Moduli Problems and Moduli Spaces in Algebraic Dynamics

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All the lonely objects, Where do they all come from?

There are lots of types of interesting mathematical objects that people like to study. For example:

- Elliptic curves | higher dimensional abelian varieties.
- Curves of genus g | K3 surfaces | other varieties.
- Subvarieties of $\mathbb{P}^n$.
- k-tuples of points in $\mathbb{P}^n$.
- Maps $f : X \to Y$ between given (varieties) X and Y.
- Maps $f : X \to X$ from a given (variety) X to itself.
- …and the list goes on and on …

Spoiler: The primary topic of this talk will be the last sort of example, and more specifically maps

$$f : \mathbb{P}^n \longrightarrow \mathbb{P}^n.$$

All the lonely objects, Where do they all belong?

There's a lot of great mathematics that has the form:

> Here's my favorite object X of type T. I'm going to prove some cool facts about X.

There's also lots of great mathematics that has the form:

> Objects of type T are fascinating, so I'm going to prove that every object of type T has these cool properties.

The second type of theorem involves looking at all of the objects of type T, so it makes sense to look at the set(?) of those objects. For example:

- The set of all abelian varieties.
- The set of all morphisms $\mathbb{P}^n \to \mathbb{P}^n$.

But these "sets" are large and unwieldy. We can look at better behaved subsets by adding restrictions.

All the lonely objects, Where do they all belong?

What sorts of restrictions? For example:

- There are "too many" abelian varieties, so look at

  principally polarized abelian varieties of a fixed dimension.

- There are "too many" subvarieties of $\mathbb{P}^n$, so look at

  subvarieties of a fixed dimension and degree.

- The are "too many" maps $\mathbb{P}^n \to \mathbb{P}^n$, so look at

  finite maps $\mathbb{P}^n \to \mathbb{P}^n$ of a fixed degree.

It's often helpful to add some structure. For example:

- The set of pairs (E, P) consisting of an elliptic curve E and a point $P \in E$ of order N.
- The set of pairs (f, P) consists of a map $f : \mathbb{P}^n \to \mathbb{P}^n$ and a fixed point $f(P) = P$.

Ah, look at all the lonely objects

One learns early in one's mathematical career that

Isomorphic objects are "the same"

So when studying the set of objects of type T, we should treat X and Y as being equivalent if

$$X \cong Y.$$

Goal: Classify the objects of type T up to equivalence

For example, we could classify . . .

- abelian varieties up to isomorphism; or up to isogeny;
- varieties up to isomorphism; or up to birational isomorphism;
- maps $f : X \to \mathbb{P}^1$ up to change of coordinates on $\mathbb{P}^1$, i.e.,

$$f \cong g \text{ if } f = \alpha \circ g \text{ for some } \alpha \in \operatorname{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2.$$

- maps $\mathbb{P}^n \to \mathbb{P}^n$ up to **dynamical equivalence**.

Dynamical Systems

An abstract **dynamical system** is a pair (X, f) consisting of an object X and a self-map

$$f : X \longrightarrow X.$$

Dynamics is the study of the **iterates of f**,

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n\text{th iterate}}.$$

Typically X is a set, the map f is a function, and we want to classify the points $x \in X$ according to the behavior of their **orbits**

$$\mathcal{O}_f(x) = \{x, f(x), f^2(x), f^3(x), \ldots\}.$$

Equivalence of Dynamical Systems

The dynamics doesn't change if we "change coordinates on X," but we need to change coordinates to be compatible with iteration.

Definition: The dynamical systems $f : X \to X$ and $g : X \to X$ are **dynamically equivalent** if

$$g = \underbrace{\phi^{-1} \circ f \circ \phi}_{\text{denote this } f^{\phi}} \quad \text{for some } \phi \in \mathrm{Aut}(X).$$

This is the "right" notion of equivalence for dynamics:

$$\begin{array}{ccc} X & \xrightarrow{f^{\phi}} & X \\ \phi \downarrow & & \downarrow \phi \\ X & \xrightarrow{f} & X \end{array}$$

$$(f^{\phi})^n = (f^n)^{\phi}$$

$$\mathcal{O}_{f^{\phi}}(\phi^{-1}(x)) = \phi^{-1}\left(\mathcal{O}_f(x)\right)$$

Undergrad Example: Classify linear operators $L : V \to V$ up to change of coordinates, i.e., classify matrices A up to conjugation $A \sim B^{-1}AB$.

All the lonely objects, Where do they all belong?

