# Old and new on the symmetry groups of K3 surfaces

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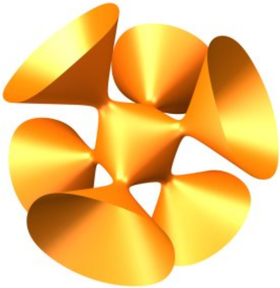


# Automorphisms of K3 surfaces

We recall briefly the definition:

- a K3 surface $S$ is a compact complex smooth manifold of dimension 2 such that:
  - $S$ is simply connected,
  - up to scalar multiplication there exists a unique global holomorphic two form without zeros, $H^0(S, \Omega^2_S) \cong \mathbb{C}\omega_S$
  - we call $\omega_S$ the period of $S$.
- Here we want to study the automorphism group of $S$

$$\mathrm{Aut}(S) = \{f : S \longrightarrow S \mid f \text{ is biholomorphic}\}$$

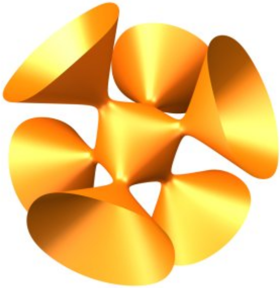
i.e. we want to study the symmetries.

# Quartics

Smooth quartics in $\mathbb{P}^3(\mathbb{C})$ are the "most easy" examples of K3 surfaces :

$$S = \{f_4(x_0, x_1, x_2, x_3) = 0\} \subset \mathbb{P}^3(\mathbb{C}), \qquad \deg f_4 = 4, \text{ homogeneous.}$$

Do you want to see a K3? A picture:



$$1 + x^4 + y^4 + z^4 + a(1 + x^2 + y^2 + z^2)^2 = 0, \quad a = -0.49$$

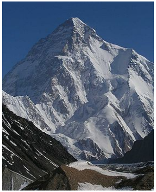
# K3 surfaces : André Weil

The name K3 was given by André Weil in 1958, in a report on a research project for his research stay in Paris.

*Dans la seconde partie de mon rapport, il s'agit des variétés kählériennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire.*

In the second part of my report, it is about certains Kähler manifolds, called K3, so named in honour of Kummer, Kähler, Kodaira and of the beautiful mountain K2 at Cachemire.

André Weil (1906-1998)



K2 mountain, 8611m
1st climb in 1954

# The Broad Peak at Cachemire

- It exists a K3 mountain in Cachemire. The name was given in 1856. It is also called Broad Peak, the 12th highest mountain in the world, 8047m.

- The first climb was "only" in 1957, at the same time as the report of André Weil.

The K3 mountain

# Involutions on quartics

- Take the Fermat quartic surface in $\mathbb{P}^3(\mathbb{C})$

$$F: \; x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$$

- it admits an involution

$$(x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : x_1 : x_2 : x_3),$$

- the 2-form in the chart $x_3 \neq 0$, $x_2 \neq 0$ is

$$\frac{dx_0 \wedge dx_1}{4x_2^3}$$

  which is multiplied by $-1$ by the involution.

# Lattices

- Lattice theory is an important tool when studying K3 surfaces.
- Recall that for a K3 surface $S$ we have

$$H^2(S,\mathbb{Z}) = U^3 \oplus E_8^2 := L, \quad \text{the K3 lattice}$$

- where $E_8$ is the lattice associated to the root system with the same name and

$$U = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right).$$

- The K3 lattice has rank 22, signature $(3,19)$, it is even and unimodular,
- where even means $(v,v) \in 2\mathbb{Z}$ for all $v \in L$.
- If $g \in \mathrm{Aut}(S)$ we will often consider

$$H^2(S,\mathbb{Z})^{g^*} = \{x \in H^2(S,\mathbb{Z}) \mid g^*x = x\} \text{ the invariant lattice}$$

# The problem

- Study properties of automorphisms of finite order on K3 surfaces.
- Some main problems: classify finite groups $G$ that can act on a K3 surface and study their action on cohomology.
- Most of the K3 surfaces have an infinite automorphism group, only a finite number have finite automorphism group and the Picard lattices are known by results of Nikulin and Vinberg.
- If a K3 surface has $\operatorname{rk}\operatorname{Pic}(S) = 20$ then Shioda and Inose in 1977 by using the elliptic fibrations show that $\operatorname{Aut}(S)$ has infinite order.

