Vanishing criteria for Ceresa cycles and examples

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Motivation

X an algebraic variety with subvarieties $Y, Y' \subset X$.

Basic questions:

- Can we algebraically "deform" Y to Y' inside X?
- How many such equivalence classes of subvarieties on X are there?

Equivalence relations on algebraic cycles

X smooth, projective, n-dimensional, algebraic variety over a field k. Define

$$Z_i(X) = \left\{ \sum n_j Y_j \quad : n_j \in \mathbb{Z} \right\},$$

the free abelian group on closed integral subschemes $Y_j \subset X$ of dimension i.

Define $Z^i(X) = Z_{n-i}(X)$ = codimension i algebraic cycles.

Equivalence relations on $Z_i(X)$: (very roughly)

- For every $W \subset X$ of dim $i+1$ and every rational map $W \xrightarrow{f} \mathbb{P}^1$, we declare the fibers to be rationally equivalent.
- Algebraic equivalence is similar, but with maps $W \xrightarrow{f} C$, for arbitrary curves C.

Example: $Z^1(X) = \mathrm{Div}(X)$ and rational equivalence is linear equivalence.

Chow groups

Definitions:

- $\mathrm{CH}_i(X) = Z_i(X)/\sim_{\mathrm{rat}}$
- $\mathrm{CH}^i(X) = \mathrm{CH}_{n-i}(X)$ as before.
- $\mathrm{CH}^i(X)_{\mathrm{alg}} \subset \mathrm{CH}^i(X)$ subgroup of algebraically trivial cycles.

Properties:

- $\mathrm{CH}(X) := \bigoplus_{i=0}^{n} \mathrm{CH}^i(X)$ forms a ring under intersection pairing.
- Functoriality: (suitably behaved) morphisms induce pullback and push-forward.

Question: How big is $\mathrm{CH}^i(X)$? How do we tell whether two cycles are equivalent?

Cycle class map

Let $\mathrm{H}^*(X) = \mathrm{H}^*(X(\mathbb{C}), \mathbb{Z})$ (if $k = \mathbb{C}$) or your favorite Weil cohomology theory.

The cycle class map $\mathrm{CH}^i(X) \xrightarrow{\mathrm{cyc}_i} \mathrm{H}^{2i}(X)(i)$ sends $[Y]$ to its homology class.

Set $\mathrm{CH}^i(X)_{\mathrm{hom}} := \ker(\mathrm{cyc}_i)$.

- $\mathrm{Im}(\mathrm{cyc}_i)$ is finitely generated. Its rank is predicted by the Hodge/Tate conjecture.

What about $\mathrm{CH}^i(X)_{\mathrm{hom}}$?

- If $i = 1$ and X has a k-point, then $\mathrm{CH}^1(X)_{\mathrm{hom}} = A(k)$ for some abelian variety A.
- If $i > 1$ an $k = \mathbb{C}$, it is typically "infinite-dimensional" in some sense (Mumford).
- If k is a number field, it is conjecturally finitely generated (Bass).
- Beilinson-Bloch conjecture: $\mathrm{rk}\, \mathrm{CH}^i(X)_{\mathrm{hom}} = \mathrm{ord}_{s=i}\, L(\mathrm{H}^{2i-1}(X), s)$.

Let's explain why $\mathrm{H}^{2i-1}(X)$ is relevant.

Abel-Jacobi maps

Classical case: For a curve $C/\mathbb{C}$, let $V = \mathrm{H}^0(C, \Omega_C)$ and $\Lambda = \mathrm{H}_1(C, \mathbb{Z})$. The Jacobian of C is $J := V^*/\Lambda$, a g-dimensional complex torus. Have

$$\mathrm{Pic}^0(C) = \mathrm{CH}^1(C)_{\mathrm{hom}} \xrightarrow{\mathrm{AJ}} J$$
$$p - q \mapsto \left[\omega \mapsto \int_p^q \omega\right]$$

Hodge theory: $\mathrm{H}^1(C, \mathbb{C}) = \mathrm{H}^{1,0} \oplus \mathrm{H}^{0,1} = V \oplus \bar{V}$, i.e. $V = \mathrm{Fil}^1\mathrm{H}^1(C, \mathbb{C})$.

