Random Models of Profinite Groups in Arithmetic Geometry and Topology

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11/04/2025

The Topological Space of Small Profinite Groups

- A profinite group Γ is small if for every finite group G the set (of continuous surjective group homomorphisms) $\mathrm{Sur}(\Gamma, G)$ is finite.
- $\mathcal{S}$ - the set of isomorphism classes of small profinite groups.
- A basis for a topology on $\mathcal{S}$ is given by the sets

  $$\mathcal{S}(G_i; H_i) = \{\Gamma \in \mathcal{S} : \mathrm{Sur}(\Gamma, G_i) \neq \emptyset,\ \mathrm{Sur}(\Gamma, H_i) = \emptyset,\ 1 \leq i \leq m\}$$

  where $G_1, \dots, G_m, H_1, \dots, H_m$ range over all finite groups.
- $\mathcal{S}$ is Hausdorff, totally disconnected, but not locally compact.
- Liu–Wood computed the closure in $\mathcal{S}$ of the set of all profinite groups with a given deficiency, inspired by Dunfield–Thurston.
- Sawin–Wood computed the closure in $\mathcal{S}$ of

  $$\left\{\widehat{\pi_1(M)} : M \text{ is a closed orientable } 3\text{-dimensional manifold}\right\}.$$

- For every finite group G, Sawin–Wood are making a conjecture regarding the closure in $\mathcal{S}$ of

  $$\{\mathrm{Gal}(K^{\mathsf{ur}}/K) : K \text{ is a Galois extension of } \mathbb{Q} \text{ with } \mathrm{Gal}(K/\mathbb{Q}) \cong G\}.$$

Closure in $\mathcal{S}$ of Profinite Completions of 3-manifold Groups

A profinite group $\Gamma \in \mathcal{S}$ lies in the closure of the set

$$\left\{ \widehat{\pi_1(M)} : M \text{ is a closed orientable } 3\text{-dimensional manifold} \right\}$$

if and only if there exists a homomorphism of (abelian) groups

$$\tau \colon H^3(\Gamma, \mathbb{Q}/\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$$

such that, for each simple Γ-module V (of finite dimension) over a finite field $\mathbb{F}$, the following four conditions are met.

- We have $\dim_{\mathbb{F}} H^1(\Gamma, V) = \dim_{\mathbb{F}} H^1(\Gamma, V^\vee)$.
- For each $0 \neq \alpha \in H^2(\Gamma, V)$ there exists β in $H^1(\Gamma, V^\vee)$ with
$$\tau(\alpha \cup \beta) \neq 0.$$
- If V is a symplectic module and $\mathbb{F}$ has odd characteristic then
$$\dim_{\mathbb{F}} H^1(\Gamma, V) \equiv 0 \bmod 2.$$
- If the characteristic of $\mathbb{F}$ is 2 and V is affine symplectic then
$$\dim_{\mathbb{F}} H^1(\Gamma, V) \equiv \tau(c_V) \bmod 2.$$

The Measurable Space of Small Profinite Groups

- $\mathcal{S}$ with its Borel σ-algebra is a measurable space.
- The set of finitely generated profinite groups is measurable.
- Is the set of finitely presented profinite groups measurable?
- How to specify a probability measure on a closed subset of $\mathcal{S}$?
- Dunfield–Thurston considered two models of a random group.
- For $u \in \mathbb{Z}$, the limit of a random group given by the presentation

$$\langle x_1, \ldots, x_n : r_1, \ldots, r_{n+u} \rangle$$

  where the relations are independent Haar-random from the free profinite group on $x_1, \ldots, x_n$. Convergence is due to Liu–Wood.
- The limit as $g \to \infty$ of a random group given by the presentation

$$\langle x_1, y_1, \ldots x_g, y_g : [x_1, y_1] \cdots [x_g, y_g], y_1, \ldots, y_g, \varphi(y_1), \ldots, \varphi(y_g) \rangle$$

  where φ is sampled from the mapping class group of a genus g surface. Convergence is due to Sawin–Wood.

Putative Examples of Measures on (Closed Subsets of) $\mathcal{S}$

- The étale fundamental group of a random curve over a finite field:
$$\lim_{g\to\infty}\lim_{q\to\infty}\frac{1}{\#\mathcal{M}_g(\mathbb{F}_q)}\sum_{X\in\mathcal{M}_g(\mathbb{F}_q)}\delta(\pi_1^{\text{ét}}(X)).$$
- A distribution conjectured to exist by Boston–Ellenberg:
$$\lim_{N\to\infty}\frac{1}{\#\{p : p \text{ is a prime number, } p<N\}}\sum_{p<N}\delta(\pi_1^{\text{ét}}(\operatorname{Spec}\mathbb{Z}[p^{-1}])).$$
- The limit as $n\to\infty$ of a random group given by the presentation
$$\langle x_1,\dots,x_n : \varphi(x_1)=x_1,\dots,\varphi(x_n)=x_n\rangle$$
where φ is sampled from a (coset of a) subgroup of $\mathrm{Aut}(F_n)$.

Does a Limit of Probability Measures on $\mathcal{S}$ Exist?

- A sequence of measures μ_n on $\mathcal{S}$ (weakly) converges to a measure μ on $\mathcal{S}$ if

$$\mu_n(\mathcal{S}(G_1, \ldots, G_m; H_1, \ldots, H_m)) \to \mu(\mathcal{S}(G_1, \ldots, G_m; H_1, \ldots, H_m))$$

  as $n \to \infty$, for all finite groups $G_1, \ldots, G_m, H_1, \ldots, H_m$.
- Our measures μ_n are probability measures, and we want the limit measure μ to be a probability measure as well.
- For a finite group G, we define the G-moment of a measure μ on $\mathcal{S}$ to be the expectation of $\#\operatorname{Sur}(\Gamma, G)$ where Γ is a random variable on $\mathcal{S}$ distributed according to μ.
- The Sawin–Wood strategy consists of showing that the G-moment of μ_n converges as $n \to \infty$, and the limit does not 'grow too fast' as we vary G. They also keep track of the support of μ_n.
- What is the probability that a random profinite group (in any of our models) is finitely generated?

Moments of the Function Field Boston–Ellenberg Model

- Consider the étale fundamental group of the complement of a random irreducible divisor on the affine line over $\mathbb{F}_q$:

$$\lim_{n\to\infty} q^{-n} \sum_{\alpha\in\mathbb{F}_{q^n}} \delta(\pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{F}_q} \setminus \{\alpha, \alpha^q, \ldots, \alpha^{q^{n-1}}\})).$$

- The G-moments count regular Galois G-extensions of $\mathbb{F}_q(t)$ ramified at a single place.
- Branched G-covers of $\mathbb{P}^1_{\mathbb{F}_q}$ by smooth projective geometrically connected curves correspond to $\mathbb{F}_q$-points of Hurwitz spaces.
- Landesman–Levy computed the homology of Hurwitz spaces.
- Joint work in progress with Ellenberg studies the twisted homology of Hurwitz spaces by representations of S_n.
- Himes–Miller–Wilson also obtain results in this direction.