Our mission is now clear:

Describe the objects of type T up to isomorphism, i.e., describe the equivalence classes

- That's fine, we get a set of equivalence classes.
- But it would be nice if the set of equivalence classes itself had some nice structure.

For example:

- The set of isomorphism classes of elliptic curves is naturally identified with $\mathbb{A}^1$ via the j-invariant.
- The set of isomorphism classes of principaly polarized abelian varieties of dimension g is naturally identified with an algebraic variety $\mathfrak{A}_g$ of dimension $\frac{1}{2}g(g+1)$.
- The set of isomorphism classes of degree d dynamical systems $f : \mathbb{P}^n \to \mathbb{P}^n$ is naturally identified with **...**

And thus our tale begins...

Dynamics in Dimension 1

Rational Maps of $\mathbb{P}^1$

We look at rational functions

$$f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \quad \text{of degree } d \geq 2.$$

Thus f has the form

$$f(z) = \frac{a_0 z^d + \cdots + a_d}{b_0 z^d + \cdots + b_d} \in \mathbb{C}(z).$$

First Observation: We get the same f if we multiply the numerator and the denominator by a non-zero constant, so f is determined by a point in projective space

$$f_{\boldsymbol{a},\boldsymbol{b}} \longleftrightarrow [a_0, \ldots, a_d, b_0, \ldots, b_d] \in \mathbb{P}^{2d+1}.$$

Second Observation: f has exact degree d if and only if its numerator and denominator have no common roots, i.e.,

$$\boxed{\deg(f) = d \iff \operatorname{Res}(a_0 z^d + \cdots + a_d, b_0 z^d + \cdots + b_d) \neq 0.}$$

Dynamical Equivalence for Maps of $\mathbb{P}^1$

Third Observation: The underlying dynamics remains the same if we simultaneously change change coordintaes on $\mathbb{P}^1$.

Thus if we take a linear fractional transformation

$$\phi(z) = \frac{\alpha z + \beta}{\gamma z + \delta} \in \mathrm{PGL}_2,$$

then

$$f(z) \text{ and its conjugate } f^{\phi}(z) := \phi^{-1} \circ f \circ \phi(z)$$

have equivalent dynamics.

- The linear map ϕ acts on the coefficients of f, so we may view it as acting on the points of $\mathbb{P}^{2d+1}$.
- It easy to see that the action on $\mathbb{P}^{2d+1}$ is linear, and indeed we get a homomorphism

$$\rho_\phi : \mathrm{PGL}_2 \longrightarrow \mathrm{PGL}_{2d+2} \quad \text{satisfying} \quad f^{\phi}_{\boldsymbol{a},\boldsymbol{b}} = \rho_\phi\bigl([\boldsymbol{a},\boldsymbol{b}]\bigr).$$

An Example: Dynamical Equivalence for Degree 2 Maps of $\mathbb{P}^1$

Example: For $\deg(f) = 2$, the linear transformation

$$\phi = \frac{\alpha z + \beta}{\gamma z + \delta}$$

acts on the space of degree 2 maps via a homomorphism

$$\rho_\phi : \mathrm{PGL}_2 \longrightarrow \mathrm{PGL}_6\,.$$

$$\rho_\phi = \begin{pmatrix} \alpha^2\delta & \alpha\gamma\delta & \gamma^2\delta & -\alpha^2\beta & -\alpha\beta\gamma & -\beta\gamma^2 \\ 2\alpha\beta\delta & \alpha\delta^2 + \beta\gamma\delta & 2\gamma\delta^2 & -2\alpha\beta^2 & -\alpha\beta\delta - \beta^2\gamma & -2\beta\gamma\delta \\ \beta^2\delta & \beta\delta^2 & \delta^3 & -\beta^3 & -\beta^2\delta & -\beta\delta^2 \\ -\alpha^2\gamma & -\alpha\gamma^2 & -\gamma^3 & \alpha^3 & \alpha^2\gamma & \alpha\gamma^2 \\ -2\alpha\beta\gamma & -\alpha\gamma\delta - \beta\gamma^2 & -2\gamma^2\delta & 2\alpha^2\beta & \alpha^2\delta + \alpha\beta\gamma & 2\alpha\gamma\delta \\ -\gamma\beta^2 & -\beta\gamma\delta & -\gamma\delta^2 & \alpha\beta^2 & \alpha\beta\delta & \alpha\delta^2 \end{pmatrix}.$$

What a mess!! And that's the simplest non-trivial case!!