General case: Griffiths defines the **intermediate Jacobian**

$$J^i(X) := \frac{\mathrm{H}^{2i-1}(X, \mathbb{C})}{\mathrm{Fil}^i + \mathrm{H}^{2i-1}(X, \mathbb{Z})} \simeq \frac{\mathrm{Fil}^{n-i+1}\mathrm{H}^{2n-2i+1}(X, \mathbb{C})^*}{\mathrm{H}_{2n-2i+1}(X, \mathbb{Z})}$$

and the "higher" Abel-Jacobi map

$$\mathrm{CH}^i(X)_{\mathrm{hom}} \xrightarrow{\mathrm{AJ}} J^i(X)$$
$$Z \mapsto \left[\omega \mapsto \int_{\partial^{-1}Z} \omega\right]$$

Beilinson-Bloch conjecture

Analogously, for k a number field, we have the ℓ-adic Abel-Jacobi map

$$\mathrm{CH}^i(X)_{\mathrm{hom}} \xrightarrow{\mathrm{AJ}_\ell} H^1(\mathrm{Gal}_k, \mathrm{H}^{2i-1}_{et}(\overline{X}, \mathbb{Z}_\ell(i))),$$

whose kernel is conjecturally torsion (Beilinson-Bloch).

Griffiths group

Consider the chain of subgroups

$$\{0\} \subset \mathrm{CH}^i_{\mathrm{alg}} \subset \mathrm{CH}^i(X)_{\mathrm{hom}} \subset \mathrm{CH}^i(X)$$

- Fact: $\mathrm{CH}^1_{\mathrm{alg}}(X) = \mathrm{CH}^1(X)_{\mathrm{hom}}$.
- Griffiths: not true for $i \geq 2$! The difference of two lines $[L] - [L'] \in \mathrm{CH}^2(X)_{\mathrm{hom}}$ on a very general quintic threefold $X \subset \mathbb{P}^4$ over $\mathbb{C}$ is not algebraically trivial.
- Define the Griffiths group

$$\mathrm{Gr}^i(X) := \mathrm{CH}^i(X)_{\mathrm{hom}} / \mathrm{CH}^i(X)_{\mathrm{alg}}.$$

- $\mathrm{Gr}^i(X)$ is countable but can have infinite rank (Clemens).

Ceresa cycle

Let C be a curve of genus $g \geq 2$ over k, with a degree one divisor $e \in \mathrm{Div}(C)$.

Let $C \overset{\iota}{\hookrightarrow} J$ be the Abel-Jacobi embedding $x \mapsto x - e$.

The Ceresa cycle (based at e) is

$$\kappa_e(C) = [\iota(C)] - (-1)^*[\iota(C)] \in \mathrm{CH}_1(J)_{\mathrm{hom}}.$$

Homologically trivial since $\mathrm{H}^{2g-2}(J) = \bigwedge^{2g-2} \mathrm{H}^1(J)$ and $(-1)^*$ acts as -1 on $\mathrm{H}^1(J)$.

Its image $\kappa_{\mathrm{Gr}}(C) \in \mathrm{Gr}_1(J)$ is independent of the choice of e.

Theorem (Ceresa)

For very general C over $\mathbb{C}$ of genus $g \geq 3$, $\kappa_{\mathrm{Gr}}(C)$ has infinite order.

For proof of something stronger, see Dick's talk.

Clemens' question

So most curves have infinite order Ceresa cycle, even modulo $\sim_{\text{alg}}$. On the other hand:

Example: If C is hyperelliptic and e is a Weierstrass point, then $\kappa_e(C) = 0 \in Z_1(J)$.

