The freeness alternative to thin sets in Manin's conjecture

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Manin's conjecture

Let X be a smooth projective Fano variety (i.e. the anticanonical bundle K_X^{-1} is ample) defined over $\mathbb{Q}$.

We can define the height of $x \in X(\mathbb{Q})$ by choosing an embedding $i\colon X \to \mathbb{P}^N_{\mathbb{Q}}$ and setting $h_i(x) = \max(|x_0|, \ldots, |x_N|)$ when $i(x) = (x_0 : \cdots : x_N)$ for $x_0, \ldots, x_n \in \mathbb{Z}$ with no common factors. If $i^*\mathcal{O}(d) \cong K_X^{-e}$ then we say $H(x) = h_i(x)^{\frac{d}{e}}$ is the anticanonical height.

Manin's conjecture gives beautiful predictions for the number of $x \in X(\mathbb{Q})$ of height $h(X) < T$, as well as the distribution of those x in $X(\mathbb{Q}_p)$ and $X(\mathbb{R})$.

But for these predictions to come true, we must first remove a thin set of rational points.

Thin sets and their discontents

A thin set of $X(\mathbb{Q})$ is a finite union of subsets of $X(\mathbb{Q})$ of the following two types:

- $Y(\mathbb{Q})$ for a subvariety $Y \subset X$ of X.
- The image of $Z(\mathbb{Q})$ for $f \colon Z \to X$ a generically finite map of degree ≥ 2.

Using cutting-edge algebraic geometry tools, Lehmann, Sengupta, and Tanimoto found a good thin set to remove.

But this is all very strange. To decide whether a rational point is good or bad, you first have to go looking for bad subvarieties of your variety, or bad coverings.

Is there a way to tell whether a rational point is good or bad by looking at just that point?

Counting lattice points

Let's think about a (seemingly) overly specific special case – integer points on a variety defined by linear equations, or, in other words, lattice points.

Let $y_1, \dots, y_k$ be vectors in $\mathbb{Z}^N$. Let

$$X = \{x \in \mathbb{A}^N \mid x \cdot y_1 = \cdots = x \cdot y_k = 0\}.$$

How many $x \in \mathbb{Z}^N$ with $||x|| < R$ lie in X?

This is a special case of the affine analogue of Manin's conjecture, with integer points instead of rational points, and $||x||$ playing the role of height.

It is also a crucial step of the proof of pretty much every case of Manin's conjecture, starting with projective space.

Counting lattice points

How many $x \in \mathbb{Z}^N$ with $||x|| < R$ satisfy $x \cdot y_1 = \cdots = x \cdot y_k = 0$?

Let $\Lambda = \{x \in \mathbb{Z}^N \mid x \cdot y_1 = \cdots = x \cdot y_k = 0\}$ be a lattice in the vector space $V = \{x \in \mathbb{R}^N \mid x \cdot y_1 = \cdots = x \cdot y_k = 0\}$. Let n be the dimension of this vector space / rank of this lattice. Let C_n be the volume of the unit ball of dimension n.

A natural guess:

$$\#\{x \in \Lambda \mid ||x|| < R\} \approx \frac{C_n R^n}{\mathrm{vol}(V/\Lambda)}$$

The lattice points are evenly distributed in V, with density $\frac{1}{\mathrm{vol}(V/\Lambda)}$. Therefore, we expect the number of lattice points in any region to be approximately proportional to the volume of that region. The volume of the ball of radius R is $C_n R^n$.

When is this true?

Successive minima

An answer can be provided using the successive minima of the lattice Λ.

Let λ_r be the least λ such that Λ contains r linearly independent vectors of length $\leq \lambda$. So λ_1 is the length of the shortest nonzero vector v_1, λ_2 is the length of the shortest vector v_2 not a multiple of v_1, etc. We call λ_r the rth successive minimum.

The most important one for us is λ_n.

- If λ_n/R is small then

  $$\#\{x \in \Lambda \mid x\cdot, ||x|| < R\} \approx \frac{C_n R^n}{\mathrm{vol}(V/\Lambda)}$$

  is true with error term proportional to λ_n/R.
- If $\lambda_n > R$ then all points in the ball of radius R lie in the sublattice generated by $v_1, \dots, v_{n-1}$. A "thin set". Moreover, if λ_n/R is large then the number of integer points is much more than predicted.

