Heavenly elliptic curves over quadratic fields

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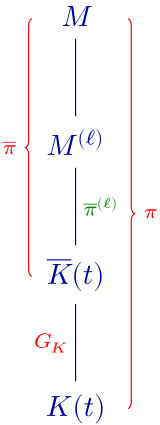
Outline

1. Heavenly abelian varieties: What and Why
2. Finiteness results: old (K fixed, ℓ varies) and new (ℓ fixed, K varies)
3. **heavenly**: strikingly similar to **complex multiplication**
4. Characterization of "heavenly" among elliptic curves with CM



K number field

ℓ rational prime ($\ell \neq 2$ for convenience)

M maximal extension of $\overline{K}(t)$ unramified outside $t = 0, 1, \infty$

$M^{(\ell)}$ maximal pro-ℓ subextension

$$1 \longrightarrow \overline{\pi} \longrightarrow \pi \longrightarrow G_K \longrightarrow 1$$

$$\star\colon \qquad \sigma \mapsto \left(\eta \mapsto (\tilde{\sigma}\eta\tilde{\sigma}^{-1})\right)$$

$$\begin{tikzcd} G_K \arrow[r, "\star"] \arrow[rr, bend right, dashed, "\Phi_K"'] & \operatorname{Out}\overline{\pi} \arrow[r] & \operatorname{Out}\overline{\pi}^{(\ell)} \end{tikzcd}$$

$$\begin{tikzcd}[row sep=small]
 & \overline{\mathbf{Q}} \arrow[d, dash] & \\
\text{“\textit{ten}”} & \text{天} \arrow[d, dash, "?"] & \text{ker } \Phi_K \\
\text{“\textit{yama}”} & \text{山} \arrow[d, dash] & \\
 & K(\mu_{\ell\infty}) \arrow[d, dash] & \\
 & K &
\end{tikzcd}$$

Definition

- 天: maximal pro-ℓ extension of $K(\mu_{\ell\infty})$ unramified away from ℓ
- 山: fixed field of ker Φ_K

Question (Ihara 1986)

For $K = \mathbf{Q}$, does 山 $=$ 天? (*Does the mountain reach the heavens?*)

Theorem (Brown 2012 (+ Sharifi 2002))

For $K = \mathbf{Q}$ and ℓ an odd regular prime, 山 $=$ 天.

Theorem (Anderson-Ihara (1988))

Suppose X/K is a smooth projective curve and $f\colon X \to \mathbf{P}^1$ is a morphism such that:

- $K \subseteq$ 山 $=$ 山$(\mathbf{Q}, \ell)$,
- *f branches over only $\{0, 1, \infty\}$, and*
- *the galois closure $f^{\mathit{gal}}: X^{\mathit{gal}} \to \mathbf{P}^1$ has degree ℓ^N.*

If $J :=$ Jac(X), then $K(J[\ell^\infty]) \subseteq$ 山.

But such $f\colon X \to \mathbf{P}^1$ are rare (in bounded degree). Both to understand this rarity, and to study the extension 天/山, we can search for a more general object: which abelian varieties A/K satisfy $K(A[\ell^\infty]) \subseteq$ 天?

Definition

An abelian variety A/K is called **heavenly** (at ℓ over K) if $K(A[\ell^\infty]) \subseteq$ 天.

$$\mathcal{H}(K,g,\ell) := \{[A]_K : \dim A = g,\ A \text{ heavenly at } \ell\},$$
$$\mathcal{H}(K,g) := \{([A]_K,\ell) : [A]_K \in \mathcal{H}(K,g,\ell)\}.$$

$K(A[\ell^\infty])/K$ unramified away from ℓ

- A/K **good reduction outside** $\{\ell\}$
  (Serre-Tate)
- $\sharp\mathcal{H}(K,g,\ell) < \infty$
  (Shafarevich Conjecture / Faltings / Zarhin)

$K(A[\ell^\infty])/K(\mu_\ell)$ pro-ℓ

- $[K(A[\ell]) : K(\mu_\ell)] = \ell^m$
  as $K(A[\ell^\infty])/K(A[\ell])$ is *always* pro-ℓ
- $[K(E[\ell]) : K(\mu_\ell)] = 1$ or ℓ
  (for elliptic curve case / $g = 1$)

- Suppose A/K is heavenly at ℓ, of dimension g.
- χ: ℓ-adic cyclotomic character, modulo ℓ
- Form of $\rho_{A,\ell}\colon G_K \to \mathrm{GL}_{2g}(\mathbf{F}_\ell)$ is constrained:

$$\rho_{A,\ell} \sim \begin{pmatrix} \chi^{i_1} & \star & \cdots & \star \\ & \chi^{i_2} & \cdots & \star \\ & & \ddots & \vdots \\ & & & \chi^{i_{2g}} \end{pmatrix}, \qquad \begin{cases} \det \rho_{A,\ell} = \chi^g, \\ \sum_r i_r = g. \end{cases}$$

- $g = 1$: E/K heavenly $\implies$ E/K admits K-rational ℓ-isogeny.