The Moduli Space of Dynamical Systems on $\mathbb{P}^1$

The space that classifies degree d maps $\mathbb{P}^1 \to \mathbb{P}^1$ up to dynamical equivalence is the quotient space

$$\frac{\left\{\text{maps } \mathbb{P}^1 \xrightarrow{\deg d} \mathbb{P}^1\right\}}{\text{action of } \mathrm{PGL}_2} = \frac{\mathbb{P}^{2d+2} \setminus \{\mathrm{Res} = 0\}}{\text{action of } \mathrm{PGL}_2 \text{ via } \rho_\phi}.$$

Definition:

$$\mathcal{M}_d^1 = \frac{\mathbb{P}^{2d+2} \setminus \{\mathrm{Res} = 0\}}{\text{action of } \mathrm{PGL}_2 \text{ via } \rho_\phi}$$

Theorem. [Levy, Milnor, JS]
(a) $\mathcal{M}_d^1$ has a natural structure as an algebraic variety.*
(b) $\mathcal{M}_2^1 \cong \mathbb{A}^2$.
(c) For all $d \geq 2$, the variety $\mathcal{M}_d^1$ is a rational variety.

* $\mathcal{M}_d^1$ exists as a geometric quotient over $\mathbb{Z}$, in the sense of geometric invariant theory.

Adding Level Structure

Adding Level Structure: A Classical Example

Problem: Classify isomorphism classes of pairs

$$(E, P) \quad \text{such that} \quad \begin{cases} E \text{ is an elliptic curve,} \\ P \text{ is a point of exact order } n. \end{cases}$$

This set of (E, P) is classified by the points of the

elliptic modular curve $Y_1(n)$.

It is hard to overstate the importance of elliptic modular curves. They play a fundamental role in theorems ranging from Mazur's uniform boundedness result to Wiles' proof of Fermat's last theorem.

More generally, $\mathfrak{A}_g(n_1, \ldots, n_r)$ classifies p.p. abelian varieties A with points $P_1, \ldots, P_r$ of order $n_1, \ldots, n_r$.

The geometry of moduli spaces is very important:

Theorem. (a) genus $Y_1(n) \to \infty$ as $n \to \infty$.
(b) (Tai 1982) $\mathfrak{A}_g$ is of general type for $g \geq 9$.

Periodic and Preperiodic Points

Let $f : X \to X$ be a dynamical system and let $P \in X$.

P is **preperiodic** if its orbit $\mathcal{O}_f(P)$ is finite.

P is **periodic** if $f^n(P) = P$ for some $n \ge 1$.

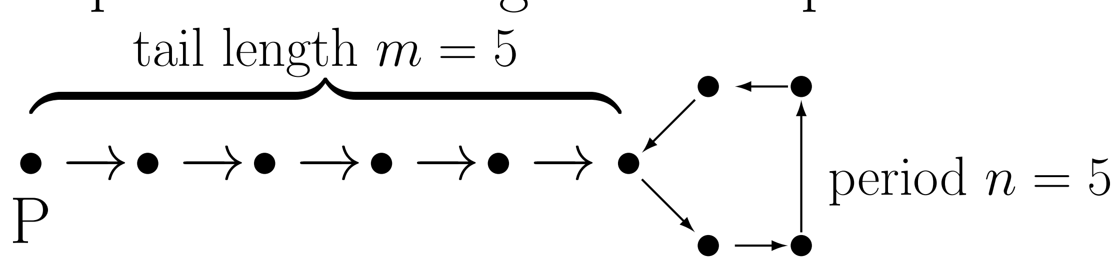
Periodic and preperiodic points are dynamcal analogues of torsion points.

Easy Exercise:

$$P \in E_{\text{tors}} \text{ iff } P \text{ is preperiodic for } E \xrightarrow{2} E.$$

Fundamental Problem: Classify isomorphism classes of pairs (f, P) such that:

- $f : \mathbb{P}^1 \to \mathbb{P}^1$ is a rational map of degree d;
- P is a point of period n for f; or more generally,
- P is a point of tail length m and period n.



Adding Dynamical Level Structure

Definition: $\mathcal{M}_d^1(n)$ classifies maps with a marked point of period n,

$$\mathcal{M}_d^1(n) = \frac{\left\{(f,P) : f : \mathbb{P}^1 \xrightarrow{\deg d} \mathbb{P}^1,\ P \text{ period } n\right\}}{\text{PGL}_2\text{-equivalence}}.$$

Note that these are finite covers

$$\mathcal{M}_d^1(n) \longrightarrow \mathcal{M}_d^1, \quad (f,P) \longmapsto f.$$

Fundamental Problem:

Describe the geometry of $\mathcal{M}_d^1(n)$.

