Points on Shimura varieties modulo primes

Sug Woo Shin (UC Berkeley)

VaNTAGe Seminar, May 2, 2023

Plan

Part I. Path to Langlands–Rapoport (LR) conjecture

1. Hilbert's 12th problem, Kronecker's Jugendtraum
2. Shimura varieties
3. Langlands: early investigation
4. Langlands–Kottwitz–Rapoport (LKR) method

Part II. On mod p points of Shimura varieties

1. LR conjecture
2. Recent results

Hilbert's 12th problem (1900)

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Give a family of "special" functions and their values generating **maximal abelian extensions** of a number field F.

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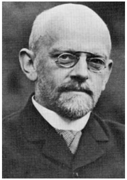

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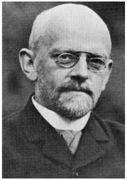

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Still open, cf. Dasgupta–Kakde's Thm over totally real fields.

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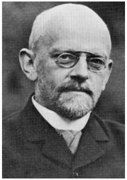

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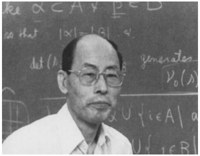
... how this result can be generalized for the number fields of higher degree, making thereby an introduction to the theory of automorphic functions and abelian varieties.
– Shimura, *Automorphic Functions and Number Theory* (1968)

Shimura 1/2



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algebraic ↓

CM theory for elliptic curves

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CM theory for abelian varieties (Shimura–Taniyama, 1961)

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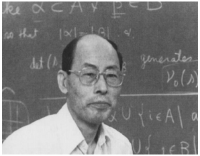

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→ algebraic → CM theory for elliptic curves → CM theory for abelian varieties (Shimura–Taniyama, 1961)

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The modular varieties M_g turn out to be canonical models/$\mathbb{Q}$.

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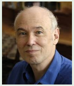

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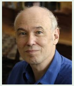

2. **Langlands** proposed and developed a program (next topic).

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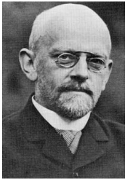

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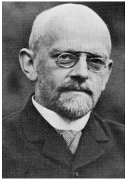

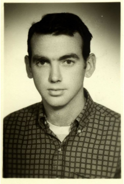

Langlands reciprocity/functoriality conjectures (according to Tate)

In 1967, Langlands was studying the analytic theory of automorphic forms on general reductive algebraic groups and saw a formal relation between Artin's L-series and some Euler products arising in the theory of Eisenstein series. This led him to some **general conjectures**, ...

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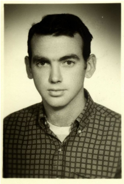

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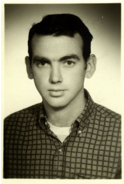

For various reasons it is to be expected that

the *L*-functions associated to these Shimura varieties

can be expressed in terms of

the *L*-functions associated to automorphic forms

on the group defining the variety and on certain related groups.

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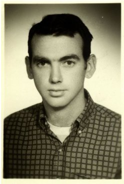

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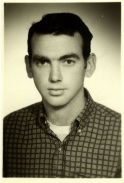

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$\rightsquigarrow$ many instances of Langlands reciprocity (Hilbert's 9th problem)

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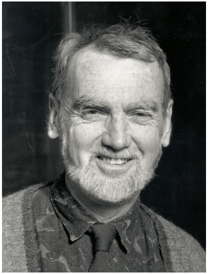
$$\text{Prove :} \quad \boxed{\zeta(s, \mathrm{Sh}) = \prod L(s, \pi, r)^a},$$

where π is an auto. rep. on some group H, r is a rep of ${}^L H$, $a \in \mathbb{Z}$.



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Langlands–Kottwitz–Rapoport (LKR) method:

[① LR conj.] —(fixed pt formula)→ [trace formula for Sh] —(②,③)→ [ζ-function, ℓ-adic coh. of Sh]

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Point

- Frob_p acts on $X_p(\phi)$; Hecke (away from p) acts on $X^p(\phi)$.
- RHS is group-theoretic. NO reference to abelian varieties.

LR Conjecture: relation to Jugendtraum/CM theory

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- In this case, class field theory + CM theory $\Rightarrow$ LR conj.

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Key Principle (part of Honda-Tate theory)

Every isogeny class $[\phi]$ contains **mod p reduction of CM points (special pts)**.

