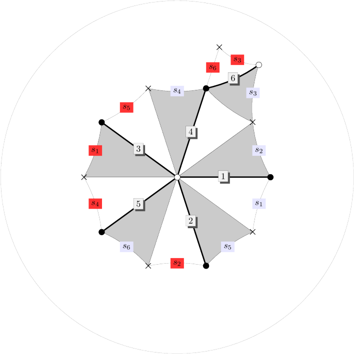
# Belyi maps: computation and data

Sam Schiavone



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# Outline

(I) Background
(II) Computation
(III) Data

# Background

# Belyi's Theorem

$$\{ \text{ Curves over } \overline{\mathbb{Q}} \} \hookrightarrow \{ \text{ Curves over } \mathbb{C} \} \overset{\sim}{\leftrightarrow} \{ \text{ Riemann surfaces } \}$$

**Q**: Which curves over $\mathbb{C}$ arise as base changes of curves over $\overline{\mathbb{Q}}$?

Theorem (Belyi, 1979)

*A smooth projective curve $X$ over $\mathbb{C}$ can be defined over $\overline{\mathbb{Q}}$ if and only if there exists a nonconstant morphism of algebraic curves $\varphi : X \to \mathbb{P}^1_{\mathbb{C}}$ unramified outside $\{0, 1, \infty\}$.*

Such a map is called a *Belyi map*.

# Belyi maps

## Definition

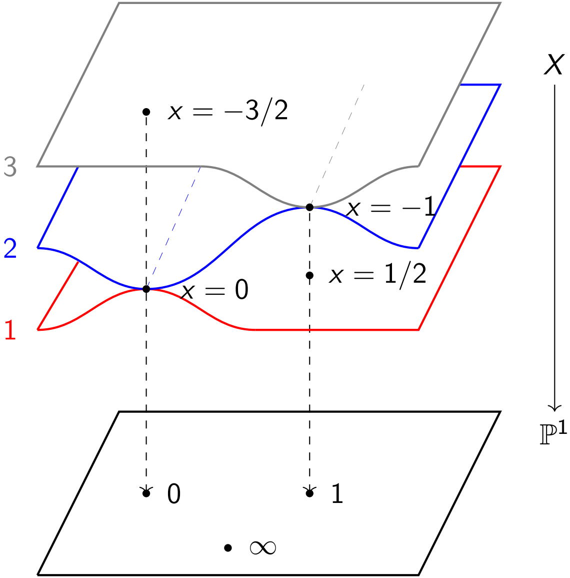
A *Belyi map* over $\mathbb{C}$ is a nonconstant morphism of algebraic curves $\varphi : X \to \mathbb{P}^1_{\mathbb{C}}$ that is unramified outside $\{0, 1, \infty\}$.

## Example

Consider the map $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ given by $\varphi(x) = 2x^3 + 3x^2$. Since $\varphi'(x) = 6x^2 + 6x = 6x(x+1)$, $\varphi$ is only ramified above $0, 1, \infty$.

$$\varphi(x) = 2x^3 + 3x^2 = x^2(2x+3)$$
$$\varphi(x) - 1 = 2x^3 + 3x^2 - 1 = (2x-1)(x+1)^2.$$

# Belyi maps



$$\varphi(x) = 2x^3 + 3x^2$$

# Belyi maps and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Of Belyi's theorem, Grothendieck said, "Never, without a doubt, was such a deep and disconcerting result proved in so few lines!" Using this theorem, Grothendieck described a faithful action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of isomorphism classes of Belyi maps.

**Goal:** Understand $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by studying this action.

We compute explicit equations for Belyi maps in order to get a concrete view of the Galois action.

# Big bijective picture

The players:

- Belyi maps
- Transitive permutation triples
- Subgroups of triangle groups
- Dessins d'enfants
- Function field extensions (in one variable)

# Permutation triples

A *transitive permutation triple* of degree $d$ is a triple $\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$ such that

- $\sigma_\infty \sigma_1 \sigma_0 = 1$, and
- $\langle \sigma_0, \sigma_1, \sigma_\infty \rangle \leq S_d$ is a transitive subgroup.