The Geometry of $\mathcal{M}_d^1(n)$

$$\mathcal{M}_d^1(n) = \frac{\left\{(f,P) : f : \mathbb{P}^1 \xrightarrow{\deg d} \mathbb{P}^1,\ P \text{ period } n\right\}}{\mathrm{PGL}_2\text{-equivalence}}.$$

Theorem. (Blanc–Canci–Elkies)

(a) For $1 \le n \le 5$, $\mathcal{M}_2^1(n)$ is a rational surface.

(b) $\mathcal{M}_2^1(6)$ is a surface of general type.

Conjecture.

$\mathcal{M}_2^1(n)$ is a surface of general type for all $n \ge 6$.

Conjecture. Let $d \ge 2$. There is an $n_0(d)$ such that

$\mathcal{M}_d^1(n)$ is a variety of general type for all $n \ge n_0(d)$.

A Brief Foray into Number Theory

We recall a weak form of Mazur's theorem.

Uniformity Theorem. (Mazur) There is a C such that for all elliptic curves $E/\mathbb{Q}$ and all torsion points $P \in E(\mathbb{Q})$,

$$\operatorname{Order}(P) \le C.$$

An alternative formulation, fundamental for the proof:

$$Y_1(n)(\mathbb{Q}) = \emptyset \quad \text{for all } n > C.$$

Here is a (special case of a) dynamical analogue.

Uniform Boundedness Conjecture. (Morton–Silverman) There is a $C(d)$ so that for all degree d maps $f : \mathbb{P}^1_{\mathbb{Q}} \to \mathbb{P}^1_{\mathbb{Q}}$ and all f-periodic points $P \in \mathbb{P}^1(\mathbb{Q})$,

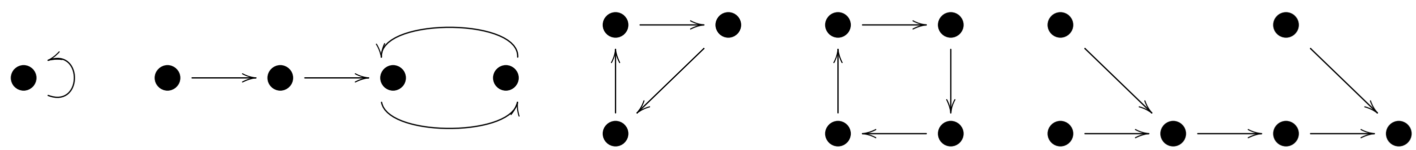
$$\operatorname{Period}_f(P) \le C(d).$$

The moduli-theoretic formulation says:

$$\mathcal{M}^1_d(n)(\mathbb{Q}) = \emptyset \quad \text{for all } n > C(d).$$

Portrait Level Structure / Multiplicities

Instead of marking one periodic point, we can mark lots of points and specify what their orbits look like. This is done using a **portrait**, which is a bunch of points and arrows. A typical example:



This portrait has points of period 1, 2, 3, and 4, a point of tail length 2, and 6 points with no specified periodicity. It is often useful to assign weights (multiplicities) to the vertices. A **weighted portrait** is a 4-tuple

$$\boxed{\mathcal{P} = (\mathcal{V}, \mathcal{W}, \Phi, \omega)}$$

$\mathcal{V}$ = finite set of vertices,

$\mathcal{W}$ = a subset of $\mathcal{V}$,

Φ = a function $\Phi : \mathcal{W} \to \mathcal{V}$,

ω = a weight function $\omega : \mathcal{W} \to \mathbb{N}$.

Portrait Moduli Spaces

For $f : \mathbb{P}^1 \to \mathbb{P}^1$ and $P \in \mathbb{P}^1$, we denote the **multiplicity (ramification index)** by $e_f(P)$.

Let $\mathcal{P} = (\mathcal{V}, \mathcal{W}, \Phi, \omega)$ be a portrait, say with $n = \#\mathcal{V}$.

Goal: Classify maps f and points $P_1, \ldots, P_n \in \mathbb{P}^1$ so that

$$\boxed{(f, P_1, \ldots, P_n) \text{ “looks like } \mathcal{P}\text{.”}}$$

We can make this precise by looking at pairs

$$(f, \iota) \quad \text{with} \quad f : \mathbb{P}^1 \xrightarrow{\deg d} \mathbb{P}^1 \text{ and } \iota : \mathcal{V} \hookrightarrow \mathbb{P}^1$$

satisfying

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\iota} & \mathbb{P}^1 \\ \Phi \downarrow & & f \downarrow \\ \mathcal{V} & \xrightarrow{\iota} & \mathbb{P}^1 \end{array} \quad \text{and} \quad e_f\bigl(\iota(v)\bigr) \ge \omega(v) \text{ for all } v \in \mathcal{W}.$$