# A useful exact sequence

- If $G$ is a finite group acting on a K3 surface $S$, the group $G$ induces an action on the vector space $H^0(S, \Omega_S^2) \cong \mathbb{C}\omega_S$
- Let $g \in G$ then $g^*\omega_S = \alpha(g)\omega_S$, $\alpha(g) \in \mathbb{C}^*$
- We get a map:
$$\alpha : \; G \longrightarrow \mathbb{C}^*, \;\; g \mapsto \alpha(g)$$
- If $g$ has finite order $n$ then $\alpha(g)$ is a $n$th-root of unity
- Since $G$ is finite then $\mathrm{im}(\alpha)$ is a cyclic group of some $m$th-roots of unity $\mu_m$.
- We get an exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1$$

# Symplectic automorphisms

- We have an exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1$$

- where $G_0$ are those automorphisms that act as the identity on $\omega_S$ (the symplectic automorphisms),
- observe that if $G_0 = \{\mathrm{id}\}$ then $G$ is cyclic.

# Non–symplectic automorphisms

- An automorphism $\sigma$ is said to act non–symplectically if its action is non–trivial on the 2-form.
- If $\sigma$ has order $n$ and it acts on $\omega_S$ by multiplication by a primitive $n$th–root of unity then one says that $\sigma$ acts purely non–symplectically.
- In particular by the exact sequence

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1$$

if a group $G$ acts purely non–symplectically then $G$ is cyclic.

# Important tools

- Nikulin started the study of automorphisms in the 80's, by using lattice theory. His results together with Torelli's theorem are powerful tools in the study of automorphisms:

**Theorem (Torelli Theorem)(Piatetski-Shapiro and Shafarevich, 1971)**

*Let $S$ and $S'$ be K3 surfaces and let $\varphi : H^2(S,\mathbb{Z}) \longrightarrow H^2(S',\mathbb{Z})$ be an isometry of lattices if*

1. $\varphi(\mathbb{C}\omega_S) = \mathbb{C}\omega_{S'}$ *(Hodge isometry)*
2. $\varphi$ *sends a Kähler class to a Kähler class (effective isometry)*

*then there exists a unique isomorphism $f : S' \longrightarrow S$ such that $f^* = \varphi$.*

# How to attack the problem of classification

- Split first the study between symplectic and (purely) non–symplectic automorphisms,
- then use the exact sequence to study $G$ in all generality.
- First questions : how big can be $G$, $G_0$ and $m$?

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1$$

# Symplectic automorphisms: finite abelian groups

In his famous paper *Finite groups of automorphisms of Kählerian surfaces of type K3* in 1976, Nikulin classifies all finite abelian groups acting symplectically on a K3 surface. These are 14 cases (different from the identity):

- $\mathbb{Z}/n\mathbb{Z}, 2 \leq n \leq 8,$
- $(\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}),\ m = 2, 3, 4,$
- $(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}),\ (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}),$
- $(\mathbb{Z}/2\mathbb{Z})^i,\ i = 3, 4$

# Unicity of the action in cohomology

- An important problem is to determine the action in cohomology,
- an important result of Nikulin says that if one fixes $G_0$ one of the previous abelian groups, then up to conjugacy by an element in $O(L)$ where $L$ is the K3 lattice, the action on $L$ is unique.
- So it is enough to compute the action for one example, then one gets the action in general.
- Morrison in 1984 computed the action for a symplectic involution: it exchanges the two copies of $E_8$ in the K3 lattice $L$ and it preserves the rest.
- In several papers, Garbagnati–S. 2007–2009, we computed the invariant sublattice of the K3 lattice for the remaining abelian groups.

