Challenges and usefulness of creating a database of groups in the LMFDB

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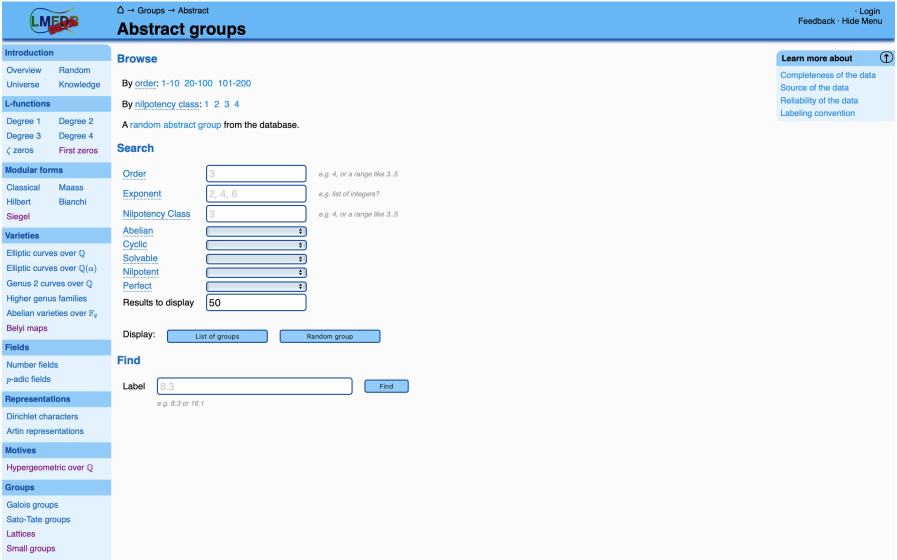
Creating a database for the finite abstract groups in the L-functions and Modular Forms Database (LMFDB).

https://groups.lmfdb.xyz/Groups/Abstract/

Database for abstract groups



Questions

- Isn't there a database for finite groups already?
- Why is it important to have a database for finite groups in the LMFDB?
- What are some of the challenges in creating this database?
- What else can we do in the future?

Isn't there a database for finite groups?

Yes! There is a page by Tim Dokchitser for small groups of order ≤ 500:

http://groupnames.org

Search [? for help] Go | Home | About | C_n | D_n | S_n | A_n | Degree | Linear | Irr Reps | Characters

		d	ϱ	Label	ID
	Groups of order 1	d	ϱ	Label	ID
C_1	Trivial group	1	1+	C1	1,1
	Groups of order 2	d	ϱ	Label	ID
C_2	Cyclic group	2	1+	C2	2,1
	Groups of order 3	d	ϱ	Label	ID
C_3	Cyclic group; = A_3 = triangle rotations	3	1	C3	3,1
	Groups of order 4	d	ϱ	Label	ID
C_4	Cyclic group; = square rotations	4	1	C4	4,1
C_2^2	Klein 4-group V_4 = elementary abelian group of type [2,2]; = rectangle symmetries	4		C2^2	4,2
	Groups of order 5	d	ϱ	Label	ID
C_5	Cyclic group; = pentagon rotations	5	1	C5	5,1
	Groups of order 6	d	ϱ	Label	ID
C_6	Cyclic group; = hexagon rotations	6	1	C6	6,2
S_3	Symmetric group on 3 letters; = D_3 = $GL_2(\mathbb{F}_2)$ = triangle symmetries = 1^{st} non-abelian group	3	2+	S3	6,1
	Groups of order 7	d	ϱ	Label	ID
C_7	Cyclic group	7	1	C7	7,1
	Groups of order 8	d	ϱ	Label	ID
C_8	Cyclic group	8	1	C8	8,1
D_4	Dihedral group; = He_2 = $A\Sigma L_1(\mathbb{F}_4)$ = 2_+^{1+2} = square symmetries	4	2+	D4	8,3
Q_8	Quaternion group; = $C_4.C_2$ = Dic_2 = 2_-^{1+2}	8	2-	Q8	8,4
C_2^3	Elementary abelian group of type [2,2,2]	8		C2^3	8,5
$C_2 \times C_4$	Abelian group of type [2,4]	8		C2xC4	8,2
	Groups of order 9	d	ϱ	Label	ID
C_9	Cyclic group	9	1	C9	9,1
C_3^2	Elementary abelian group of type [3,3]	9		C3^2	9,2
	Groups of order 10	d	ϱ	Label	ID
C_{10}	Cyclic group	10	1	C10	10,2
D_5	Dihedral group; = pentagon symmetries	5	2+	D5	10,1
	Groups of order 11	d	ϱ	Label	ID
C_{11}	Cyclic group	11	1	C11	11,1
	Groups of order 12	d	ϱ	Label	ID

GroupNames

Finite groups of order ≤500, group names, extensions, presentations, properties and character tables.