Two permutation triples $\sigma, \sigma'$ are *simultaneously conjugate* if there exists $\rho \in S_d$ such that

$$(\sigma_0', \sigma_1', \sigma_\infty') = (\rho\sigma_0\rho^{-1}, \rho\sigma_1\rho^{-1}, \rho\sigma_\infty\rho^{-1}) \, .$$

# Passports

A *passport* consists of the data $(g, G, \lambda)$ where

- $g \geq 0$ is an integer,
- $G \leq S_d$ is a transitive subgroup; and
- $\lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ is a triple of partitions of $d$.

A permutation triple $\sigma \in S_d^3$ *belongs* to a passport $(g, G, \lambda)$ if

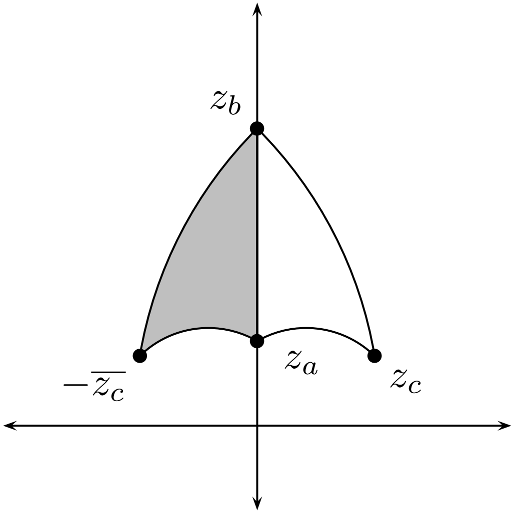
1. $2g - 2 = -2d + \sum_{s \in \{0,1,\infty\}} \sum_{\tau \text{ a cycle in } \sigma_s} (\mathrm{len}(\tau) - 1)$
2. $\langle \sigma_0, \sigma_1, \sigma_\infty \rangle = G$; and
3. $\sigma_0, \sigma_1, \sigma_\infty$ have the cycle types specified by $\lambda_0, \lambda_1, \lambda_\infty$.

The *size* of a passport is the number of permutation triples belonging to it, up to simultaneous conjugacy.

# Triangle groups

For a triple of integers $a, b, c \in \mathbb{Z}_{\geq 2}$, we define the *triangle group*

$$\Delta(a, b, c) = \langle \delta_a, \delta_b, \delta_c \mid \delta_a^a = \delta_b^b = \delta_c^c = \delta_c\delta_b\delta_a = 1 \rangle \, .$$



# Triangle groups

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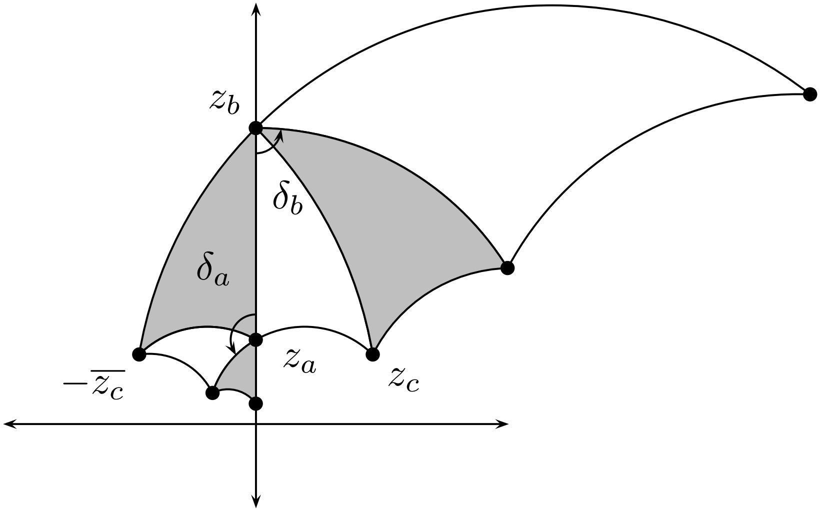


# Dessins d'enfants

A *dessin (d'enfant)* is a finite graph $D$ embedded in an oriented compact connected topological surface $X$ with the following properties.

