What examples should we ask AI for?

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Example problems

An *example problem* is a problem that asks to construct or define a mathematical object with specific properties.

A *counterexample problem* is the problem of disproving a conjecture that there does not exist a mathematical object with specific properties.

An *generative example problem* is an example problem whose solution would lead to new, interesting mathematics.

Counterexample problems may or may not be generative. Generative example problems may or may not involve counterexamples.

Question: How can we find generative example problems? Can we do better than simply listing the most interesting conjectures, and asking for counterexamples?

AI's strength at counterexamples

AI seems particularly strong, relative to humans, at finding counterexamples.

- The most impressive problems solved by AI have mostly been counterexample problems.

It seems likely to me that AI is good at finding examples, and it has found many counterexamples because people have asked AI to resolve many conjectures, some of which admit counterexamples.

- Why? Probably because AI has more time to try many different constructions.

Thus, to best use AI, we should come up with many generative example problems.

AI's strength at *counter*examples

Why *counter*examples?

- Most conjectures are universal statements like "All elliptic curves over $\mathbb{Q}$ are modular". Can't be proven with an example.

The practice of mathematics is mostly about proving universal statements.

A problem: Conjectures are also statements where mathematicians believe a proof would be particularly interesting. They are not necessarily statements where a disproof would be particularly interesting.

A challenge: Can we figure out which examples would be most interesting to find, and then ask AIs to find them?

Don't ask AI to disprove Goldbach

Goldbach conjectured that every even number $n > 2$ is a sum of two primes.

This is an interesting conjecture, and a proof would be interesting.

But it's probably not a good use of time to ask AI to disprove it:

- Goldbach is probably true, so no disproof is likely.
- Disproof seems to require ordinary computer search: AI may have no advantage.
- A counterexample, if it existed, would not tell us very much.

A counterpoint: A counterexample to Goldbach would be so surprising that it might somehow help us revise probabilistic models of primes.

An example due to Ramanujan

What is the most insight we can hope to get from an example?

Ramanujan found the function $\tau(n)$, defined as the coefficientt of q^n in the power series $q\prod_{m=1}^{\infty}(1-q^m)^{24}$, and observed that it has multiple remarkable properties:

- $\tau(nm) = \tau(n)\tau(m)$ if $\gcd(n,m) = 1$
- $\tau(p^{k+1}) = \tau(p)\tau(p^k) - p^{11}\tau(p^{k-1})$ for p prime, $k \geq 1$
- $|\tau(p)|] \leq 2p^{\frac{11}{2}}$ for p prime.

Proven by Mordell (Hecke operators) and Deligne (Galois representations attached to modular forms, Weil conjectures). Led to multiple major fields of number theory.

Probably not possible to know in advance how interesting this example would be, but one can try.

Counterexamples are pretty interesting

Finding a counterexample always has some benefits:

- One learns something surprising about the mathematical universe (at least if one believed the conjecture).
- One knows not to waste time searching for a proof.
- One can often use it to find other counterexamples.

I'm interested in examples that benefit mathematics, and in particular the quest to prove universal statements, in other ways:

- Ways to learn a subject
- Testing grounds for arguments
- Tools used in arguments
- Objects with strange properties that cry out for explanation

Curves with many rational points (suggested by David Speyer)

Problem: construct, for q a non-square prime power, a sequence of smooth projective connected curves X_i of genuses g_i tending to infinity, such that

$$\lim_{i \to \infty} \frac{\#X_i(\mathbb{F}_q)}{g_i} = \sqrt{q} - 1.$$

- $\sqrt{q} - 1$ is largest possible (Drinfeld-Vlăduț).
- Multiple constructions are known for q a perfect square.

Would be interesting because:

- The constructions when q is square are interesting.
- There are interesting applications of these constructions.

Super-conjectures (suggested by Simon L. Rydin Myerson)

Most conjectures, when disproven, we realize the conjecture was wrong

Some conjectures, we just realize we didn't state the conjecture correctly.