Portrait Moduli Spaces

As usual, we want to classify pairs (f,ι) up to equivalence:

$$(f,\iota) \sim (f^{\phi}, \phi^{-1} \circ \iota) \quad \text{for } \phi \in \mathrm{PGL}_2.$$

Theorem. (Doyle–Silverman) Let $\mathcal{P}$ be a portrait. There is a moduli space $\mathcal{M}_d^1[\mathcal{P}]$ that classifies equivalence classes of pairs (f,ι) consisting of a degree d map $f : \mathbb{P}^1 \to \mathbb{P}^1$ and an f-model ι for $\mathcal{P}$.

More precisely, the space $\mathcal{M}_d^1[\mathcal{P}]$ exists as a geometric quotient scheme over $\mathbb{Z}$, in the sense of geometric invariant theory, with the caveat that if $\mathcal{P}$ has non-periodic points, some care is needed in the choice of an ample line bundle on $\mathbb{P}^{2d+1} \times (\mathbb{P}^1)^n$.

Problem: Describe the geometry of $\mathcal{M}_d^1[\mathcal{P}]$.

Moving to Higher Dimension

Moving to Higher Dimension

More generally, we look at morphisms

$$f : \mathbb{P}^N \longrightarrow \mathbb{P}^N, \quad f = [F_0, \ldots, F_N], \quad \text{where}$$

$$F_0, \ldots, F_N \in K[X_0, \ldots, X_N]$$

$$\text{have degree } d \text{ and no common roots in } \mathbb{P}^N.$$

The coefficients of $F_0, \ldots, F_N$ determine a point

$$\xi_f \in \mathbb{P}^M \quad \text{with } M = (N+1)\binom{N+d}{d} - 1.$$

The requirement that f be a morphism gives a Zariski open subset

$$\operatorname{End}_d^N := \{\xi_f \in \mathbb{P}^M : \overbrace{\operatorname{Res}(f)}^{\text{Macauley resultant}} \neq 0\}.$$

The conjugation action of $\phi \in \mathrm{PGL}_{N+1}$ on $\mathbb{P}^N$ induces an action on $f \in \operatorname{End}_d^N$ via

$$\xi_f^{\phi} = \xi_{f^\phi} = \xi_{\phi^{-1} \circ f \circ \phi}.$$

Dynamical Moduli Spaces for Maps of $\mathbb{P}^N$

The proof is somewhat more elaborate, but the result is the same:

> **Theorem.** (Petsche–Szpiro–Tepper, Levy) The quotient space $\mathcal{M}_d^N = \operatorname{End}_d^N / \operatorname{PGL}_{N+1}$ exists as a GIT geometric quotient scheme.

More generally, we can extend the definition of models of a portrait to maps $\mathbb{P}^N \to \mathbb{P}^N$ and sets of points in $\mathbb{P}^N$.

> **Theorem.** (Doyle–Silverman) Let $\mathcal{P}$ be a portrait. Then *mutatis mutandis*, the quotient space $\mathcal{M}_d^N[\mathcal{P}] = \operatorname{End}_d^N[\mathcal{P}] / \operatorname{PGL}_{N+1}$ exists as a GIT geometric quotient scheme.

Problem: Describe the geometry of $\mathcal{M}_d^N[\mathcal{P}]$.

The Dynamical Uniform Boundedness Conjecture

Conjecture. (Morton–Silverman) Fix $D \ge 1$, $N \ge 1$, and $d \ge 2$. There is a constant $C(D,N,d)$ so that for all number fields $K/\mathbb{Q}$ of degree at most D and for all morphisms $f : \mathbb{P}^N_K \to \mathbb{P}^N_K$ of degree d,

$$\#\bigl\{P \in \mathbb{P}^N(K) : P \text{ is } f\text{-preperiodic}\bigr\} \le C(D,N,d).$$

In moduli-theoretic language, for all $[K:\mathbb{Q}] \le D$,

$$\#(\text{vertices of } \mathcal{P}) \ge C(D,N,d) \implies \mathcal{M}_d^N[\mathcal{P}](K) = \emptyset.$$

Trivial Result:

$(D,N,d) = (1,1,4)$ implies Mazur's $\#E(\mathbb{Q})_{\text{tors}} \le C$.

Theorem. (Fakhruddin) The Dynamical Uniform Boundedness Conjecture implies

$$\#A(K)_{\text{tors}} \le C\bigl([K:\mathbb{Q}], \dim(A)\bigr).$$

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