Order ≤60, ≤120, ≤250, ≤500

Orders with >300 groups of order n
n = 128, 192, 256, 288, 320, 384, 432, 448, 480.

[non]abelian, [non]soluble, supersoluble, [non]monomial, Z-groups, A-groups, metacyclic, metabelian, p-groups, elementary, hyperelementary, linear, perfect, simple, almost simple, quasisimple, rational groups.

Why do we need another database?

- Finite groups show up at many other sections in the LMFDB. They deserve their own pages which can be referred to by other pages.

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  - Galois groups

Galois group: 24T8

Group action invariants

Degree n:	24
Transitive number t:	
Group:	$C_3 : C_8$
Parity:	
Primitive:	no
Nilpotency class:	-1 (not nilpotent)
$\lvert\mathrm{Aut}(F/K)\rvert$:	24
Generators:	(1,8,13,20,2,7,14,19)(3,17,16,6,4,18,15,5)(9,23,21,12,10,24,22,11), (1,17,9)(2,18,10)(3,19,11)(4,20,12)(5,22,14)(6,21,13)(7,24,15)(8,23,16)

Low degree resolvents

IG/NI	Galois groups for stem field(s)
2:	C_2
4:	C_4
6:	S_3
8:	C_8
12:	$C_3 : C_4$

Resolvents shown for degrees ≤ 47

Subfields

Degree 2: C_2

Degree 3: S_3

Degree 4: C_4

Degree 6: S_3

Degree 8: C_8

Degree 12: $C_3 : C_4$

Low degree siblings

There are no siblings with degree ≤ 47

A number field with this Galois group has no arithmetically equivalent fields.

Properties

Label	24T8
Degree	24
Order	24
Cyclic	no
Abelian	no
Solvable	yes
Primitive	no
p-group	no
Group:	$C_3 : C_8$

Learn more about

Completeness of the data
Source of the data
Reliability of the data
Galois group labels

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One refined passport of genus 3 with automorphism group $C_4 \wr C_2$

Family Information

Genus:	3
Quotient genus:	0
Group name:	$C_4 \wr C_2$
Group identifier:	$[32, 11]$
Signature:	$[0; 2, 4, 8]$
Conjugacy classes for this refined passport:	$4, 10, 14$

The full automorphism group for this family is $C_4^2 : C_3 : C_2$ with signature $[0; 2, 3, 8]$.

Jacobian variety group algebra decomposition: $E \times E^2$
Corresponding character(s): $5, 12$

Generating vector(s)

Displaying the unique generating vector for this refined passport.

3.32-11.0.2-4-8.3.1

(1,9) (2,10) (3,11) (4,12) (5,14) (6,13) (7,16) (8,15) (17,25) (18,26) (19,27) (20,28) (21,30) (22,29) (23,32) (24,31)
(1,26,8,31) (2,25,7,32) (3,28,5,30) (4,27,6,29) (9,21,16,19) (10,22,15,20) (11,23,13,18) (12,24,14,17)
(1,24,4,22,2,23,3,21) (5,20,8,18,6,19,7,17) (9,27,12,25,10,28,11,26) (13,32,16,30,14,31,15,29)

Properties

Label	3.32-11.0.2-4-8.3
Genus	3
Quotient genus	0
Group	$C_4 \wr C_2$
Signature	$[0; 2, 4, 8]$
Generating Vectors	1

Related objects

Full automorphism 3.96-64.0.2-3-8
Family containing this refined passport

Downloads

Code to Magma
Code to Gap

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Labeling convention

Why do we need another database?

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  - $\bar{\mathbb{Q}}$-automorphism groups for genus 2 curves over $\mathbb{Q}$
  - Monodromy groups of Belyi maps.

Belyi map orbit 9T11-6.2.1_3.3.3_2.2.2.2.1-a

Introduction
Overview Random
Universe Knowledge

L-functions
Degree 1 Degree 2
Degree 3 Degree 4
ζ zeros First zeros

Modular forms
Classical Maass
Hilbert Bianchi
Siegel

Varieties
Elliptic curves over $\mathbb{Q}$
Elliptic curves over $\mathbb{Q}(\alpha)$
Genus 2 curves over $\mathbb{Q}$
Higher genus families
Abelian varieties over $\mathbb{F}_q$
Belyi maps

Fields
Number fields
p-adic fields

Representations
Dirichlet characters
Artin representations

Motives
Hypergeometric over $\mathbb{Q}$

Groups
Galois groups

Invariants

Degree:
Monodromy group: $C_3^2 : C_6$

Galois group information

$C_3^2 : C_6$ — Transitive group 9T11, order 54, parity 1, non-abelian solvable, imprimitive

CHM label: E(9):6=1/2[3^2:2]S(3)

Generators

(1,2,9)(3,4,5)(6,7,8), (1,2)(3,6)(4,8)(5,7), (3,4,5)(6,8,7), (1,4,7)(2,5,8)(3,6,9)

Subfields

Degree 3: S_3

Other low-degree representations

9T13, 18T20, 18T21, 18T22, 27T11

9T11 home page

permalink

Genus: 0
Geometry type: Euclidean

Base field

$\mathbb{Q}(\sqrt{-3})$; Generator ν, with minimal polynomial $x^2 - x + 1$.