1. $D$ is connected
2. $D$ is bicolored: each vertex is assigned the color black or white, and adjacent vertices have different colors
3. $X \setminus D$ has finitely many connected components, each of which is homeomorphic to a disc. (These are called the *faces* of $D$.)

# Big bijective picture

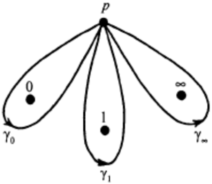
$$\begin{tikzcd}
\left\{\begin{array}{l}\text{Transitive permutation}\\ \text{triples } (\sigma_0,\sigma_1,\sigma_\infty)\in S_d^3\\ \text{with } \sigma_\infty\sigma_1\sigma_0 = 1\end{array}\right\} \arrow[dr, leftrightarrow, "\sim"] & & \left\{\begin{array}{l}\text{Index } d \text{ triangle sub-}\\ \text{groups } \Gamma \leq \Delta(a,b,c)\end{array}\right\} \arrow[dl, leftrightarrow, "\sim"'] \\
& \left\{\begin{array}{l}\text{dessins with}\\ d \text{ edges}\end{array}\right\} \arrow[dl, leftrightarrow, "\sim"'] \arrow[dr, leftrightarrow, "\sim"] & \\
\left\{\begin{array}{l}\text{Degree } d \text{ function field}\\ \text{extensions } K \supseteq \mathbb{C}(t)\\ \text{with discriminant sup-}\\ \text{ported at } 0,1,\infty\end{array}\right\} & & \{\text{Belyi maps of degree } d\}
\end{tikzcd}$$

# Belyi maps $\to$ permutation triples

Let $U = \varphi^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ and fix a base point $p \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$. The restriction of a Belyi map

$$\varphi|_U : U \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$$

is a covering space. Note that $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ is generated by loops around $0, 1$, and $\infty$.



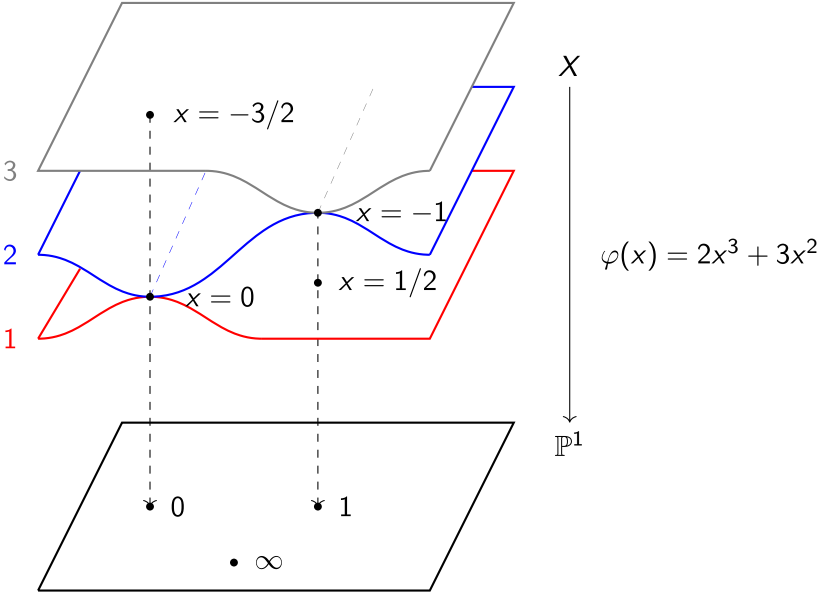
# Belyi maps → permutation triples

$\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ acts on the $d$ points in the fiber $\varphi^{-1}(p)$ via path-lifting, inducing a homomorphism $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to S_d$.