# Elliptic fibrations

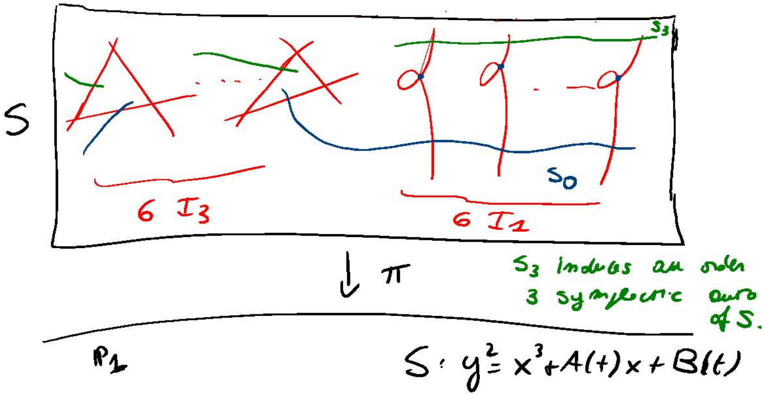
- These are very useful in the study of automorphisms.
- Take the elliptic fibration in Weierstrass equation on a K3 surface

$$y^2 = x^3 + A(t)x + B(t),\ \ A(t), B(t) \in \mathbb{C}[t]$$

  with $\deg A(t) \leq 8$ and $\deg B(t) \leq 12$.
- For a suitable choice of $A(t)$ and $B(t)$ the fibration admits a torsion section,
- this section induces a symplectic automorphism of the same order.

# Elliptic fibration with 3–torsion section

- Assume we have a 3–torsion section,
- the fibration admits 6 fibers of type $I_3$ and 6 fibers of type $I_1$, see the picture :



# Order 3 symplectic : the invariant lattice

**Theorem (Garbagnati–S. 2007)**

*Let $S$ be a K3 surface (projective or not) admitting a symplectic automorphism $\sigma$ of order three. Then*

$$H^2(S,\mathbb{Z})^{\sigma^*} = U \oplus U(3)^2 \oplus A_2^2, \ \ (H^2(S,\mathbb{Z})^{\sigma^*})^{\perp} = K_{12}(-2).$$

- where $K_{12}$ is the Coxeter–Todd lattice, a rank 12 even, lattice of discriminant $3^6$, which gives the densest sphere packing in dimension 12, known sofar.
- Last week on february 1st, there was a result of Garbagnati and Prieto computing explicitely the action on $H^2(S,\mathbb{Z})$.

# Groups of symplectic automorphisms

Remove the assumption that the groups are abelian.

- An important result of Mukai in 1988 says that

**Theorem (Mukai 1988)**

*A finite group acting symplectically on a K3 surface is a subgroup of the Mathieu group $M_{23}$.*

- where $M_{23}$ is one of the 26 sporadic groups, recall that there are 5 Mathieu groups $M_n$, for $n = 11, 12, 22, 23, 24$.
- Mukai describes 11 maximal finite groups acting symplectically on a K3 surface and gives examples of K3 surfaces with an action of each of these groups.
- Xiao in 1996 gives then the complete list of all finite groups acting symplectically, he gives a list of 81 groups.

# The Mathieu group $M_{20}$

- The Mathieu group $M_{20} = A_5 \ltimes (\mathbb{Z}/2\mathbb{Z})^4$ has order 960 and it is the biggest of the 11 groups in Mukai's list, it can also be described as the stabilizer subgroup of 21 and 22 in $M_{22}$.
- Example of Mukai of a K3 surface with a symplectic action by $M_{20}$:
$$X_{Mu} \ : \ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0x_1x_2x_3 = 0$$
- We will find this example again later when studying groups of maximum order, acting on a K3 surface (not necessarily symplectically).
- It seems that there are infinitely many K3 surfaces with a $M_{20}$ action (work in progress by Comparin–Demelle 2021).

# The maximum order of $G$

Remove the assumption that the groups act symplectically.

- Recall the exact sequence:

$$1 \longrightarrow G_0 \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1$$

- By Nikulin's results all K3 surfaces with $m > 1$ are projective.
- In 1999 Kondo shows that :

**Theorem (Kondo 1999)**

1. $|G| \leq 3840 = 4 \cdot 960.$
2. *If* $|G| = 3840$ *then* $S$ *is a Kummer surface* $\mathrm{Km}(E_{\sqrt{-1}} \times E_{\sqrt{-1}})$, *where* $E_{\sqrt{-1}} = \mathbb{C}/(\mathbb{Z} + \sqrt{-1}\,\mathbb{Z})$ *and* $G$ *is isomorphic to an extension of* $M_{20}$ *by* $\mathbb{Z}/4\mathbb{Z}$. *Moreover* $G$ *and an action of* $G$ *on* $S$ *are unique up to isomorphism.*