Freeness for affine varieties

Let $X \subseteq \mathbb{A}^N$ be a smooth variety of dimension n defined by polynomials $f_1, \ldots, f_k$. Let $x \in X(\mathbb{Z})$ be an integer point. How can we apply this dichotomy to $x \in X$?

Take the derivative to obtain a lattice! Let

$$\Lambda_x = \{y \in \mathbb{Z}^N \mid y \cdot \nabla f_1(x) = \cdots = y \cdot \nabla f_k(x) = 0.\}$$

Because X is smooth, Λ_x is a lattice of rank n.
We say x is free if

$$\lambda_n(\Lambda_x) < ||x||^{1-\epsilon}.$$

Freeness for projective varieties

Let $X \subseteq \mathbb{P}^N$ be a smooth variety of dimension n defined by homogeneous polynomials $f_1, \ldots, f_k$. Let $x \in X(\mathbb{Q})$ be an rational point with projective coordinates $(x_0 : \cdots : x_N)$.
Let

$$\Lambda_x = \{y \in \mathbb{Z}^N / \langle (x_0, \ldots, x_n) \rangle \mid y \cdot \nabla f_1(x) = \cdots = y \cdot \nabla f_k(x) = 0\}.$$

Here the norm of a vector y is the length of the projection of y to the orthogonal complement of $(x_0, \ldots, x_n)$, equivalently is $\min\{||y + t(x_0, \ldots, x_n)||_2 \mid t \in \mathbb{R}\}$.

Following Peyre, we say x is free if

$$\lambda_n(\Lambda_x) < \max(|x_0|, \ldots, |x_N|)^{1-\epsilon}.$$

Q (Peyre): Can removing the non-free rational points replace the removal of the thin set in Manin's conjecture?

Positive example: The cubic surface case

Let $X \subseteq \mathbb{P}^3$ be defined by a cubic equation, for concreteness $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$.

The anticanonical line bundle on X is $\mathcal{O}(1)$, so the height is simply $\max(|x_0|, |x_1|, |x_2|, |x_3|)$. Manin's conjecture predicts the number of points of height $< T$ is proportional to T times a power of $\log T$.

The thin set in this case consists of at most 27 lines on the cubic surface X, for example $(x_0 : x_1 : x_2 : x_3) = (a : -a : b : -b)$. The number of points of height $< T$ on a line is proportional to T^2, much too big.

We want to check that removing unfree points can substitute for removing the thin set. In particular, we need to check that almost every point on the line is not free.

Positive example: The cubic surface case

Let $X \subseteq \mathbb{P}^3$ be defined by $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$. Let $x = (a : -a : b : -b)$.

Then $\Lambda_x = \{(y_1, y_2, y_3, y_4) \in \mathbb{Z}^4/x \mid a^2(y_1 + y_2) + b^2(y_3 + y_4) = 0.\}$ The sublattice defined by $y_1 + y_2 = y_3 + y_4 = 0$ contains only one linearly independent vector (mod x). Outside this sublattice, we must have $y_1 + y_2$ a multiple of b^2 and $y_3 + y_4$ a multiple of a^2, so the minimum length is

$$\sqrt{\frac{a^4 + b^4}{2}} \approx \max(|a|, |b|)^2 > \max(|a|, |b|)^{1-\epsilon}.$$

So x is not free.

The generator of this sublattice is $(c : -c : d : -d)$ where $ad - bc = 1$, which has length

$$\lambda_1 = \sqrt{\frac{2}{a^2 + b^2}} \approx \max(|a|, |b|)^{-1}.$$

The schematic perspective

Suppose we can spread our variety X out to a proper scheme $\mathcal{X}$ of dimension n over $\mathbb{Z}$. Further, fix a Riemannian metric on the real points $X(\mathbb{R})$.

A rational point $x \in X(\mathbb{Q})$ extends to a map $i_x \colon \operatorname{Spec} \mathbb{Z} \to \mathcal{X}$.