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LR Conjecture (1987) – at primes of good reduction

1. $\exists\,!$ integral canonical models $\mathscr{S}$.
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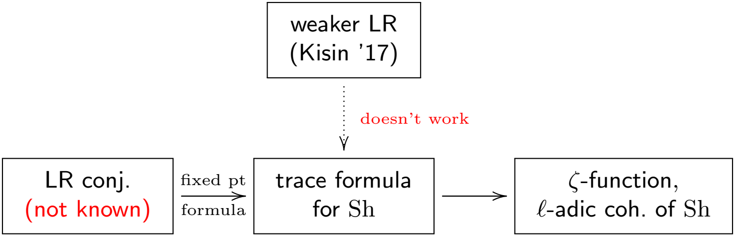
Problem:



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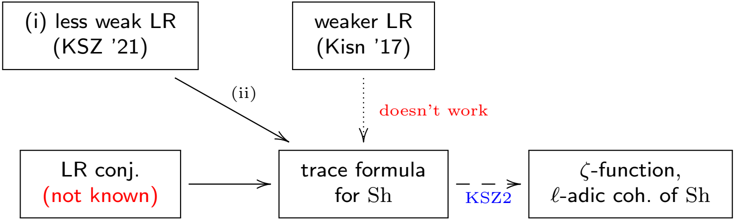

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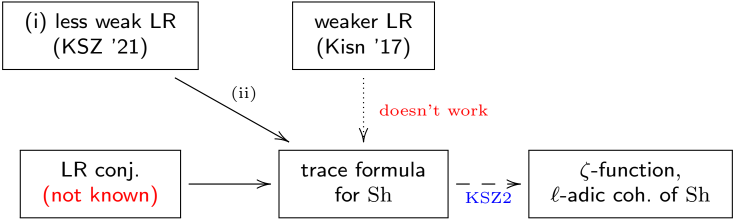
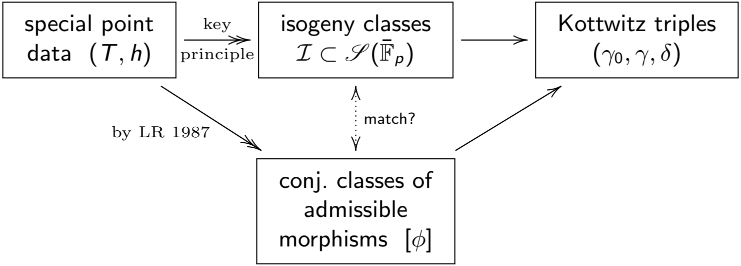


Proof by "enhanced probing" via CM theory, integral p-adic Hodge theory, ...

Probing points of Shimura varieties mod p



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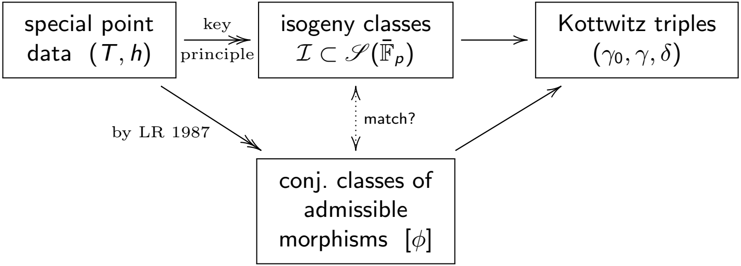

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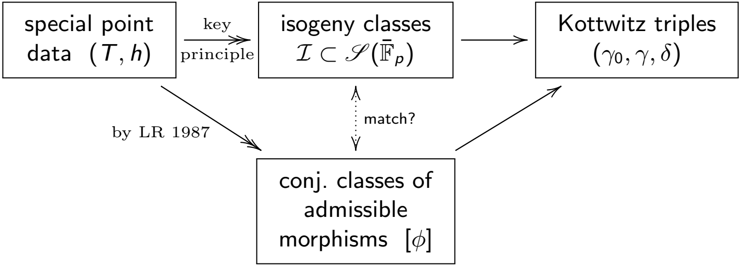

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KSZ '21

Refined matching via finer invariants. $\rightsquigarrow$ less weak LR ("rational error").

Comments, Further directions

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There are recent developments on analogues of the LKR method:

1. bad reduction of Sh: $\exists$ at least 3 approaches due to Harris–Taylor, Mantovan // Haines, Kottwitz, Rapoport // Scholze
2. intersection cohomology
3. Igusa varieties: related to first approach of ①; $\exists$ other apps.

by

1. Caraiani–Scholze, Haines–Richarz, Hamacher–Kim, Kisin–Pappas, Pappas–Rapoport, van Hoften, Youcis, Zhou, Haines–Zhou–Zhu, ...
2. ..., Morel, Y. Zhu, KSZ2
3. Bertoloni Meli–S., Kret–S., MackCrane, M. Zhang, ...

(I apologize for unintended omissions.)

Summary, main takeaways

Jugendtraum, Hilbert's 12th, CM theory

⇝ Shimura, Deligne, Langlands, ...

Shimura varieties, study of their ζ-functions

⇝ Langlands, Kottwitz, Rapoport, ...

(versions of) Langlands–Rapoport conjecture

⇝ LKR, Kisin, KSZ

stable trace formula for Shimura varieties of abelian type

⇢ very many (e.g., Kret–S, KSZ2)

cohomology, ζ-functions, Hilbert's 9th (Langlands reciprocity), ...

⇝

further applications