Manin's conjecture gave a prediction for the number of rational points on a Fano variety whose height is $< X$, as a function of X, excluding points lying on bad subvarieties like lines on a cubic surface.

Counterexamples were found.

Solution: Also remove points lying on a "thin set".

- Led to new mathematics

Similar story with Malle's conjecture on the number of number fields with a given Galois group and discriminant $< X$.

How to look for more examples

Find conjectures which we believe so strongly that we would not abandon them if a counterexample was found.

Myerson suggests:

- Malle's conjecture on counting number fields (in current form due to Loughran-Santens)
- Serre's problem on counting everywhere locally soluble members of a family of varieties
- Logarithmic Manin's conjecture on counting integer points on varieties

A variant: Rather than ask for counterexamples, ask for numerical apparent counterexamples: specific X where the number of points of height at most X is much larger or smaller than predicted.

Examples used in proofs

Sometimes the existence of an example with specific properties is used to prove a general statement.

To count number fields of degree 5 over $\mathbb{Q}$, Bhargava needed an example of a representation V of a group G such that the ring of G-invariant polynomials on V has a single generator, and the stabilizer in G of a general point of V is isomorphic to S_5.

If you need such an example, maybe ask AI.

If AI can't find it, maybe publicize the properties of the needed example and see if anyone else can find it.

Can we find examples used in proofs in advance?

That is, can we find an example which is likely to be used in future proofs, without just working on the proof to see what example is needed?

One general idea: Rational moduli spaces for use in modularity arguments:

- Wiles/Taylor-Wiles proof that all semistable elliptic curves over $\mathbb{Q}$ are modular uses a $3-5$ switch. This argument relies on the fact that the moduli space of elliptic curves with level 5 structure is rational.
- Boxer-Calegari-Gee-Pilloni proof that a positive proportion of abelian surfaces over $\mathbb{Q}$ are modular relies on the fact that the moduli space of genus 2 curves with level 3 structure and a marked Weierstrass point is rational.

Can we write down a list of moduli spaces of varieties with level structure that may be useful in future modularity theorems and ask AI to find rational examples from the list?

A question about braid groups

Braided set: A set X with a bijection $\sigma\colon X \times X \to X \times X$ such that $(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma)(\sigma \times \mathrm{id}) = (\mathrm{id} \times \sigma)(\sigma \times \mathrm{id})(\mathrm{id} \times \sigma)$.

The braid group B_n has generators $\sigma_1, \ldots, \sigma_{n-1}$, and the relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

For X a braided set, the braid group B_n acts on X^n by applying σ to adjacent pairs of coordinates.

X is a rack if $\sigma(x, y) = (y, f(x, y))$ for $f\colon X \times X \to X$. The braid group representations that arise from racks have been used in powerful recent results in number theory over function fields.

Question: Does there exist a braided set X, whose braid group representations do not come from racks, in the sense that there does not exist a rack R such that for all n, every element of B_n that acts trivially on R^n also acts trivially on X^n?

Instructive examples

Many examples are helpful when first learning a subject. We usually don't need AI to find these.

Examples are also helpful at other times, like when working on a proof. Can find a strategy that works in one case, then generalize.

One approach to find these, is when trying to prove a statement, ask AI for counterexamples to stronger statements (or related but logically incomparable ones).

Counterexamples to probabilistic conjectures

The most valued examples are those that reveal some new structure. But how to look for these before we know what structure?

One approach: Take a heuristic that some mathematical object behaves randomly. Formulate a very strong conjecture that the heuristic should always be correct. Look for counterexamples.

- Many different kinds of structure will prevent random behavior.

Example: Cramér, based on a model of the primes as a random set of numbers, conjectured that for p_n the nth prime, $p_{n+1} - p_n = O(\log(p_n)^2)$. No counterexamples known.

The same model predicts that $\pi(x+y) - \pi(x) = \frac{y}{\log x} + O(y^{1/2+\epsilon})$. Counterexamples to this are known (Maier's theorem), by some interesting structural method.

There are many randomness heuristics in number theory one can use as a starting point.