The pullback $i_x^* \mathcal{T}_{\mathcal{X}}$ of the tangent bundle $\mathcal{T}_{\mathcal{X}}$ of $\mathcal{X}$ is a vector bundle on $\operatorname{Spec} \mathbb{Z}$, which defines a lattice Λ_x. Our fixed Riemannian metric defines a metric on this lattice.

We say x is free if $\lambda_n(\Lambda_x) < H(x)^{-\epsilon}$.

The geometric perspective

There is a close analogy between rational points on varieties over $\mathbb{Q}$ and rational curves on varieties over finite fields.

The set of rational curves on a varity X is not just a set, but carries geometric structure – the moduli space of rational curves on X. The tangent bundle of the moduli space of rational curves on X can be calculated using the tangent bundle of X.

Using this, we can show that the smooth points of the moduli space of rational curves on X correspond to the rational curves that are free in Peyre's sense. (Closely related to the concepts of a free and very free rational curve).

So we have a geometric reason to study freeness, even if we like the current Manin's conjecture just fine.

Positive example: Hypersurfaces (Browning-S)

Birch: Let X be a smooth hypersurface of degree d in $\mathbb{P}^N$. If $N \geq 2^d(d-1)$, then the number of points in $X(\mathbb{Q})$ of height $< T$ is proportional to a constant times T. That is, Manin's conjecture is true, with empty thin set.

Browning-S: Let X be a smooth hypersurface of degree d in $\mathbb{P}^N$. If $N \geq 3 \cdot 2^{d-1}(d-1)$, then the number of free points in $X(\mathbb{Q})$ of height $< T$ is proportional to the same constant times T.

Method of proof: After Birch, suffices to upper bound the number of unfree points. If x is unfree, then there are many $y \in \Lambda_x$ with $||y|| < ||x||$. So it suffices to upper bound the number of solutions (x, y) to the system of equations $f(x) = 0, y \cdot \nabla f = 0$, which we do with a circle method argument, following closely the strategy of Birch.

Negative example: Hilbert schemes of projective space (S)

Let $X = \mathrm{Hilb}_2(\mathbb{P}^n)$. This is a resolution of the singularities of $\mathrm{Sym}^2(\mathbb{P}^n)$. Abstractly, it is the moduli space of ideal sheaves on $\mathbb{P}^n$ with quotient of length 2. Concretely, it is the quotient of the blow-up $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n)$ of the diagonal Δ of $\mathbb{P}^n \times \mathbb{P}^n$ by the involution switching the two copies of $\mathbb{P}^n$.

In Manin's conjecture for $\mathrm{Hilb}_2(\mathbb{P}^n)$, the thin set is the image of $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n)(\mathbb{Q})$.

Can removing unfree points substitute for removing this thin set? No! In fact, most points in this thin set are free.

Method of proof: First, following Peyre, check that most points in $(\mathbb{P}^n \times \mathbb{P}^n)(\mathbb{Q})$ are free. Then, show that the tangent lattice Λ_x doesn't change much as we pullback along the blow-up map $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \to \mathbb{P}^n \times \mathbb{P}^n$ and then follow the degree two covering $Bl_\Delta(\mathbb{P}^n \times \mathbb{P}^n) \to \mathrm{Hilb}_2(\mathbb{P}^n)$. Because the tangent lattice doesn't change much, the heights and successive minima don't change much.

Where does this leave freeness?

It is possible that unfree points do substitute for the special subvarieties in Manin's conjecture, but do not substitute for the degree ≥ 2 covers.

Peyre has another proposal, the "all the heights" approach, which should suffice to substitute for the degree ≥ 2 thin set.

It's possible that combining the two gives a good alternative formulation of Manin's conjecture.

Open questions

(1) Can we come up with new strategies to count unfree points? Currently known strategies include bounding the number of pairs (x, y) with $y \in \Lambda_x$ (Browning-S), and, for special varieties like Grassmanians, giving an explicit description of Λ_x and using equidistribution results for lattices (Browning-Horesh-Wilsch).

(2) Can we find any example where counting free points is easier than counting all rational points (or all rational points outside an explicit thin set)?

(3) What do the number and distributions of unfree points look like numerically for some interesting examples of Fano manifolds?

(4) Are there low-dimensional examples where we can calculate the cohomology of the moduli space of free curves on X and the moduli space of all curves on X and see which one better approximates the cohomology of the loop space of X?