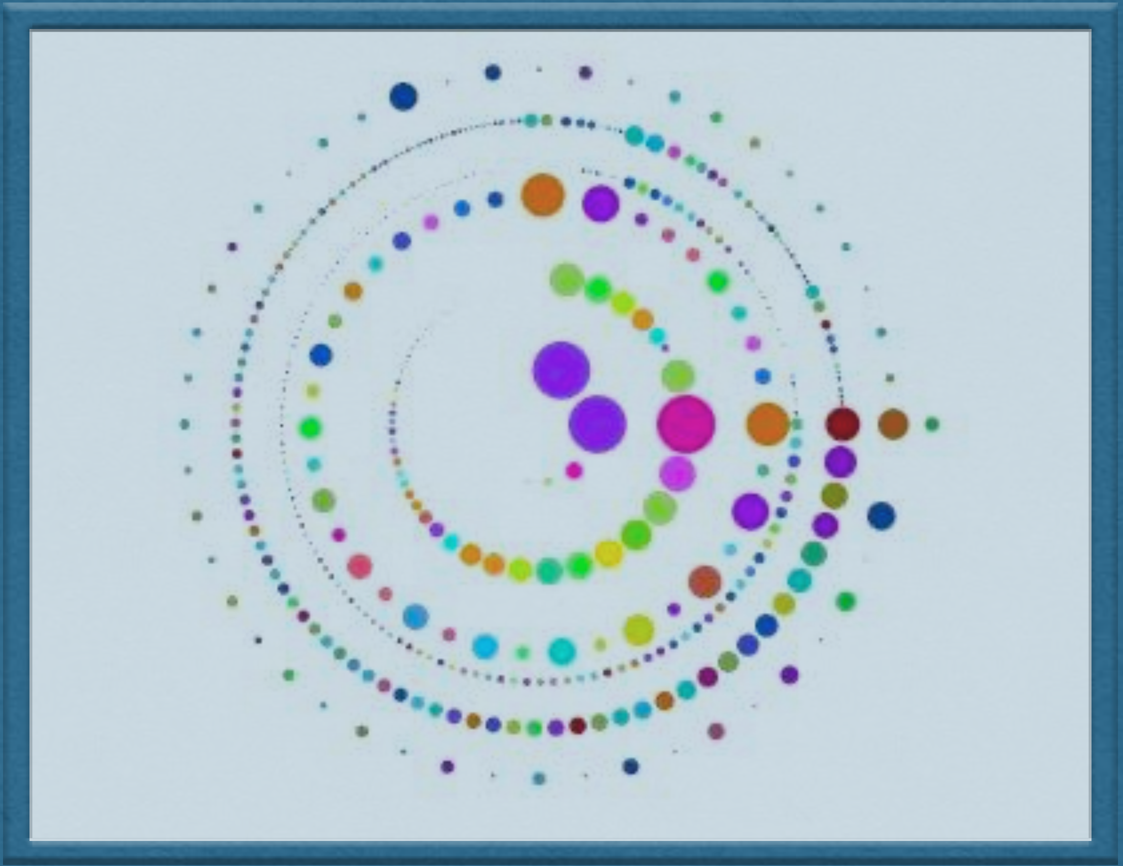

Automorphism groups of compact Riemann surfaces

Jen Paulhus
Grinnell College

Plan of Attack

- Introduction to Riemann surfaces, group actions, monodromy
- Classifying groups with almost all actions
- A brief simple group interlude
- A bit about non-normal subvarieties of $\mathscr{M}_g$

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Setup

Take a curve like $w^2 = z^6 + z^3 + 1$. The dihedral group $G = D_6$ acts on points on this curve by moving those points around as follows (ζ_3 a cube root of unity):

$$r : (z, w) \to (\zeta_3 z, -w) \text{ and } s : (z, w) \to \left(\frac{1}{z^3}, \frac{w}{z^3}\right)$$

We call this group *the automorphism group* of the curve. It is always a finite group.