This map is called the *monodromy representation* of $\varphi$ and its image the *monodromy group*.

The cycles of the permutation correspond to the points of $X$ above $0, 1, \infty$ and the length of the cycle corresponds to its ramification index.

# Belyi maps → permutation triples



# Permutation triples $\to$ triangle subgroups

Given a transitive permutation triple $\sigma = (\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$ with orders $a, b, c \in \mathbb{Z}_{\geq 2}$, let $\Delta = \Delta(a, b, c)$ be the associated triangle subgroup. Then there is a group homomorphism

$$\begin{aligned} \pi : \Delta &\to S_d \\ \delta_a, \delta_b, \delta_c &\mapsto \sigma_0, \sigma_1, \sigma_\infty \,. \end{aligned}$$

Let

$$\Gamma = \mathrm{Stab}_\Delta(1) = \pi^{-1}(\mathrm{Stab}_G(1)) = \{\delta \in \Delta : 1^{\pi(\delta)} = 1\}$$

where $G = \langle \sigma \rangle \leq S_d$. Then $[\Delta : \Gamma] = d$.

# Triangle subgroups → Belyi maps

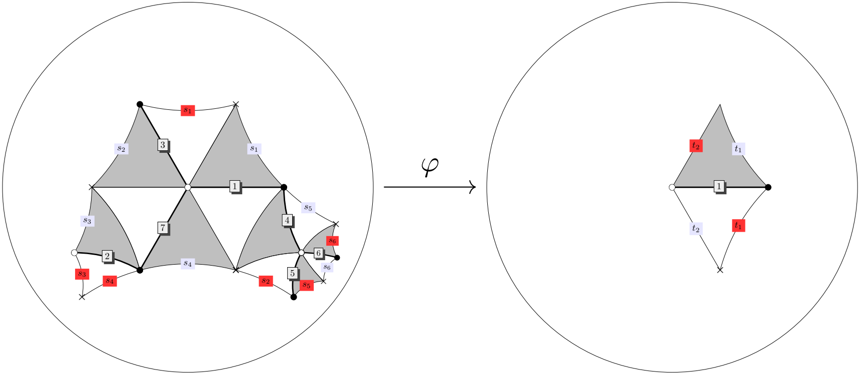
Let $\Gamma$ be a subgroup of a triangle group $\Delta = \Delta(a, b, c)$. Then $\Delta$ (and $\Gamma$) acts on its associated geometric space $\mathcal{H}$ (the sphere, Euclidean space, or hyperbolic space).

The quotient space $\Delta\backslash\mathcal{H}$ is homemorphic to a sphere and, after resolving quotient singularities, even isomorphic to $\mathbb{P}^1_{\mathbb{C}}$.

Similarly, $\Gamma\backslash\mathcal{H}$ can be given the structure of a smooth projective curve, and there is a natural map

$$\begin{aligned} \Gamma\backslash\mathcal{H} &\to \Delta\backslash\mathcal{H} \\ z \bmod \Gamma &\mapsto z \bmod \Delta. \end{aligned}$$