Conjecture (2008)

Fix K and g. Then $\sharp\mathcal{H}(K,g) < \infty$. Equivalently, $\ell \gg_{K,g} 0 \implies \mathcal{H}(K,g,\ell) = \emptyset$.

The conjecture is open, but **many** partial or conditional results are known ...

- $\sharp\mathcal{H}(K,g) < \infty$ under GRH
  [R.-Tamagawa 2017]
- $\sharp\mathcal{H}^{\text{CM}}(K,g) < \infty$
  [Ozeki 2013]
- $\sharp\mathcal{H}^{\text{pot-CM}}(K,g) < \infty$
  [Bourdon 2015; Lombardo 2018]
- $\sharp\mathcal{H}(K,1) < \infty$ for $[K:\mathbf{Q}] \leq 3$
  [R.-Tamagawa 2008, 2017]
- Uniformity ($[K:\mathbf{Q}]$ odd) under GRH
  [R.-Tamagawa 2017]
- Uniformity (in $[K:\mathbf{Q}]$) with pot-CM, $g = 1$
  [Bourdon 2015]

And more: [Arai-Momose 2014], [Melistas 2023], [Okumura 2020], [Das-Sarkar 2023], ...

Question: What finiteness results are available for fixed ℓ and varying K?

- Some control on K is required:

$$E/K \text{ good outside } \ell,\ L = K(E[\ell]) \quad \Longrightarrow \quad E \times_K L \text{ heavenly}$$

- **Solution**: require $[K : \mathbf{Q}] \le d$.

- Issues around base-change (and twists):

$$E/K \text{ heavenly},\ K'/K \text{ finite extn} \quad \Longrightarrow \quad E \times_K K' \text{ heavenly over } K'$$

- **Solution**: track $\overline{\mathbf{Q}}$-isomorphism classes (count $[A]_{\overline{\mathbf{Q}}}$, not $[A]_K$).

$$\overline{\mathcal{H}}(K,g,\ell) := \{[A]_{\overline{\mathbf{Q}}} : [A]_K \in \mathcal{H}(K,g,\ell)\}, \quad \overline{\mathcal{H}}_F(d,g,\ell) := \bigcup_{K:\,[K:F]\leq d} \overline{\mathcal{H}}(K,g,\ell).$$

Theorem (McLeman-R. (2024))

Let F be a number field. Suppose $d > 1$ and $\ell > 2d+1$. Then $\sharp\overline{\mathcal{H}}_F(d,1,\ell) < \infty$.

Corollary

Fix $\ell \geq 7$. $\sharp\overline{\mathcal{H}}_{\mathbf{Q}}(2,1,\ell) < \infty$.

Remark

$\overline{\mathcal{H}}_{\mathbf{Q}}(2,1,\ell_0)$ is infinite for $\ell_0 = 2$; likely infinite for $\ell_0 = 3,5$.

Theorem (McLeman-R. (2024))

Let F be a number field. Suppose $d > 1$ and $\ell > 2d+1$. Then $\sharp\overline{\mathcal{H}}_F(d,1,\ell) < \infty$.

Sketch of Proof.

- Set $\mathcal{J}_{S,d} :=$ set of "degree at most d" S-integral points on $Y_1(\ell)$
- Known: $\mathcal{J}_{S,d}$ is finite for fixed S and "sufficiently many cusps on $X_1(\ell)$".
  Application of [Siegel 1929 / Corvaja-Zannier 2004 / Levin 2009, 2016]
- Identify $\overline{\mathcal{H}}_F(d,1,\ell)$ as a subset of $\mathcal{J}_{S,d}$, by choosing $S = \{\mathfrak{l} \subseteq \mathcal{O}_F : \mathfrak{l} \mid \ell\}$.
- (Unfortunately, the argument is not effective.)

$\square$

- Could we hope for a finiteness result when both ℓ and K vary?
- Literally? No:
  - hopeless if we allow $\ell < 7$
  - hopeless unless we exclude trivial base change constructions
  - More serious: fix $A_0/\mathbf{Q}$; there could be *many* A/K with

$$A \ncong A_0 \times_{\mathbf{Q}} K, \quad \text{but} \quad A \times_K \overline{\mathbf{Q}} \cong A_0 \times_{\mathbf{Q}} \overline{\mathbf{Q}}.$$

- One approach: Let $\mathcal{H}^\circ$ be the set of pairs $([A]_K, \ell)$ for which
  - $\ell \geq 7$ and $[K : \mathbf{Q}] = 2$
  - A/K is heavenly at ℓ
  - there does not exist $A_0/\mathbf{Q}$, heavenly at ℓ, with $A \times_K \overline{\mathbf{Q}} \cong A_0 \times_{\mathbf{Q}} \overline{\mathbf{Q}}$.