Proof: Let $\tau \in \text{Aut}(C)$ be the hyperelliptic involution. Then

$$(-1)^*(x - e) = \tau^*(x - e) = \tau(x) - e,$$

so that $(-1)^*(\iota(C)) = \iota(C)$.

Question (Clemens '87)

$\kappa_{\text{Gr}}(C) = 0$ *if and only if* C *is hyperelliptic?*

I suspect this is not true, but the spirit of the question is:

The Ceresa class vanishes only if there is a good geometric reason.

Vanishing in the Chow group: choosing the base point

Let $K_C \in \mathrm{Pic}(C)$ be the canonical class.

Lemma

If $\kappa_e(C)$ is torsion then $(2g-2)e \equiv K_C$ in $\mathrm{Pic}(C) \otimes \mathbb{Q}$.

Upshot: We should and do assume $(2g-2)e \equiv K_C$ in $\mathrm{Pic}(C) \otimes \mathbb{Q}$.
(Let's also assume $\mathrm{char}(k) = 0$ from this point on...)

There are $(2g-2)^{2g}$ choices of such $\kappa_e(C)$, differing only by torsion.

Thus, there is a canonical Ceresa class in $\kappa(C) \in \mathrm{CH}_1(J) \otimes \mathbb{Q}$.

Note: $\kappa_e(C)$ is torsion if and only if $\kappa(C) = 0$.

Possible counterexamples to Clemens' question

Recently, non-hyperelliptic curves with $\mathrm{AJ}(\kappa(C)) = 0$ were found. Let $G = \mathrm{Aut}(C)$.

- (Bisogno-Li-Litt-Srinivasan) Fricke-Macbeath curve: $g = 7$ and $G \simeq \mathrm{PSL}_2(\mathbb{F}_8)$.
- (Beauville) The genus 3 curve $C : y^3 = x^4 + x$ with $G \simeq C_9$.

Proof.

AJ is G-equivariant and $\kappa(C)$ is canonical and hence G-invariant. One computes that the target $J^{g-1}(J)^G \otimes \mathbb{Q} = 0$, hence $\mathrm{AJ}(\kappa(C)) = 0$. □

This proof was then upgraded to a statement about $\kappa(C)$ itself:

Theorem (Qiu-Zhang)

If $\mathrm{H}^1(C)^{\otimes 3}$ has no G-invariants, then $\kappa(C) = 0$.

Chow-vanishing criterion

It turns out, it is enough to check a smaller G-representation.

Theorem (Laga-S)

If $\mathrm{H}^3(J)^G = 0$ *then* $\kappa(C) = 0$.

- Note that $\mathrm{H}^3(J) = \bigwedge^3 \mathrm{H}^1(C)$.
- Even $\mathrm{H}^3(J)^G_{\mathrm{prim}} = 0$ is enough, but if $g(C/G) = 0$, this is equivalent.

Examples:

- $y^3 = x^4 + 1$ (Qiu-Zhang, $g = 3$, $\#G = 48$)
- $y^3 = x^4 + x$ (Beauville-Schoen, $g = 3$, $G \simeq C_9$)
- $y^3 = x^5 + 1$ (Lilienfeldt-S, $g = 4$, $G \simeq C_{15}$)
- $y^3 = (x^3 + t)^2(tx^3 - 1)$ (Qiu-Zhang, $g = 4$, $G \simeq S_3^2$)
- 2-dimensional family of Humbert-Edge curves (Laterveer, $g = 5$, $G \simeq C_2^4$)
- If $\kappa(C) = 0$ and $C \to D$, then $\kappa(D) = 0$.

Griffiths-vanishing criterion

Recall $V = \mathrm{H}^0(C, \Omega_C)$, so $\mathrm{H}^1(C)_{\mathbb{C}} \simeq V \oplus \bar{V}$.