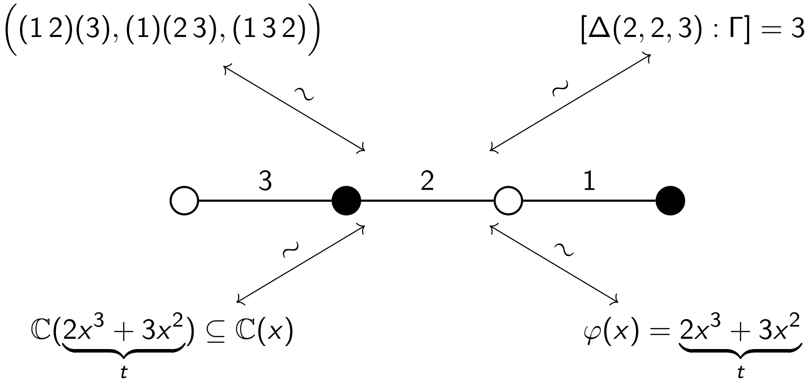
# Triangle subgroups → Belyi maps



# Navigating the bijections

- Belyi maps $\to$ permutation triples: monodromy
- Permutation triples $\to$ triangle subgroups: $\Gamma = \mathrm{Stab}_{\Delta}(1)$
- Triangle subgroups $\to$ Belyi maps: $\Gamma \backslash \mathcal{H} \to \Delta \backslash \mathcal{H}$
- Belyi maps $\to$ dessins: $\varphi^{-1}([0,1])$
- Belyi maps $\to$ function field extensions: rational (meromorphic) function fields

# Example



$\big((1\,2)(3), (1)(2\,3), (1\,3\,2)\big)$

$[\Delta(2,2,3) : \Gamma] = 3$

$\mathbb{C}(\underbrace{2x^3+3x^2}_{t}) \subseteq \mathbb{C}(x)$

$\varphi(x) = \underbrace{2x^3+3x^2}_{t}$

# Questions?

# Belyi maps and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

> 'A. Grothendieck and his students developed a combinatorial description ("maps") of finite coverings [of $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$]... It has not helped in understanding the Galois action. We have only a few examples of non-solvable coverings whose Galois conjugates have been computed.'
>
> —Pierre Deligne

**Goal:** Compute many examples of Belyi maps.

# Computation

Sijsling and Voight (2014) give a survey of the various techniques used to compute Belyi maps. They roughly fall into 3 categories:

- Direct method (Gröbner bases)
- *p*-adic methods
- Complex analytic methods

# Direct method

Consider the passport $(0, S_3, (2+1, 2+1, 3)$ as in the first example.

Since the cycles of length $2, 2, 3$ are unique in their respective fibers above $0, 1, \infty$, we can apply an automorphism of $\mathbb{P}^1$ so that corresponding points are again $0, 1, \infty$ without extending the field of definition. Then the Belyi map is a degree 3 polynomial of the form

$$\varphi(x) = cx^2(x-a) = 1 + c(x-1)^2(x-b)$$

for some $a, b, c \in \overline{\mathbb{Q}}$. Expanding and equating coefficients, we have

$$0 = bc - 1 = (2b+1)c = (a-b-2)c$$

which has the unique solution

$$a = 3/2, \qquad b = -1/2, \qquad c = -2\,.$$

In general, one can try to use Gröbner bases to try to solve the resulting system of polynomial equations in the coefficients.

# KMSV (2014) numerical method

- **Input:** A permutation triple $\sigma = (\sigma_0, \sigma_1, \sigma_\infty)$
- **Output:** Equations for the corresponding curve $X$ and Belyi map $\varphi : X \to \mathbb{P}^1$ with monodromy group $\langle \sigma \rangle$

1. Form the triangle subgroup $\Gamma \leq \Delta(a, b, c)$ associated to $\sigma$ and compute its coset graph.
2. Use a reduction algorithm for $\Gamma$ and numerical linear algebra to compute numerical power series expansions of modular forms $f_i \in S_k(\Gamma)$ for an appropriate weight $k$.
3. Use numerical linear algebra and Riemann–Roch to find polynomial relations among the series $f_i$, yielding equations for the curve $X$ and $\varphi$.
4. Normalize the equations of $X$ and $\varphi$ so that the coefficients are algebraic and recognize these coefficients as elements of a number field $K \subseteq \mathbb{C}$.
5. Verify that $\varphi$ has the correct ramification and monodromy.

# Newton's Method

We can combine the direct method and our numerical method.

1. Compute a moderate-precision numerical approximation using KMSV.
2. Use this as an initial approximate solution to the system of polynomial equations from the direct method.
3. Apply Newton iteration to obtain massive gains in precision.

The passport $(0, S_8, (5+2+1, 4+3+1, 4+2+1+1)$ has size 28 and turns out to consist of one Galois orbit. Using Newton iteration to compute to $26,000$ digits of precision, we are able to recognize the coefficients of the corresponding Belyi map as elements of a degree 28 number field.

