Algorithms for certifying nontriviality of Ceresa cycles

Joint work with Jordan Ellenberg &
Adam Logan &
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(X, p) "nice" genus g pointed curve$/K$

$p \in X(K)$

g-dim'l group variety/k

$\rightsquigarrow i_p : X \hookrightarrow J := \mathrm{Jac}(X)$

$x \longmapsto [x - p]$

$$J := \frac{Z_0^{\deg=0}(X)}{Z_0^{\mathrm{rat}}(X)}$$

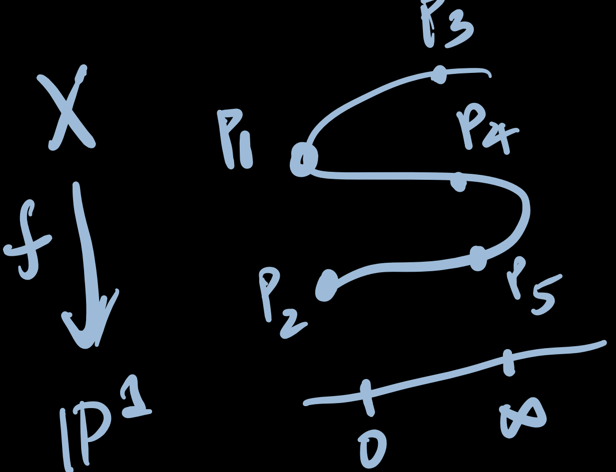
$Z_0^{\deg=0}(X) \rightsquigarrow$ Degree 0 divisors, Group of "homologically trivial" 0 cycles

Subgroup $\subset Z_0^{\deg=0}(X)$ of "ratly trivial" 0 cycles $= \{\mathrm{div}(f)\}$

$/\mathbb{C}$ $\{\sum_{P\in X} n_P P \mid \sum n_P = 0\}$

$= \ker\left[Z_0(X) \xrightarrow{\mathrm{cyc}} H^2(X,\mathbb{Z})\right]$, where $H^2(X,\mathbb{Z}) = \mathbb{Z}$

$\sum n_P P \mapsto \sum n_P$



$$\mathrm{div}(f) = 2P_1 + P_2 - P_3 - P_4 - P_5$$

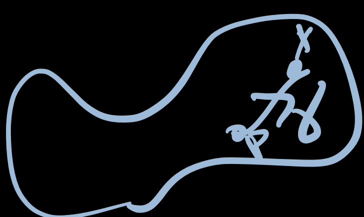
MOTIVATING QUESTION:

How large is $J(K)$?

A: Depends on K.

• $\underline{K = \mathbb{C}}$, $J(\mathbb{C}) \xrightarrow[\sim]{\text{Abel-Jacobi}} \dfrac{\mathbb{C}^g}{\mathbb{Z}^{2g}} = \dfrac{H^0(X,\Omega^1)^\vee}{H_1(X,\mathbb{Z})}$

$\int$



$$[x-p] = \partial\gamma \longmapsto [\omega \mapsto \int_\gamma \omega]$$

• K <u>number field</u>: Mordell-Weil

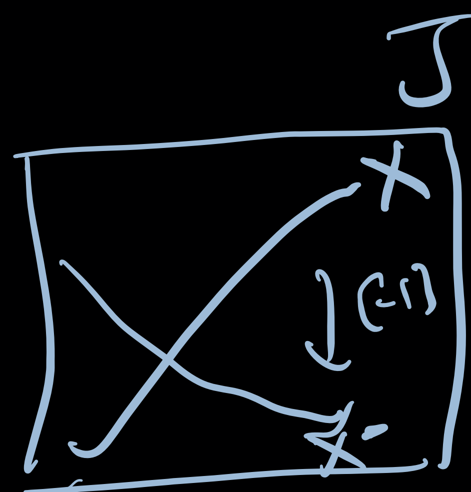
$J(K)$ is fintely generated.

$$J(K) \xrightarrow{\delta} Sel(J) \subset H^1(G_K, H^1(X_{\bar{K}}, \mathbb{Z}_\ell))$$

TODAY: 1-dim'l subvarieties on $\mathrm{Jac}(X)$

$Z_1(J) =$ Free ab. grp. on 1-dim'l subvar of J (defined over K)

Example/Defn:



The Ceresa cycle

$$C(X,p) := i_p(X) - i_p^-(X) \in Z_1(J)$$

$$= X - X^-$$

$$X \xhookrightarrow{i_p} J$$
$$x \longmapsto [x-p]$$

$$X \xhookrightarrow{i_p^-} J$$
$$x \longmapsto [p-x]$$

A natural filtn. on $Z_1(J)$

Potentially new!

(A)

$$Z_1^{rat}(J) \subset Z_1^{alg}(J) \subset Z_1^{hom}(J) \subset Z_1(J)$$

$Z_1^{rat}(J) = \{\mathrm{div}(f)\}$

$Z_1^{hom}(J) = \ker(\mathrm{cyc})$

$$Z_1(J) \xrightarrow{\mathrm{cyc}} H^{2g-2}(J_{\bar{k}}, \mathbb{Z}_\ell(g-1))$$

like deg 0 div.

dim 2

$$W \subset J$$
$$\downarrow f$$

$\mathbb{P}^1$

$0 \qquad \infty$

$2C_1 + C_2 - C_3 - C_4 - C_5$

NEW MOTIVATING QUESTION:

~~How large is~~ $\cancel{\dfrac{Z_1^{hom}(J)}{Z_1^{rat}(J)}}$ ~~?~~

~~(Beilinson–Bloch conjectures)~~

How deep in (★) does $C(X,p)$ lie?