# Classify all maximal $G$

- We say that a finite group $G$ acting on a K3 surface $S$ is maximal if the following holds: if $H$ is another finite group acting on $S$ then $G$ is not contained in $H$.
- Problem classify all such groups: Kondo's result gives already the biggest one.
- But one can show that there is another exact sequence :

$$1 \longrightarrow M_{20} \longrightarrow G \xrightarrow{\alpha} \mu_2 \longrightarrow 1$$

which produces also two such non–isomorphic groups and unique K3 surfaces (Bonnafé–Sarti/Brandhorst–Hashimoto 2020)

# More results

- Brandhorst and Hashimoto classify in fact all pairs $(S, G)$ where $S$ is a K3 surface, $G \subset \mathrm{Aut}(S)$, such that the symplectic part $G_0 \subset G$ is one of the 11 maximal subgroups in Mukai's list:
  - they find 42 pairs,
  - and explicit equations for $S$ in 25 cases (maybe some more).
- In 2009 Frantzen classifies the groups $G_0 \times \mu_2$, where $G_0$ is one of the 11 maximal groups of Mukai.

## The degree 2 extensions of $M_{20}$

- Let us consider the exact sequence

$$1 \longrightarrow M_{20} \longrightarrow G \xrightarrow{\alpha} \mu_2 \longrightarrow 1$$

- with the previous notation we have :

**Theorem** (Bonnafé–Sarti 2020, Brandhorst–Hashimoto 2020)

*There are two non isomorphic groups $G$ such that the $G$-invariant Picard group of the corresponding K3 surface $S$ and its transcendental lattice are :*

1. $\langle 4 \rangle, \qquad \begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$
2. $\langle 8 \rangle, \qquad \begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$

*The K3 surfaces are Kummer surfaces.*

# Some steps in our proof and in Kondo's proof

- Let $\mathrm{Pic}(S)$ denote the Picard group of a K3 surface and $T_S$ be the transcendental lattice which is

$$T_S = \mathrm{Pic}(S)^{\perp} \cap H^2(S, \mathbb{Z}),$$

- if $g$ is an automorphism acting on $S$ then recall

$$H^2(S, \mathbb{Z})^{g^*} = \{x \in H^2(S, \mathbb{Z}) \,|\, g^* x = x\} \text{ the invariant lattice,}$$

- if $g$ is a symplectic automorphism by using the 2-form one can show

$$T_S \subset H^2(S, \mathbb{Z})^{g^*}, \; (H^2(S, \mathbb{Z})^{g^*})^{\perp} \subset \mathrm{Pic}(S)$$

- Let $\mathbb{L}_{20}$ be the following lattice of signature $(3,0)$:

$$\mathbb{L}_{20} = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 12 \end{pmatrix}$$

- observe that $\mathbb{L}_{20} = L'(2)$ for $L'$ some even lattice.
- If $G_0 = M_{20}$ acts symplectically on a K3 surface $S$, Kondo computed the invariant lattice :

$$H^2(S,\mathbb{Z})^{M_{20}} = \mathbb{L}_{20}$$

- hence $\mathrm{rk}((H^2(S,\mathbb{Z})^{M_{20}})^{\perp}) = 19$, so that the Picard group of the K3 surface must be of rank at least 19,
- again by Nikulin's results: $(H^2(S,\mathbb{Z})^{M_{20}})^{\perp}$ is a negative definite lattice.

- Since the K3 surface $S$ is projective, this tells us that $\operatorname{rk}\operatorname{Pic}(S) = 20$ which is the maximum possible for K3 surfaces,
- observe that the K3 surface contains an $M_{20}$–invariant ample class $L$, with $L^2 = 4t$ for some positive integer $t$.
- We have $\mathbb{Z}L \oplus T_S \subset \mathbb{L}_{20}$ with finite index, where recall:

$$\mathbb{L}_{20} = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 12 \end{pmatrix}$$

# The transcendental lattice

- If $v \in \mathbb{L}_{20}$ then $(v, v) \in 4\mathbb{Z}$
- this means that the transcendental lattice:

$$T_S = \begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}$$

- with some condition on $a, b, c \in \mathbb{Z}_{\geq 0}$ by using results of Shioda–Inose to classify K3 surfaces of Picard number 20:

**Theorem (Shioda–Inose 1977)**

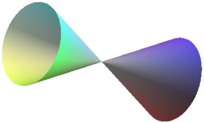
*There is a one-to-one correspondence from the set of singular (i.e. Picard number 20) K3 surfaces to the set of equivalence classes of positive definite even integral binary quadratic forms with respect to $SL_2(\mathbb{Z})$.*

- Since here $T_S = T'(2)$ with $T'$ even, by a result of the same authors one gets that $S$ is a Kummer surface.