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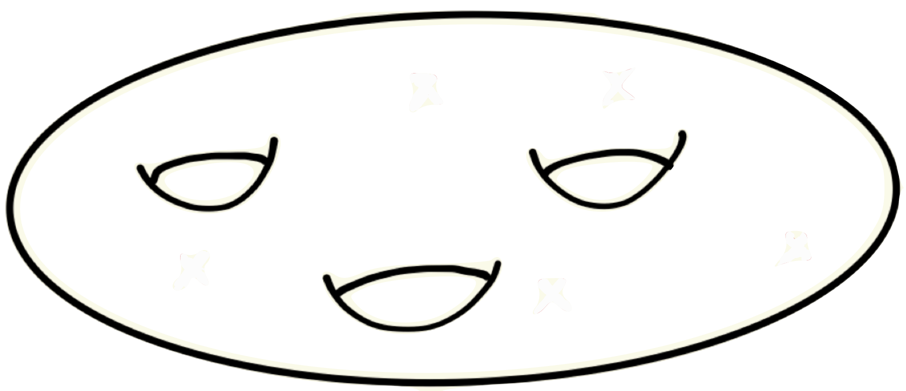
We call this group *the automorphism group* of the curve. It is always a finite group.

Projective nonsingular algebraic curves (defined over $\mathbb{C}$) are equivalent to compact Riemann surfaces.

Setup

A *Riemann surface* is a one dimensional complex manifold (a topological space that looks like complex plane locally).



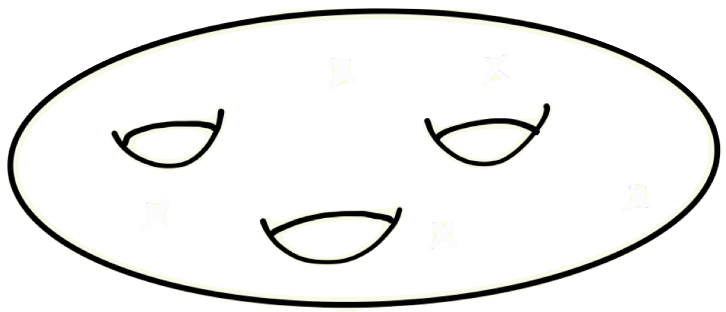
We will focus on *compact* Riemann surfaces.

The *genus* of a Riemann surface is the number of holes in the surface.

Setup

X a compact *Riemann surface* of genus g with $G = \mathrm{Aut}(X)$ (finite)

X



conformal homeomorphisms

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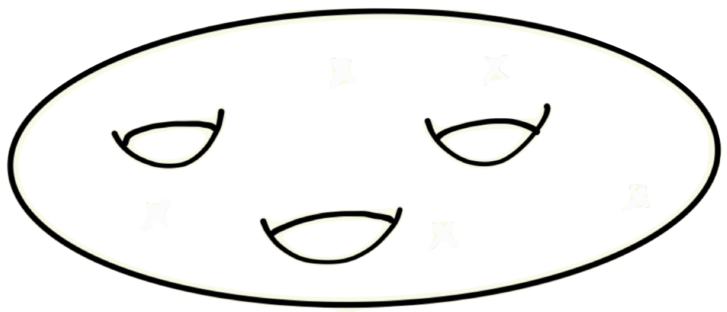
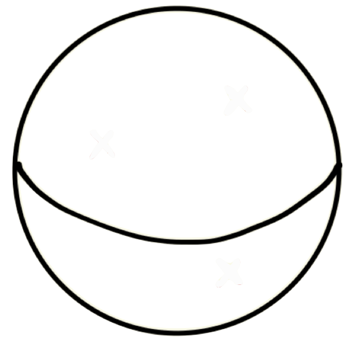


We call the set of orbits of the action X/G and this quotient is also a compact Riemann surface.

