VaNTAGe math

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Density of rational points near/on compact manifolds with certain curvature conditions

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Dimension growth conjecture

Let $n \geq 2$, $V \subseteq \mathbb{P}^n_{\mathbb{Q}}$ an irreducible variety of degree d.

For $\underline{x} = (x_0 : x_1 : \ldots : x_n) \in \mathbb{P}^n_{\mathbb{Q}}(\mathbb{Q})$ with $x_0, \ldots, x_n \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$ define

$$H(\underline{x}) := \max_{0 \leq i \leq n} |x_i|$$

(2) Define for $B \in \mathbb{R}_{\geq 0}$ the counting function

$$N_V(B) := \{ \underline{x} \in V(\mathbb{Q}) : H(\underline{x}) \leq B \}$$

→ growth of $N_V(B)$?

→ upper bounds for $N_V(B)$?

First upper bound:

$$N_V(B) \ll_V B^{\dim V + 1 + \varepsilon}$$

→ Proof by induction

→ is this optimal?

$\boxed{\text{Ex 1}}$ Let $V \subseteq \mathbb{P}^n_{\mathbb{Q}}$ be given by

$$a_0 x_0 + a_1 x_1 + \ldots + a_n x_n = 0$$

with $a_i \in \mathbb{Z}$ $0 \leq i \leq n$

and $a_0 \neq 0$.

Then $N_V(B) \sim B^n$

$\boxed{\text{Ex 2}}$ Let $a_0, \ldots, a_n \in \mathbb{Z} \setminus \{0\}$ and

$V \subseteq \mathbb{P}^n_{\mathbb{Q}}$ be given by

$$a_0 x_0^2 + a_1 x_1^2 + \ldots + a_n x_n^2 = 0.$$

For $n \geq 4$ we have

$$N_V(B) \sim c B^{n-1}$$

Ex 3 Let $F_0(\underline{x}), F_1(\underline{x}) \in \mathbb{Z}[x_0, \ldots, x_n]$ and $V \subseteq \mathbb{P}^n_{\mathbb{Q}}$ be given by

$$x_0 F_0(\underline{x}) - x_1 F_1(\underline{x}) = 0$$

Then

$$N_V(B) \geq \frac{1}{2} \# \left\{ (x_2, \ldots, x_n) \in \mathbb{Z}^{n-1} : \begin{array}{l} \max_{2 \leq i \leq n} |x_i| \leq B, \\ \gcd(x_2, \ldots, x_n) = 1 \end{array} \right\}$$

$$\gg B^{n-1}$$

<u>Conjecture</u> (weak dimension growth conjecture)

Let $V \subseteq \mathbb{P}^n_{\mathbb{Q}}$ be an irreducible, projective variety of degree $\deg(V) \geq 2$. Then

$$N_V(B) \ll_V B^{\dim V + \varepsilon}$$

for any $\varepsilon > 0$.

solved in a series of articles by Browning, Heath-Brown and Salberger

more precisely, for $d \geq 4$ Salberger obtains

$$N_V(B) \ll_{d,n,\varepsilon} B^{\dim V + \varepsilon}$$

Castryck, Cluckers, Dittmann, Nguyen 2019

Let $V \subseteq \mathbb{P}^n_{\mathbb{Q}}$, $n > 1$, irreducible of degree $d \geq 5$

Then

$$N_V(B) \leq c d^e B^{\dim V}$$

with $c, e \in \mathbb{R}$ independent of V.

② counting points on/close to manifolds/varieties ?

Set-up:

Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a bounded submanifold with $\dim \mathcal{M} = m$ (immersed submanifold)

Let $Q > 1$, $0 \leq \delta < \frac{1}{2}$

Define

$$N_{\mathcal{M}}(Q, \delta)$$

$$= \# \left\{ \frac{p}{q} \in \mathbb{Q}^n : \begin{array}{l} 1 \leq q \leq Q \\ \operatorname{dist}\left(\frac{p}{q}, \mathcal{M}\right) \leq \frac{\delta}{q} \end{array} \right\}$$

with $p \in \mathbb{Z}^n$.

② Upper/Lower bounds, asymptotics for $N_{\mathcal{M}}(Q,\delta)$?

First upper bound

$$N_{\mathcal{M}}(Q,\delta) \ll Q^{m+1}$$

Ex 1 Let $\mathcal{M}$ be a piece of a rational linear subspace, then

$$N_{\mathcal{M}}(Q,\delta) \gg Q^{m+1}$$

typical expectation for $N_M(Q,\delta)$?

"volume computation"

Set $k = n - m$, then one might expect

$$N_M(Q,\delta) \sim \left(\frac{\delta}{Q}\right)^k Q \, Q^n$$

$$\sim \delta^k Q^{m+1}$$

for what size of δ could this be realistic to expect?