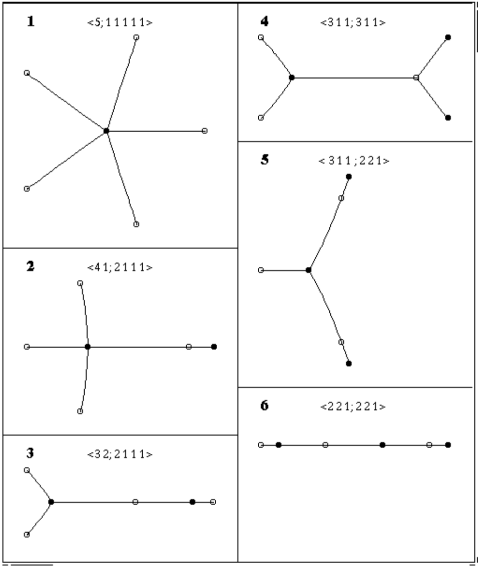
# Data

# Plane trees and Shabat polynomials, 1996

Bétréma, Péré, and Zvonkin created a catalogue of dessins that are plane trees with at most 8 edges, and computed the corresponding Belyi maps.

`https://www.labri.fr/perso/betrema/arbres/`

# Plane trees and Shabat polynomials, 1996



$$p_1 = z^5$$

$$p_2 = z^4(1 - z)$$

$$p_3 = z^3(z - 1)^2$$

$$p_4 = z^5 - \frac{10}{3}z^3 + 5z + \frac{8}{3}$$

$$p_5 = z^3\left(z^2 - 2z + \frac{32}{5}\right)$$

$$p_6 = 16z^5 - 20z^3 + 5z$$

# Elkin database, 2012

Arsen Elkin created a database of Belyi maps written in *Magma*. It includes Belyi maps of degree up to 8.

`https://github.com/arsenelkin/Belyi-Maps`

# Elkin database, 2012

```
> load "database.m";
...
> database := BelyiMapDatabase();
> passport1 := [[5],[1,3,1],[5]];
> NumberOfBelyiMaps(database, passport1);
14
> phi1 := BelyiMap(database, passport1, 5);
> phi1;
(1/5*(4*w^3 + 36*w)*x + 1/72*(31*w^3 + 434*w))*y - 5*x^2 + 1/432*(-35*w^2 -
    420)*x + 63425/124416
> BelyiPassportToString(BelyiPassport(phi1));
[[5],[3,1,1],[5]]
> Parent(phi1);
Function Field of Hyperelliptic Curve defined by y^2 = x^3 + 1345/41472*x +
    1/53747712*(237025*w^2 + 2844300) over Number Field with defining polynomial
t^4 + 24*t^2 + 150 over the Rational Field
> curve1 := Curve(Parent(phi1));
> curve1;
Hyperelliptic Curve defined by y^2 = x^3 + 1345/41472*x + 1/53747712*(237025*w^2
    + 2844300) over Number Field with defining polynomial t^4 + 24*t^2 + 150
over the Rational Field
> Genus(curve1);
1
```

# BelyiDB, 2018

A database of Belyi maps in *Magma* computed as a part of Musty, Sijsling, Schiavone, Voight (2019).

`https://github.com/michaelmusty/BelyiDB`

# Belyi maps in the LMFDB, 2018 (-present)

`beta.lmfdb.org/Belyi/`

New(ish) features:

- Downloads
- Portraits
- Links to curves
- More search features
- More statistics
- Primitivizations (joint work with Alexandra Hoey, Katherine Gravel, and Ian Limarta)

# Belyi maps (not yet) in the LMFDB, present

`pink.lmfdb.xyz/Belyi/`

Brand new features:

- Even more maps! 1,111 Galois orbits of Belyi maps and 1,007 passports (thanks to Edgar Costa)
- Embedded Belyi map pages

Thank you!