X/G



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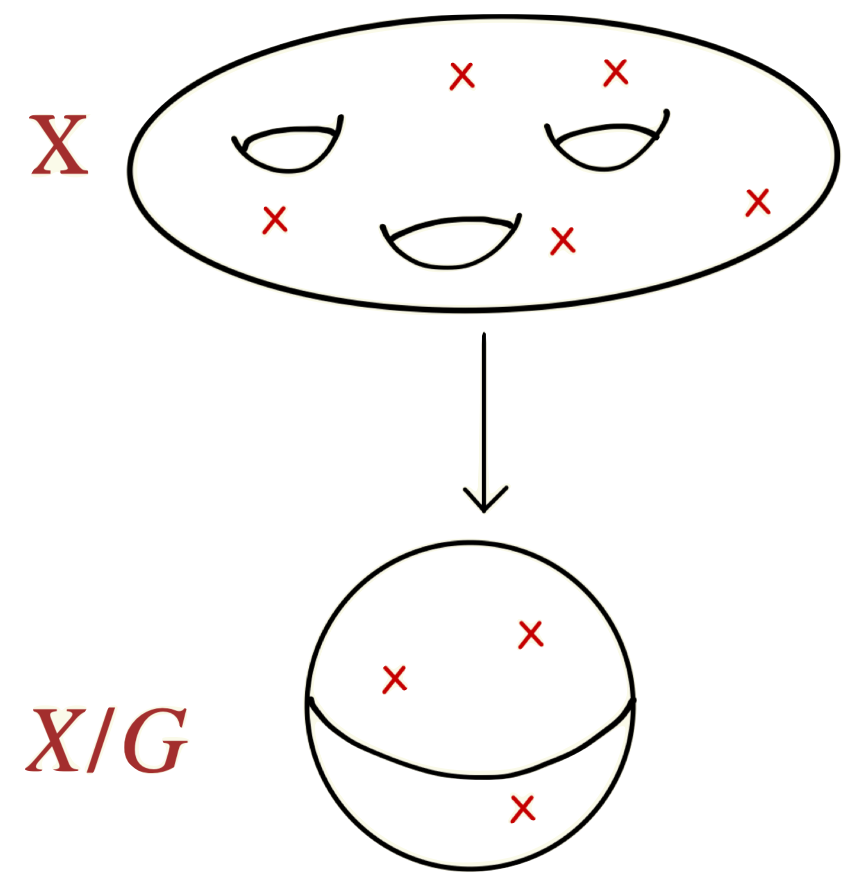
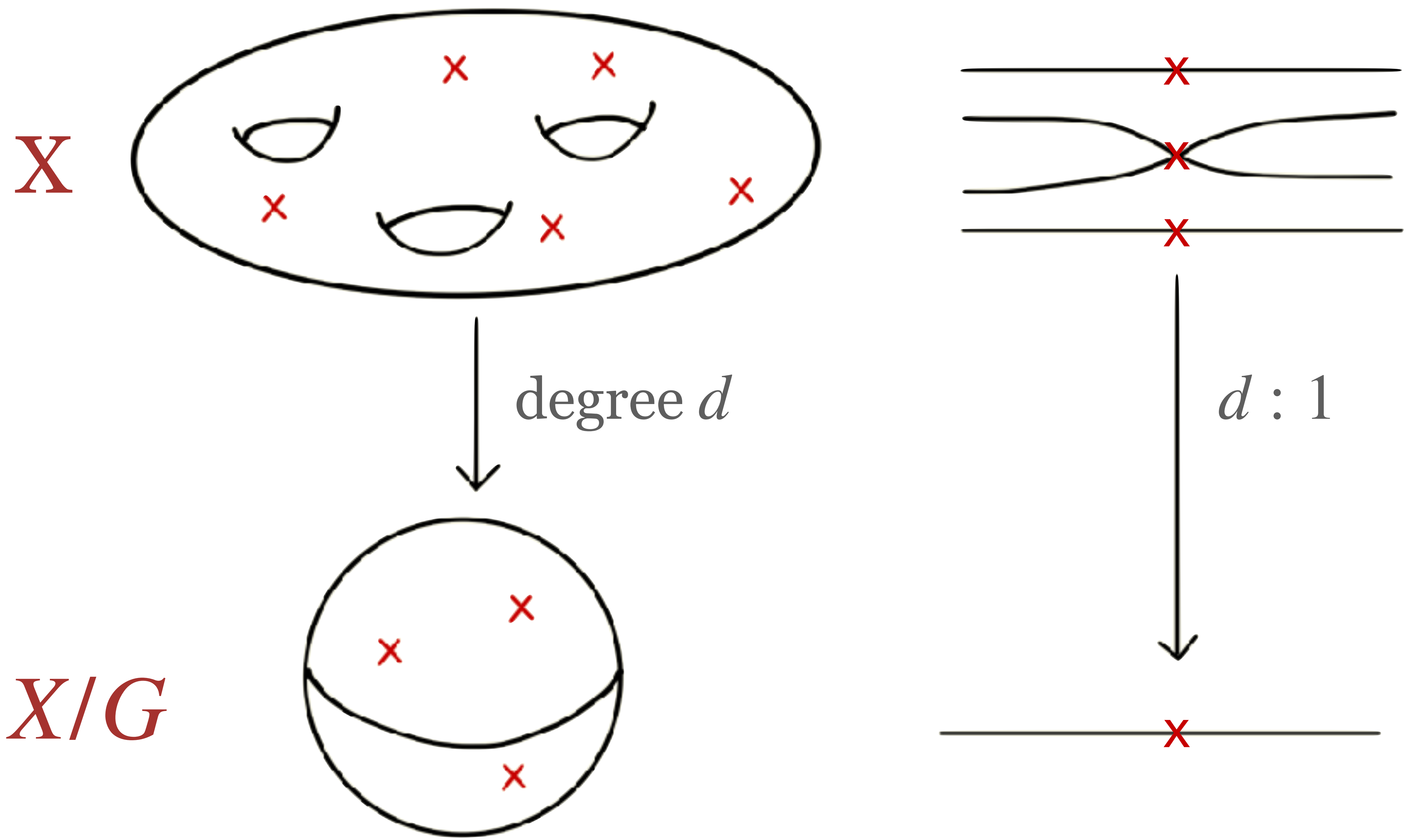