Conjecture

The set $\mathcal{H}^\circ$ is finite.

Another approach: let $\mathcal{R} \subseteq \mathbb{N} \times \mathbb{N}$ be the set of pairs (ℓ, Δ) for which:

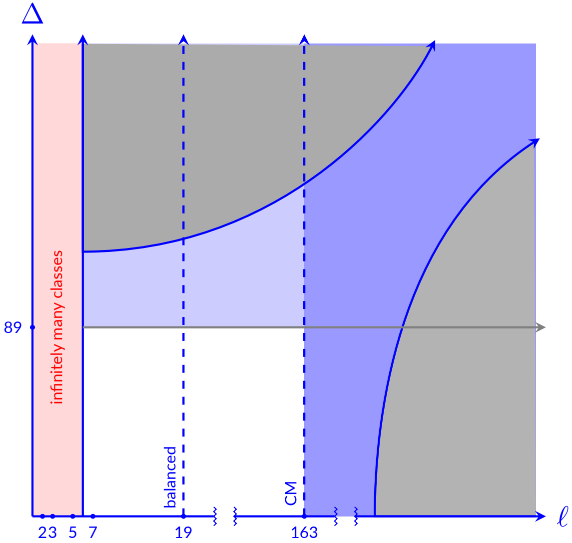
- $\ell \geq 7$ is prime,
- there exists quadratic K with $|\Delta_K| = \Delta$,
- there exists A/K heavenly at ℓ, and
- there does not exist $A_0/\mathbf{Q}$, heavenly at ℓ, with $A \times_K \overline{\mathbf{Q}} \cong A_0 \times_{\mathbf{Q}} \overline{\mathbf{Q}}$.

Conjecture

The set $\mathcal{R}$ is finite.

Proposition (McLeman-R. 2025)

The set $\mathcal{H}^\circ$ is finite if and only if $\mathcal{R}$ is finite.



Why believe $\mathcal{R}$ is finite?

- Horizontal fibers are finite:
  - $\sharp\mathcal{H}(K, 1) < \infty$.
- Vertical fibers are finite:
  - $\sharp\overline{\mathcal{H}}_{\mathbf{Q}}(2, 1, \ell) < \infty$.
- E/K heavenly with $\ell > 163$ are non-CM [Bourdon 2015]
- E/K heavenly with $\ell > 19$ are balanced [McLeman-R. 2024]
- Significant evidence suggests that balanced curves *always* have CM.

$$\begin{tikzcd} \overline{K}_{\mathfrak{l}} \arrow[d, dash] \\ K_{\mathfrak{l}}^{\mathrm{ss}} \arrow[d, dash, "e"'] \\ K_{\mathfrak{l}}^{\mathrm{unr}} \arrow[d, dash] \\ K_{\mathfrak{l}} \end{tikzcd}$$

$J_{\mathfrak{l}}$ $I_{\mathfrak{l}}$

- K number field, ℓ prime, $\mathfrak{l} \mid \ell$, A/K heavenly at ℓ

  (For simplicity, assume ℓ is unramified in $K/\mathbf{Q}$)
- $K_{\mathfrak{l}}^{\mathrm{ss}}$: minimal extension with $A \times_K K_{\mathfrak{l}}^{\mathrm{ss}}$ semistable
- $(\pi) = \mathfrak{m} \subseteq \mathcal{O} \subseteq K_{\mathfrak{l}}^{\mathrm{ss}}$
- Fix $\xi \in \overline{K}_{\mathfrak{l}}$ such that $\xi^{\ell-1} = \pi$
- Fundamental character:

$$\psi_{\mathfrak{l}} \colon J_{\mathfrak{l}} \to \mathbf{F}_\ell^\times \cong \mu_{\ell-1}, \qquad \sigma \mapsto \frac{\xi^\sigma}{\xi} \bmod \mathfrak{m}.$$

- $\psi_{\mathfrak{l}}^e = \chi$ [Serre, 1972]