Ex 1 Let M be given by

$$z_1^2 + \_\_\_ + z_n^2 = 1$$

Then $N_\mathcal{M}(Q, 0) \sim Q^{n-1}$

might expect the term δQ^n

to dominate for

$$\delta \gg Q^{-1}$$

② How large can $N_\mathcal{M}(Q, 0)$ in general be?

Idea: compare to dimension growth,

then might expect

$$N_\mathcal{M}(Q, 0) \ll_{\varepsilon, \mathcal{M}} Q^{m+\varepsilon}$$

$m = \dim \mathcal{M}$

if $\mathcal{M}$ is "not flat"

comparing the exp bound

$$N_M(Q,0) \ll Q^{m+\varepsilon}$$

and

$$N_M(Q,\delta) \sim \delta^{k} Q^{m+1}$$

may obtain the following

<u>Conjecture 1</u> (Huang)

Let $M \subseteq \mathbb{R}^n$ be a bdd immersed submanifold with boundary, $m = \dim M$, $k = n-m$.
If M satisfies 'proper' curvature conditions, then

$\exists\, c_M > 0$ st

$$N_M(Q,\delta) \sim c_M\, \delta^{k} Q^{m+1}$$

for $\delta = Q^{-\frac{1}{k}+\varepsilon}$

Rk ① By monotonicity in δ obtain

$$N_M(Q,\delta) \ll \delta^{k} Q^{m+1} + Q^{m+\varepsilon}$$

for all suitable M and $\delta > 0$.

② Some curvature condition is needed

→ linear subspaces

→ example of the Fermat curve

(also locally)

② What is known towards Conjecture 1?

Curves

$\mathcal{C}$ curve with non-vanishing curvature (C^2)

Huxley 1994

$$N_{\mathcal{C}}(Q, \delta) \ll \delta^{1-\varepsilon} Q^2 + Q \log Q$$

Vaughan and Velani 2006

$$N_{\mathcal{C}}(Q, \delta) \ll \delta Q^2 + Q^{1+\varepsilon}$$

Beresnevich, Dickinson & Velani
2007

$$N_c(Q,\delta) \gg \delta Q^2$$

for $\delta \gg Q^{-1}$

general manifolds

Beresnevich 2012

M non-degenerate, analytic

$$N_M(Q,\delta) \gg \delta^k Q^{m+1}$$

for $\delta \gg Q^{-\frac{1}{k}}$

Thm (Huang 2017)

Let $S \subseteq \mathbb{R}^n$ be a smooth compact hypersurface, C^ℓ with

$$\ell = \max \left(\lfloor \frac{n-1}{2} \rfloor + 6, n+2 \right)$$

and Gaussian curvature bdd away from 0.

Then

$$N_S(Q, \delta)$$

$$= c_S \delta Q^{n+1} + O\left(E_n(Q) Q^n \right)$$

with

$$E_n(Q) = \begin{cases} e^{c_1 \sqrt{\log Q}} & , n=2 \\ (\log Q)^{c_2} & n \geq 3 \end{cases}$$

c_1, c_2 constants dep on S.

? – examples in higher codimension?

– assuming even stronger curvature conditions?

Let $\ell \geq 2$, $f_r \in C^\ell(\mathbb{R}^n)$,

$1 \leq r \leq R$

$\underline{x}_0 \in \mathbb{R}^n$, $\varepsilon > 0$

Let $M \subseteq \mathbb{R}^{n+R}$ be given by

$$M = \{ (\underline{x}, f_1(\underline{x}), \dots, f_R(\underline{x})) \in \mathbb{R}^{n+R} : \underline{x} \in \overline{B_\varepsilon(\underline{x}_0)} \}$$

Consider $\omega \in C_0^\infty(\mathbb{R}^n)$

$\omega \geq 0$ with $\underline{x_0} \in \operatorname{supp} \omega$

and $\operatorname{supp} \omega$ suff. small

compactness

$Q \in \mathbb{N}$, $0 \leq \delta \leq \frac{1}{2}$

Define

$$N_\omega(Q, \delta) = \sum_{\substack{a \in \mathbb{Z}^n \\ q \leq Q \\ \|q f_r(\frac{a}{q})\| \leq \delta \\ 1 \leq r \leq R}} \omega\left(\frac{a}{q}\right)$$

Set $N_0 := \sum_{\substack{\underline{a} \in \mathbb{Z}^n \\ q \leq Q}} \omega\left(\frac{a}{q}\right)$

Case $R = 1$

M having non-zero Gaussian curvature in $\underline{x}_0$

$\Updownarrow$

$$\det H_{f_1}(\underline{x}_0) \neq 0$$

Condition $(*)$

$$\det H_{t_1 f_1 + \dots + t_R f_R}(\underline{x}_0) \neq 0$$

$$\forall (t_1, \dots, t_R) \in \mathbb{R}^R \setminus \{\underline{0}\}.$$

Thm 1 (S. – Yamagishi 2020)

$n \geq 2$, $\quad \ell > \max\{n+1, \frac{n}{2}+4\}$

Suppose that Con(ϵ) holds and that $\varepsilon > 0$ is suff. small.