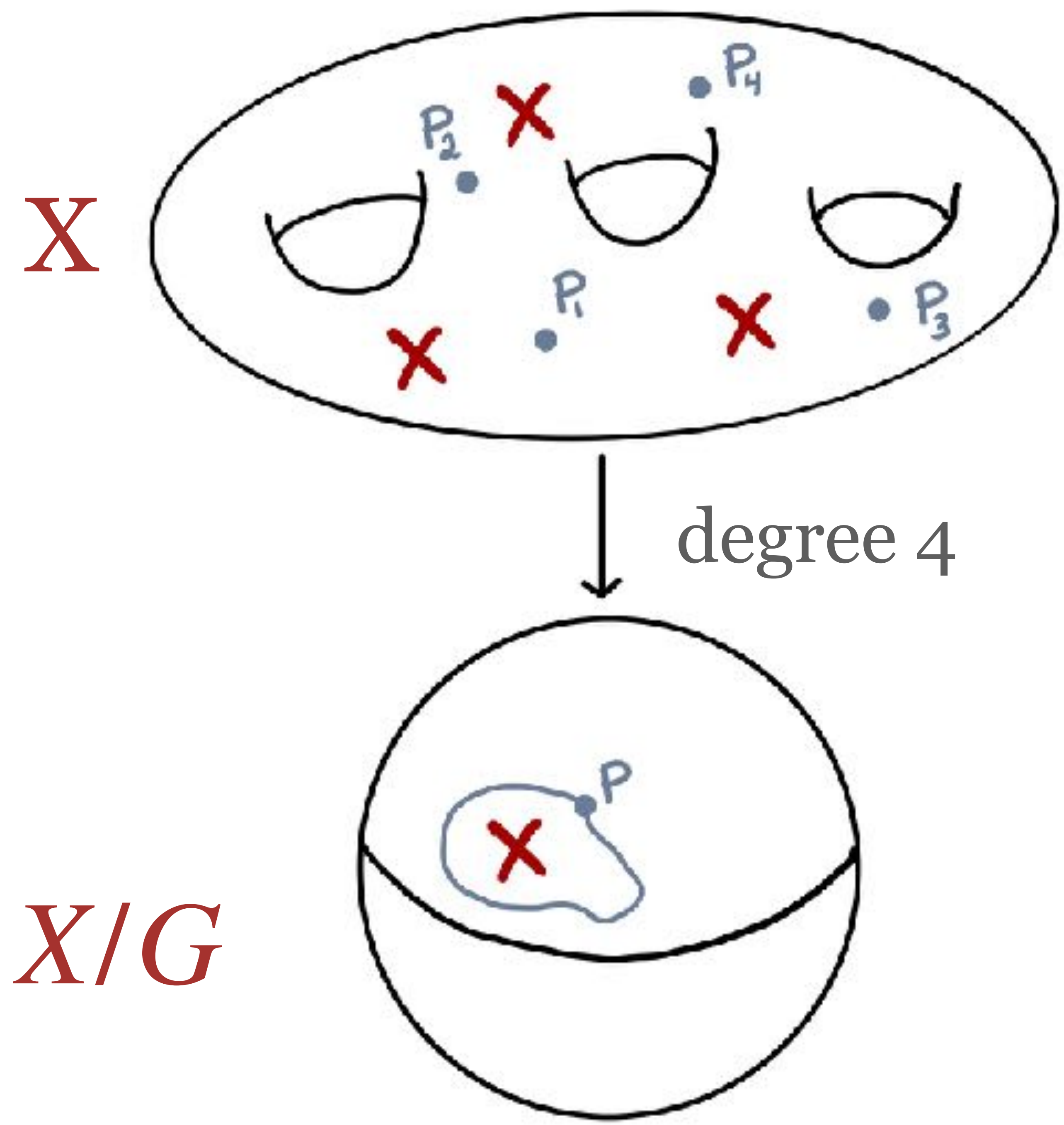
We call the set of orbits of the action X/G and this quotient is also a compact Riemann surface.