Theorem (Laga-S)

Assume Hodge conjecture for abelian varieties. If $(\bigwedge^3 V)^G = 0$ *then* $\kappa_{\mathrm{Gr}}(C)$ *is torsion.*

- The hypotheses imply $\mathrm{H}^3(J)^G \simeq \mathrm{H}^1(A)(-1)$. Need Hodge for $J \times A$.

Examples:

- Picard curves: $y^3 = x^4 + ax^2 + bx + c$. HC proved by Schoen.
- A Teichmuller curve in $\mathcal{M}_4$: $y^5 = x^3(x-1)^2(x-t)$. HC easy.
- Torelli locus in $\mathrm{U}(2,2)$-Shimura variety: $y^3 = x^2(x-1)^2(x^3 + ax^2 + bx + c)$. HC?
- Families in genus $5, 6, 8$.

Proof sketches

Use fundamental properties/structures of Chow motives of abelian varieties, especially Chow-Kunneth and Beauville decompositions.

Proof of Chow vanishing.

$\kappa(C)$ is controlled by the Chow motive $\mathfrak{h}^3(J)^G$, [a] which, by assumption, has trivial cohomological realization. By Kimura[b], $\mathfrak{h}^3(J)^G = 0$, hence $\kappa(C) = 0$. □

[a] More precisely, $\kappa(C) = 0$ if and only if $[C]_1 = 0$ and $[C]_1 \in \mathrm{CH}(\mathfrak{h}^{2g-3}(J)^G) \simeq \mathrm{CH}(\mathfrak{h}^3(J)^G)$.

[b] S.-I. Kimura. Chow groups are finite dimensional, in some sense. Math. Ann., 2005.

Proof of Griffiths vanishing.

$\mathfrak{h}^3(J)^G$ need not be 0, but its Hodge structure is weight 1. So Hodge conjecture implies $\mathfrak{h}^3(J)^G \simeq \mathfrak{h}^1(A)(-1)$ and hence $\kappa(C)$ is algebraically trivial. □

Can $\kappa(C)$ vanish for non-group theoretic reasons? Yes!

For $f(x) = x^4 + ax^2 + bx + c$, let $C_f : y^3 = f(x)$ be the corresponding genus 3 curve.

Recall the I- and J-invariants of f:

- $I(f) = a^2 + 12c$
- $J(f) = 72ac - 2a^3 - 27b^2$

Classical observation: $J(f)^2 = 4I(f)^3 - 27\mathrm{Disc}(f)$.

In other words, $P_f := (I(f), J(f))$ lies on the elliptic curve $E_f : y^2 = 4x^3 - 27\mathrm{Disc}(f)$.

Theorem (Laga-S)

$\kappa(C_f) = 0$ *if and only if* $P_f \in E_f(\mathbb{C})$ *is torsion.*

In fact: $\langle \kappa(C_f), \kappa(C_f)\rangle_{\mathrm{BB}} = N^2 \langle P_f, P_f\rangle_{\mathrm{NT}}$ for some constant N.

Proof sketch

More generally, suppose $\mathcal{C} \to S$ is a family of curves with fiber-wise algebraically trivial Ceresa cycle.

Then the holomorphic normal function $S \xrightarrow{\sigma} \mathcal{J}^{g-1}(\mathcal{J})$ to the family of intermediate Jacobians should factor through an algebraic section of an abelian scheme $\mathcal{A} \to S$.

Suppose the Mordell-Weil group $\mathcal{A}(S)$ is free of rank 1 (absent further information, this is what we should expect!). In our case, we compute $\mathcal{A}(S)$ via Shioda-Tate.

If $\sigma \neq 0$, then it must be a multiple of this generator. To prove this, simply specialize and verify for a single example! (See Padma's talk.)

The Ceresa vanishing locus in $\mathcal{M}_g$

We can consider the torsion locus of either $\kappa(C)$ or $\kappa_{\mathrm{Gr}}(C)$ in $\mathcal{M}_g$. A priori, this is just some countable union of proper closed subvarieties.