Then

$$|N_w(Q,\delta) - (2\delta)^{\ell} N_0|$$

$$\ll \begin{cases} \delta^{\frac{(\ell-1)(n-2)}{n}} Q^n \, \mathcal{E}_n(Q) & \delta \geq Q^{-\frac{n}{n+2(\ell-1)}} \\ Q^{n - \frac{(n-2)(\ell-1)}{n+2(\ell-1)}} \, \mathcal{E}_n(Q) & \delta < Q^{-\frac{n}{n+2(\ell-1)}} \end{cases}$$

where

$$\mathcal{E}_n(Q) = \begin{cases} e^{c_1\sqrt{\log Q}} & n=2 \\ (\log Q)^{c_2} & n \geq 3 \end{cases}$$

for $c_1, c_2 > 0$ constants.

<u>Rk</u> • $N_0 = \sigma Q^{n+1} + O(Q^n)$

for some $\sigma > 0$

• $$Q^{-\frac{n}{n+2(R-1)}} \leq Q^{-\frac{1}{R}}$$

Corollary 2 Asspts as in Thm 1.

Thm $\exists\ c_\mu > 0$ st

$$N_\mu(Q,\delta) \sim c_\mu\ \delta^R\ Q^{n+1}$$

for $Q \to \infty$ and

$$\delta \geq Q^{-\frac{n}{n+2(R-1)}+\varepsilon}$$

compare to $Q^{-\frac{1}{R}+\varepsilon}$

in the main conjecture

Corollary 3 $\quad n \geq 3$

Assypts as in Thm 1. Then

$$N_M(Q, 0) \ll Q^{n - \frac{(n-2)(R-1)}{n+2(R-1)}} (\log Q)^c$$

for some constant $c > 0$

$\Updownarrow$

- compare to $N_M(Q, 0) \ll Q^n$

  in the main conjecture

- for n very large comp to R

  s.t

  $$N_M(Q, 0) \ll Q^{n+1-R+\delta}$$

? Expectations in the dimension growth conjecture in higher codimension assuming "suitable" curvature conditions??

Ex: System of R generic quadrics in $\mathbb{P}^{n+R}$

$$Q_1(x_1, \ldots, x_{n+R+1}) = 0$$

$$\vdots$$

$$Q_R(x_1, \ldots, x_{n+R+1}) = 0$$

count integer solns $x_1 \sim x_{n+R+1}$ with $|x_i| \leq Q$ $\forall i$.

circle method exp.

$$N_M(Q,0) \sim c\, Q^{n+R+1-2R}$$

$$\sim c\, Q^{n+1-R}$$

Examples for which Condition (*) holds

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find real symmetric $n \times n$ matrices $A_1, \dots, A_R$ such that

$$(+) \begin{cases} \det(t_1 A_1 + \dots + t_R A_R) \neq 0 \\ \forall (t_1, \dots, t_R) \in \mathbb{R}^R \setminus \{0\}. \end{cases}$$

then take

$$f_r(\underline{x}) = \frac{1}{2} \sum_{i,j} A_{r\,ij}\, x_i x_j + g_r(\underline{x})$$

with $Hg_r(\underline{x_0}) = \underline{0}$

Ex $n=2$ (Suslin 1977)

Set $M_2(t_1, t_2) = \begin{pmatrix} t_2 & t_1 \\ t_1 & -t_2 \end{pmatrix}$

and set

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$n \geq 3$ define

$$M_n(t_1, \dots, t_n) = \begin{pmatrix} t_n I_{2^{n-2}} & M_{n-1}(t_1, \dots, t_{n-1}) \\ M_{n-1}(t_1, \dots, t_{n-1}) & -t_n I_{2^{n-2}} \end{pmatrix}$$

then show inductively

$$M_n(t_1, \ldots, t_n)^2$$

$$= (t_1^2 + \ldots + t_n^2) I_{2^{n-1}}$$

Set

$$A_1 = M_n(1, 0, \ldots, 0)$$

$$A_2 = M_n(0, 1, 0, \ldots, 0)$$

$$\vdots$$

$$A_n = M_n(0, \ldots, 0, 1)$$

alt use es work of Netzer & Thom 2012 on determinantal representations