The natural map $X \to X/G$ gives us a branched covering branched at r places.

Setup



Monodromy



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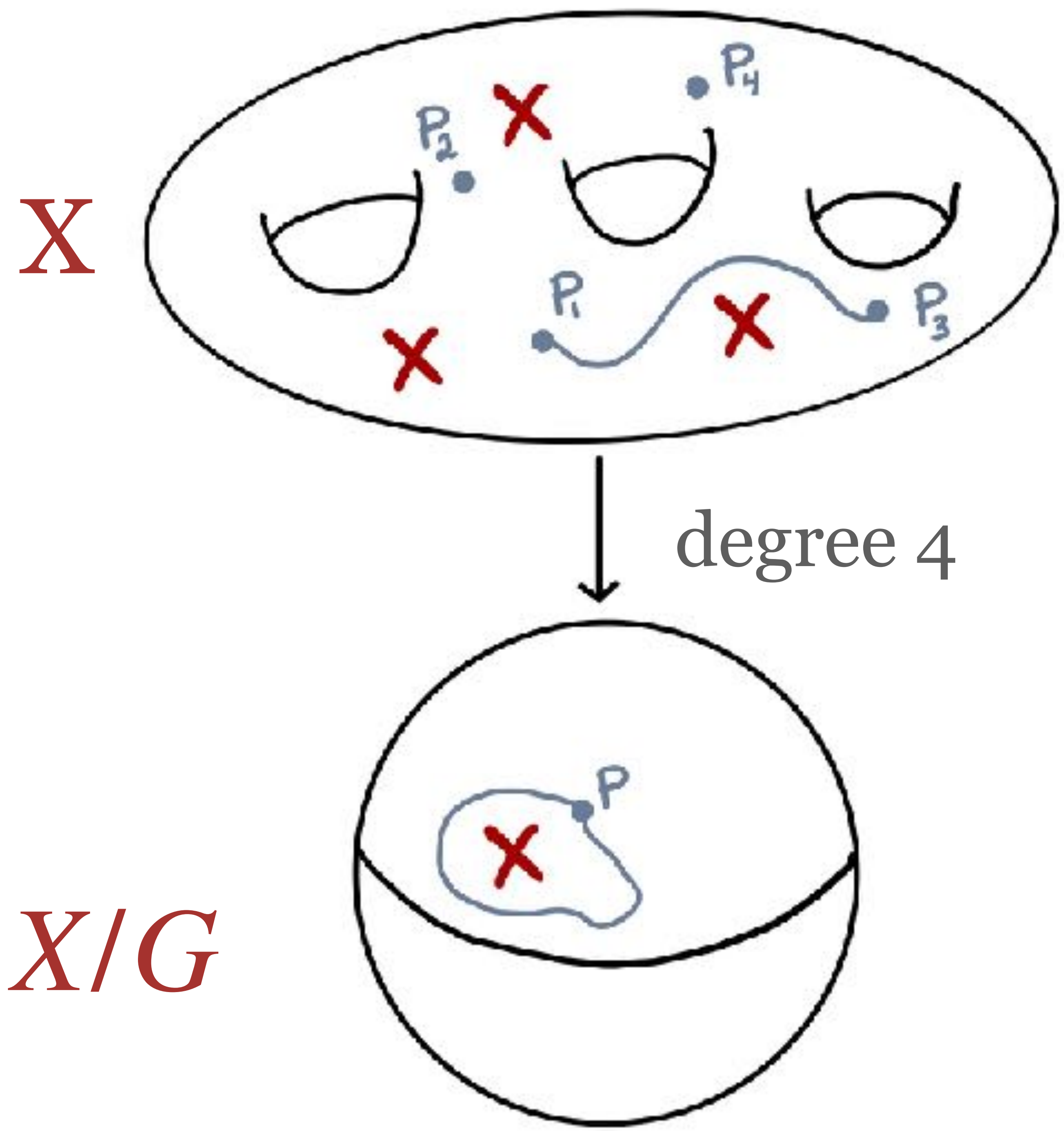
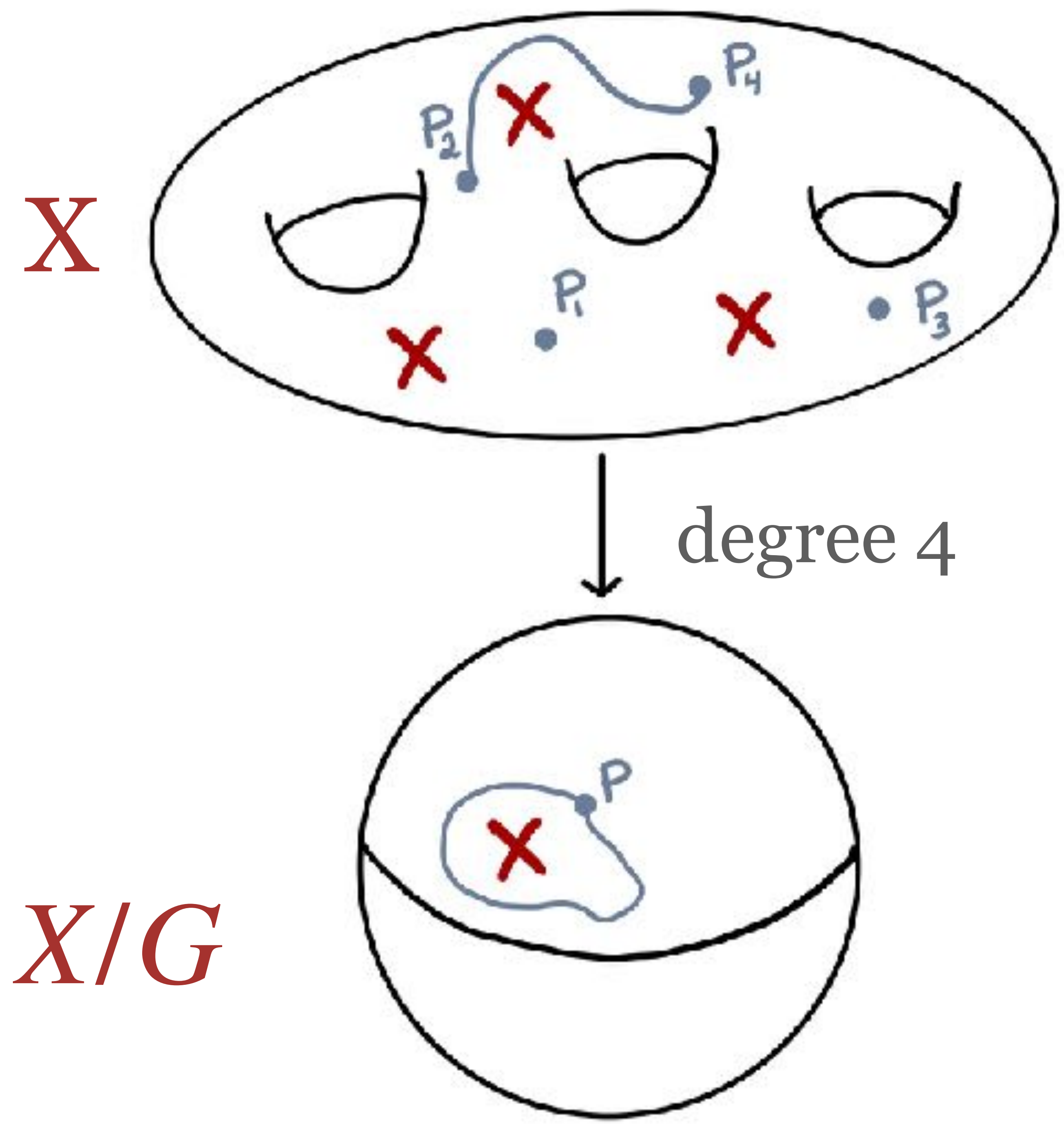