- Characters of components of $A[\ell]$ (as a group scheme) *must* be powers of $\psi_{\mathfrak{l}}$. [Tate-Oort 1970]
- So there exist $\{j_r\}$ such that

$$\begin{pmatrix} \psi_{\mathfrak{l}}^{j_1} & \star & \cdots & \star \\ & \psi_{\mathfrak{l}}^{j_2} & \cdots & \star \\ & & \ddots & \vdots \\ & & & \psi_{\mathfrak{l}}^{j_{2g}} \end{pmatrix} \sim \rho_{A,\ell} \sim \begin{pmatrix} \chi^{i_1} & \star & \cdots & \star \\ & \chi^{i_2} & \cdots & \star \\ & & \ddots & \vdots \\ & & & \chi^{i_{2g}} \end{pmatrix}.$$

- The $\{j_r\}_r$ partition into pairs satisfing give $j_s + j_t = e$.
- We say A/K is **balanced** at $\mathfrak{l}$ if $j_r = \frac{e}{2}$ for all r.

Proposition (McLeman-R. (2024))

- *For any $g \geq 1$ and any $n \geq 1$, there exists a constant $B(n, g)$ with the following property. If $[K : \mathbf{Q}] = n$ and A/K is a g-dimensional abelian variety which is heavenly at $\ell > B(n, g)$, then A/K is balanced at every $\mathfrak{l} \mid \ell$.*
- $B(2, 1) \leq 19$.

Sketch of Proof.

If $\ell \gg 0$ and A/K is *not* balanced, it is possible to demonstrate a Frobenius element whose trace violates the Weil bound. $\square$

Suppose ℓ is odd, $\mathfrak{p} \nmid \ell$, $a_{\mathfrak{p}} = \operatorname{tr} \mathrm{Frob}_{\mathfrak{p}}$, $L = \mathbf{Q}(\sqrt{-\ell})$.

Proposition (Classical)

Suppose E/K has complex multiplication by $\mathcal{O}_L$ and good reduction away from ℓ. The splitting behavior of $\mathfrak{p}$ in KL/K is related to $a_{\mathfrak{p}}$ as follows:

$$\begin{aligned} \mathfrak{p} \text{ splits} &\implies a_{\mathfrak{p}}^2 \equiv 4 \cdot \mathbf{N}\mathfrak{p} \bmod \ell, \\ \mathfrak{p} \text{ inert} &\implies a_{\mathfrak{p}} = 0. \end{aligned}$$

Proposition (McLeman-R. (2024))

Suppose E/K is heavenly at ℓ, and is balanced at $\mathfrak{l}$ for at least one $\mathfrak{l} \mid \ell$. The splitting behavior of $\mathfrak{p}$ in KL/K is related to $a_{\mathfrak{p}}$ as follows:

$$\begin{aligned} \mathfrak{p} \text{ splits} &\implies a_{\mathfrak{p}}^2 \equiv 4 \cdot \mathbf{N}\mathfrak{p} \bmod \ell, \\ \mathfrak{p} \text{ inert} &\implies a_{\mathfrak{p}} \equiv 0 \bmod \ell. \end{aligned}$$

- A tempting idea appears to fall short ...

  ...the proposition is *not* strong enough to imply that a balanced and non-CM elliptic curve gives a violation of Sato-Tate.
- We *can* characterize, among E/K with complex multiplication and good reduction away from ℓ, which ones are heavenly.

Theorem (McLeman-R. (2024))

Suppose E/K has complex multiplication and good reduction away from ℓ, and assume $K = \mathbf{Q}(j(E))$. Then

- *E is heavenly at ℓ if and only if* $\mathrm{tr}(\rho_{E,\ell}(G_K)) \neq \mathbf{F}_\ell$.
- *In this case,* $\mathrm{tr}(\rho_{E,\ell}(G_K)) = \left(\frac{2}{\ell}\right) \cdot \mathbf{F}_\ell^{\times 2} \cup \{0\}$.
- *In this case, if $\ell > 7$, then E is balanced at every $\mathfrak{l} \mid \ell$.*

- We determined a finite set $\mathcal{X}$ that contains all pairs $([E]_K, \ell)$, where
  - $K = \mathbf{Q}(j(E))$ is quadratic,
  - E/K has complex multiplication,
  - E/K is heavenly at ℓ.
- The set $\mathcal{X}$ has 240 pairs. In principle, $\mathcal{X}$ may contain "false positives," but if one believes traces of Frobenius are "independent," this is *extremely* unlikely.
- Inside $\mathcal{X}$, one isogeny class over $K = \mathbf{Q}(\sqrt{6})$, contains curves with everywhere good reduction, which are heavenly at both $\ell = 2$ and $\ell = 3$.
- Assuming ERH, we extended a calculation of Karpisz to show $|\Delta_K| < 5 \cdot 10^5$ and $\ell > 163$ implies $\mathcal{H}(K, 1, \ell) = \emptyset$.

Thank you!