# A Kummer surface

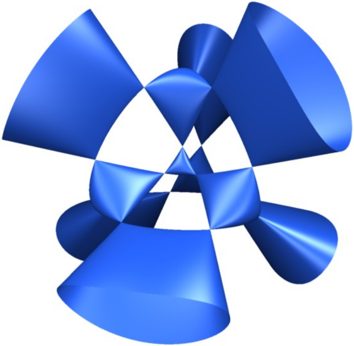
- Let $A = \mathbb{C}^2/\Lambda$, abelian surface, $\Lambda$ rank 4 lattice and let $\iota : (x, y) \mapsto (-x, -y)$ be an involution acting on it.
- The quotient $A/\langle\iota\rangle$ has 16 singularities of type $A_1$.
- Locally the equation of such a singularity is

$$\{z_0 z_1 - z_2^2 = 0\} \subset \mathbb{C}^3$$



An $A_1$ singularity

- The minimal resolution $\mathrm{Km}(A)$ contains 16 rational curves and it is called a Kummer surface.



# Back to the proof : bounding $m$

- Recall the exact sequence:

$$1 \longrightarrow M_{20} \longrightarrow G \xrightarrow{\alpha} \mu_m \longrightarrow 1$$

- By Nikulin's result the Euler's totient function of $m$ divides $\operatorname{rk} T_S = 2$,
- hence $m \in \{1, 2, 3, 4, 6\}$ and assume $m > 1$ to have a genuine extension of $M_{20}$.
- The group $\mu_m$ induces an isometry of $\mathbb{L}_{20}$ and Kondo computes

$$|O(\mathbb{L}_{20})| = 16$$

- so that $m \in \{2, 4\}$.

# Extensions of $M_{20}$

- So we get two exact sequences

$$1 \longrightarrow M_{20} \longrightarrow G \xrightarrow{\alpha} \mu_4 \longrightarrow 1$$

$$1 \longrightarrow M_{20} \longrightarrow G \xrightarrow{\alpha} \mu_2 \longrightarrow 1$$

- The first is the case described by Kondo,
- by using Xiao's list one sees that it is not possible to get order bigger than $960 \cdot 4 = 3840$.
- The second case is "our" case, here the order of $G$ is $960 \cdot 2 = 1920$.
- In both cases the sequence splits, so that $G = M_{20} \rtimes \mu_m$, $m = 2, 4$.

# Determine the polarization and the transcendental lattice

- One has to study how a lattice $\mathbb{Z}L$ with $L^2 = 4t$ can be embedded in $\mathbb{L}_{20}$ (to get the invariant ample class),
- use the fact that we have an action of $\mu_2$ on $\mathbb{L}_{20}$,
- one can compute the polarization and $T_S$ as in the theorem.

$$1 \longrightarrow M_{20} \longrightarrow G \xrightarrow{\alpha} \mu_2 \longrightarrow 1$$

**Theorem** (Bonnafé–Sarti 2020, Brandhorst–Hashimoto 2020)

*There are two non isomorphic groups $G$ such that the $G$-invariant Picard group of the corresponding K3 surface $S$ and its transcendental lattice are :*

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*The K3 surfaces are Kummer surfaces.*

- The groups $G$ are maximal!

# The two K3 surfaces

- The divisors $L$ with $L^2 = 4$ and 8 are $G$–invariant,
- hence the linear system $|L|$ allows to realize the K3 surfaces in some projective space,
- the action of $G$ can be linearized and comes from an action on the projective space.
- This allows to find examples in $\mathbb{P}^3$ and $\mathbb{P}^5$.
- The Kondo's surface has $L^2 = 40$ so that it lives in $\mathbb{P}^{21}$! We will see a singular model.