Theorem (Gao-Zhang, '24)

For $g \geq 3$, there is an open dense subset $U_g \subset \mathcal{M}_g$ on which the height of the Ceresa/modified diagonal cycle satisfies a Northcott property for $\overline{\mathbb{Q}}$-points.

See also work of Hain and Kerr-Tayou. Our previous theorem implies:

Corollary

The 2-dimensional locus of Picard curves in $\mathcal{M}_3$ is contained in the complement of U_3

This is despite the fact that the normal function is not constant on this locus!

Proof

- The Picard locus is (essentially) the universal elliptic curve $\mathcal{E}$.
  - The map sends $y^3 = f(x)$ to $y^2 = f(x)$.
- The elliptic fibration $\mathcal{E}$ has base $S = \mathbb{P}(2,3)$.
- S can be thought of as the elliptic curve $\hat{E}\colon y^2 = x^3 + 1$.
- Our result: the torsion Ceresa cycles are the fibers of $\mathcal{E}$ above $\hat{E}_{\mathrm{tors}}$.
- In the coarse space, the fibers are of the form $E/\langle \pm 1\rangle \simeq \mathbb{P}^1$.
- So Picard locus contains infinitely many rational curves on which $\kappa(C) = 0$.
- Hence the Picard locus cannot be in U_3.

Many open questions about the subset of $\mathcal{M}_g$ with $\kappa(C) = 0$.

For example, what is its Zariski closure? All of $\mathcal{M}_g$?

Thank you!


Mercé
ありがとう
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Some open questions related to Ceresa vanishing

Let $G = \mathrm{Aut}(C)$.

- Are there non-hyperelliptic curves of arbitrary large genus with $\mathrm{H}^3(J)^{\mathrm{Aut}(C)} = 0$?
- With $\left(\bigwedge^3 \mathrm{H}^0(C, \Omega_C)\right)^G = 0$?
- Which genus 3 curves C have a cover with these properties?
- When $\mathrm{H}^3(J)^G(1)$ is pure of weight 1, what is the corresponding abelian variety?

Chow motives cheat sheet

The category of pure Chow k-motives is obtained from the category of smooth projective k-varieties by taking $\mathrm{Hom}(X, Y) = \mathrm{CH}(X \times Y)$, i.e. morphisms are algebraic correspondences. Such a hom induces $\mathrm{CH}(X) \to \mathrm{CH}(Y)$ and $\mathrm{H}^*(X) \to \mathrm{H}^*(Y)$ via "pullback-intersect-pushforward".

- Chow motives are tuples $(X, e, \ldots)$ with $e \in \mathrm{End}(X) = \mathrm{CH}(X \times X)$ idempotent.
- The object corresponding to X itself, denoted $h(X)$, is $(X, \Delta, \ldots)$.
- One expects a canonical Chow-Kunneth decomposition $h(X) = \bigoplus_{i=0}^{2\dim(X)} h^i(X)$, analogous to the decomposition of cohomology.
- An abelian variety has such a decomposition, which is even multiplicative.
- A curve C has a multiplicative Chow-Kunneth decomposition iff $\kappa(C) = 0$.

Does $\kappa(C)$ ever vanish for non-group theoretic reasons?

Yes!

Example: Let $C_{a,b}: y^3 = x^4 + ax^2 + b$, a plane quartic with $C_6 \hookrightarrow \mathrm{Aut}(C)$.

Alternate description: double cover of $y^2 = x^3 + k$, branched along ∞ and μ_3-orbit.

This gives an interesting involution $C_{a,b} \leftrightarrow C_{8a,16(a^2-4b)} = \hat{C}_{a,b}$ called bigonal duality.

Theorem (Laga-S)

$\kappa(C_{a,b}) = 0$ *if and only if* $\hat{C}_{a,b}$ *is branched along torsion points.*

This geometric criteria does not hold for general bielliptic plane quartics.

Does it generalize to other bielliptic curves with extra structure?