Generators

Order	Partition
6	6, 2, 1
3	3, 3, 3

Properties

Label	9T11-6.2.1_3.3.3_2.2.2.2.1-a
Group	9T11
Orders	$[6, 3, 2]$
Genus	0
Size	1

Related objects

Passport

Downloads

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Code to SageMath
All data to text

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    - Inertia groups of p-adic fields.

p-adic field 5.10.10.11

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Groups
Galois groups

Defining polynomial

$x^{10} + 5x^6 + 10x^5 + 25x^2 + 25x + 25$

Invariants

Base field:	$\mathbb{Q}_5$
Degree d:	10
Ramification exponent e:	5
Residue field degree f:	2
Discriminant exponent c:	10
Discriminant root field:	$\mathbb{Q}_5(\sqrt{*})$
Root number:	-1
$\lvert\mathrm{Aut}(K/\mathbb{Q}_5)\rvert$:	1

This field is not Galois over $\mathbb{Q}_5$.

Intermediate fields

$\mathbb{Q}_5(\sqrt{*})$

Fields in the database are given up to isomorphism. Isomorphic intermediate fields are shown with their multiplicities.

Unramified/totally ramified tower

Unramified subfield:	$\mathbb{Q}_5(\sqrt{*}) \cong \mathbb{Q}_5(t)$ where t is a root of $x^2 + 2$
Relative Eisenstein polynomial:	$x^5 + (5t + 15)\,x + 5 \in \mathbb{Q}_5(t)[x]$

Invariants of the Galois closure

Galois group:	$D_5 : F_5$ (as 10T17)
Inertia group:	Intransitive group isomorphi $C_5^2 : C_4$
Unramified degree:	2
Tame degree:	4
Wild slopes:	[5/4, 5/4]
Galois mean slope:	123/100
Galois splitting model:	$x^{10} - 4x^5 - 4$

Properties

Label	5.10.10.11
Base	$\mathbb{Q}_5$
Degree	10
e	5
f	2
c	10
Galois group	$(C_5^2 : C_4) : C_2$ (as 10T17)

Related objects
Galois group
Unramified subfield
Discriminant root field

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Local field labels

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    - Monodromy groups of Belyi maps.
    - Inertia groups of p-adic fields.
- Dokchitser's group pages are loaded from a static file, whereas the LMFDB group pages are generated dynamically from a database, which allows us to update data as necessary.

Time for a demo

https://groups.lmfdb.xyz/Groups/Abstract/

What are some of the challenges?

- What is the best presentation for an abstract group?
- Writing the rational and complex character tables.
- Getting a good presentation for the subgroup diagram.
- How to label subgroups and the conjugacy classes of a group?

Future plan and application

- Add more groups of larger orders including the groups $\mathrm{GL}(n, q)$ and $\mathrm{SL}(n, q)$.
- Add other groups to the page which are not there yet, like subgroups of $\mathrm{GL}(n, \mathbb{Z})$, $\mathrm{GL}(n, \mathbb{Q})$ and $\mathrm{GL}(n, \mathbb{C})$. Add groups beyond the small groups.
- Add more data on higher genus curves page using the group page.
- Include the automorphism groups of Belyi maps to the LMFDB using the group page.
- Database for algebraic tori and integral Galois representations on the LMFDB will also use some data from the group page. Specifically, it will use the finite subgroups of $\mathrm{GL}(n, \mathbb{Z})$ and their representations.

Future plan and application

A non-CM elliptic curve $E/\mathbb{Q} : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.

$$E/\mathbb{Q} \longrightarrow \text{an automorphic representation } \pi \cong \bigotimes_p \pi_p \text{ on } \mathrm{GL}(2, \mathbb{A}_\mathbb{Q})$$

[R., 2019]: There is an explicit algorithm which computes the local representations π_p of $\mathrm{GL}(2, \mathbb{Q}_p)$ in terms the Weierstrass coefficients a_i of the $E/\mathbb{Q}$.

Goal: Include this data in the LMFDB with some linked to the group page.

Questions for everyone

- Where else would the data on the abstract groups page be useful?
- What data would be good addition to the abstract groups page?

Thank you!