# An equation for $X_{Mu}$

- In one case the ample class (i.e. the polarization) has $L^2 = 4$ and we get a quartic in $\mathbb{P}^3$.
- Consider again the example of Mukai, of a quartic K3 surface with a symplectic $M_{20}$–action:

$$X_{Mu} \ : \ x_0^4 + x_1^4 + x_2^4 + x_3^4 + 12x_0x_1x_2x_3 = 0$$

- one can perform a change of coordinates and write

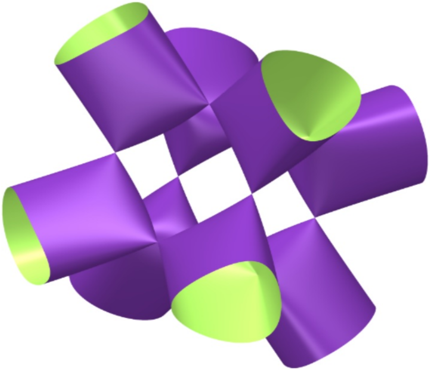
$$X'_{Mu}\ :\ x_0^4+x_1^4+x_2^4+x_3^4-6(x_0^2x_1^2+x_0^2x_2^2+x_0^2x_3^2+x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2)=0$$

- clearly $X'_{Mu}$ is invariant by the non–symplectic involution

$$\iota : (x_0 : x_1 : x_2 : x_3) \mapsto (-x_0 : x_1 : x_2 : x_3).$$

- The finite group $G_{Mu}$ is generated by $M_{20}$ and $\iota$ and has order $2 \cdot 960 = 1920$.
- It is a complex reflection group, called $G_{29}$ in Shepard–Todd classification.

# A singular deformation of $X'_{Mu}$



$$1 + x^4 + y^4 + z^4 - (x^2y^2 + x^2z^2 + x^2 + y^2z^2 + y^2 + z^2) = 0$$

# An equation for $X_{BH}$

- In the second case the polarization has $L^2 = 8$, so the K3 surface is a complete intersection of three smooth quadrics in $\mathbb{P}^5$.
- We study some central extensions of $M_{20}$ to find irreducible representations on $\mathbb{C}^6$,
- Take $G_{BH}$ be the subgroup of $\mathrm{GL}_6(\mathbb{C})$ generated by:

$$t = \mathrm{diag}(-1, 1, 1, 1, 1, 1),$$

$$u = \begin{pmatrix} i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

- The projective group $PG_{BH}$ is the group we were looking for, it contains $M_{20}$ with index 2.
- One gets the equations for $X_{BH}$ (MAGMA computations):

$$\begin{cases} x_0^2 + x_3^2 - \phi x_4^2 + \phi x_5^2 = 0, \\ x_1^2 - \phi x_3^2 + x_4^2 - \phi x_5^2 = 0, \\ x_2^2 + \phi x_3^2 - \phi x_4^2 + x_5^2 = 0, \end{cases}$$

  with $\phi = (1+\sqrt{5})/2$ the golden ratio.
- This answers a question by Brandhorst–Hashimoto, about equations for this surface.

# A singular equation of $X_{Ko}$

- Consider again the Fermat quartic surface

$$F: \ x^4 + y^4 + z^4 + t^4 = 0$$

- take the quotient by the symplectic involution

$$j : (x : y : z : t) \mapsto (-x : -y : z : t),$$

- the transcendental lattice of $F$ is

$$T_F = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

- then the transcendental lattice of the minimal resolution $F'$ satisfies $2T_{F'} = T_F$ (Inose 1976) which gives :

$$T_{F'} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

- Kondo showed that $T_{X_{Ko}} = T_{F'}$ which by Shioda-Inose result allows us to identify $S'$ with $X_{Ko}$.
- By putting $z_0 = z, z_1 = t, z_2 = x^2, z_3 = y^2, z_4 = xy$ we have the equations of a singular model of $X_{Ko}$ in the weighted projective space $\mathbb{P}(1,1,2,2,2)$ :

$$X_{Ko}: \; z_0^4 + z_1^4 + z_2^2 + z_3^2 = 0, \; z_4^2 = z_2 z_3$$

- This surface has 8 $A_1$ singularities coming from the fixed points of $j$ on $F$.