}3^{22}}j+\frac{1758389698176000094 09}{2^{11}3^{22}}$	-47	2	1	
$j^2-\frac{12167}{2^2 3}j+\frac{148035889}{2^2 3}$	-47	2	1	
$j^2-\frac{405551387718992219}{2^{22}3^{11}}j+\frac{1080278796400376469 44689}{2^{22}3^{11}}$	-47	2	1	
$j^2+\frac{4261}{2\cdot 3^2}j+\frac{24137569}{2\cdot 3^2}$	-47	2	1	
$j^2-9476266420414j-29164271838685291043$	33	2	1	
$j^2-2050534j-1672446203$	33	2	1	

TABLE 0.1. Isolated points on $X_0(22)$

What's next / Open problems:

- Continue computations, look for patterns.
  - Torsion subgroups
  - Faster Riemann-Roch - (seiving for $d < g$)

- Higher rank case:
  - Use p-adic methods (Chabauty) to cut out $\mathbb{P}^1$-param + sporadic locus, positive rank AV-translates

- Is $\{\text{deg } \underline{\underline{d}} \text{ sporadic } j\text{-invariants}\}$ on $X_0(N)$ bounded a.la. Bourdon - Ejder - Liu - Odumodu - Viray for $X_1(N)$