Some Answers:

- $X - X^- \in Z_1^{hom}(J)$ since $[-1]^*$ acts trivially on H^{2g-2}

Answer depends on X & p!

- X hyperelliptic, b Weierstrass pt

$\Rightarrow$ $X = X^-$ in $Z_1(J)$

$\Rightarrow$ $C(X, b) = 0$.

- Ceresa (1983): X generic curve, genus $g \geq 3$

$n\, C(X,p) \notin Z_1^{alg}(J)$ for every nonzero n

Tool: Degeneration + $\mathbb{C}$-Hodge thy.

☆ Betts–Li: p-adic Hodge-theoretic analogue (Ongoing)

• B. Harris (1983)

X Fermat quartic $x^4+y^4=z^4_{\mathbb{C}\mathbb{P}^2}$

$$C(X,p) \notin Z_1^{alg}(J)$$

Tool: $\int$ Integration:

A certain period integral $\notin \mathbb{Z}$.

• Bloch (1984)

X Fermat quartic

$n\, C(X,p) \notin Z_1^{alg}(J)$ for every nonzero n

Tool: $\delta : Z_1^{hom}(J) \to H^1(G_K, H^{2g-3}(J_{\bar{K}}))$

ℓ-adic

• Extending Bloch + B. Harris

to low degree Fermat curves

+ Fermat quotients

Top, Kimura, Tadokoro, Otsubo

1989 2000 2003 2012

- Eskandari-Murty, 2021

Fermat curve X_m

s.t. $\exists$ prime $p > 7$ s.t. $p \mid m$

$n.C(X_m, p) \notin Z_1^{rat}(J)$

for every nonzero n.

Tool: • Dimension redn. using symmetries

$$Z_1^{hom}(J) \longrightarrow Z_0^{hom}(X) \twoheadrightarrow J(\mathbb{Q})$$

- Arithmetic of J:
  $J(\mathbb{Q})$ is infinite
  (Gross-Rohrlich)

Last time:

★ Laga-Shnidman (2024)
Picard curves $\circlearrowright$ $\mathbb{Z}/3\mathbb{Z}$
$g=3$

(Also used dimn. redn., using Chow motives.)

☆ Kerr-Li-Qiu-Yang (2024):
Modular curves
(Dimn. redn. using special cycles)

ALL TOOLS SO FAR APPLY
TO CURVES WITH SYMMETRIES!

MOTIVATING QUESTION:

What can you say abt. the Ceresa cycle of a random curve?

Guess: Infinite order.

Certify?

TODAY'S MAIN THM. / ALGORITHM:

[Joint work with Jordan Ellenberg, Adam Logan & Akshay Venkatesh

INPUT: A "nice" curve X/K

OUTPUT: A certificate that $C(X,b)$ has ∞ order, or does not terminate

For e.g. when run on 254 704 ht-1 smooth plane quartics/$\mathbb{Q}$ (i.e. $X^4 + X^3 y + \dots$)

Coeff. in $\{-1,0,1\}$

In MAGMA, the algo. failed to terminate in 102 cases i.e. remaining 254602 have provably no order in the Chow group

$$CH_1(J) = Z_1^{hom}(J) / Z_1^{rat}(J)$$

Rmks:

1) All 102 exceptions have known symmetries, includes known torsion examples, $y^3 = x^4 - 1$
$y^3 = x^4 - x$.

2) TOOL: ℓ-adic AJ map.

☆ Ongoing [Besser–S]:

explicitly compute p-adic AJ image using iterated Coleman intn., can certify nontriviality in

$$\mathrm{Gr}(J) = Z_1^{hom}(J) / Z_1^{alg}(J)$$

MANTRA:

CYCLES ARE HARD,

CYCLE CLASSES ARE EASIER!

TOOL: Bloch's (refined) ℓ-adic cycle class map

$$Z_1^{hom}(J) \longrightarrow \prod_{\ell} H^1(G_K, H^{2g-3}(J_{\bar{K}}, \mathbb{Z}_\ell(g-1)))$$

$$C(X,b) \longmapsto (\nu_\ell(X,b))_\ell = \nu(X)$$

=: Ceresa class

EASIER GOAL: Certify $\nu(X)$ has ∞ order.

TWO STEPS APPROACH:

STEP 1: Compute upper bound $\boxed{N}$ on $\# \operatorname{im}(\mathrm{cyc})_{\mathrm{tors}}$

(Idea: Kummer. seq + weights)

STEP 2) Choose an auxiliary good prime p & compute a lower bound $\boxed{N_\ell}$ on $\operatorname{ord}(\nu(x))$.

[IDEA: Curves acquire symmetries on redn.!]

ALGORITHM: Compute $\boxed{N}$ & $\boxed{N_p}$ for $p \leq$ chosen bound

If $\exists$ p with $N_p \nmid \underline{N}$, output $v(x)$ has ∞ order.

<u>Thm:</u> (Chebotarev):

If $G_K \curvearrowright H^1$ is maximal,

& $\gamma(X)$ has ∞ order,

then N_p is unbounded as

p varies.