Monodromy



Monodromy

Lifts of the blue loop correspond to a *permutation* of the points P_1, P_2, P_3, and P_4:

send P_i to the endpoint of the lift which starts at P_i.

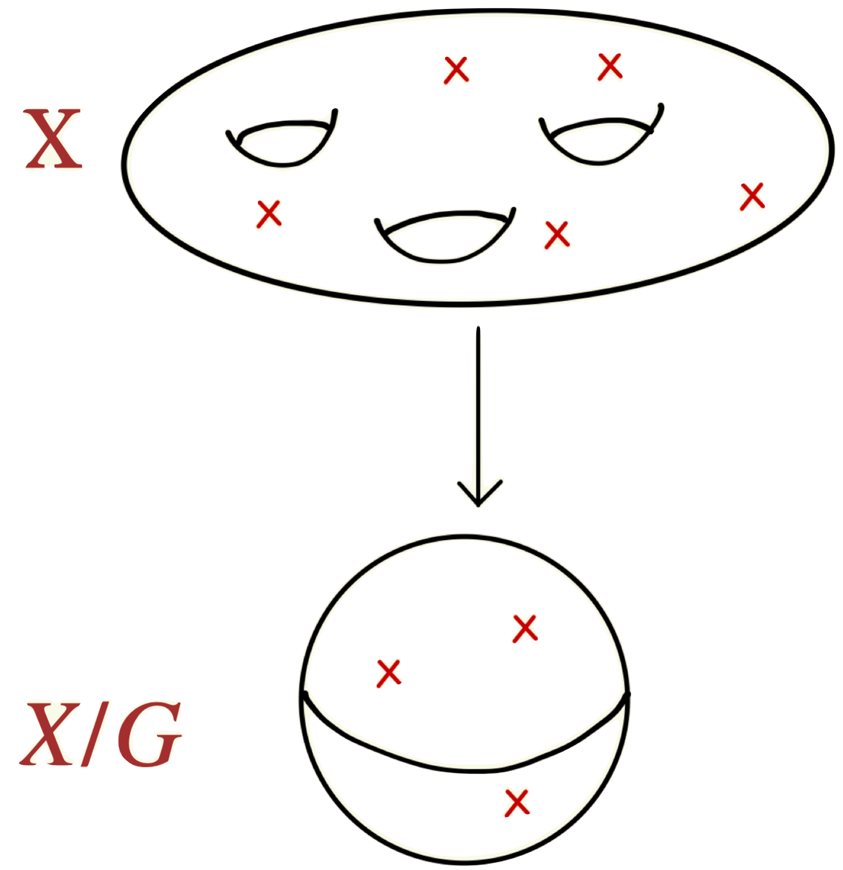
For our example, P_1 goes to P_3 in this permutation and P_2 goes to P_4 in this permutation.

Repeat for each branch point to create a permutation group which we call the *monodromy*.

The group is a subgroup of S_d where d is the degree of the cover.

Setup

X a compact Riemann surface of genus g with $G = \mathrm{Aut}(X)$ (finite) gives us a branched covering $X \to X/G$, branched at r places.



If X/G has genus h and those branch points have monodromy of order $m_1, \ldots, m_r$, respectively, then

$$[h; m_1, \ldots, m_r]$$

is the *signature* of the action of G on X.

Question

Which groups can be the automorphism group of a Riemann surface of a particular genus, and with what signature (or monodromy) and what quotient genus?

Motivation

There is an idea called “principle of finite extensions” which roughly says that if a result is true over some extension field, it is true over a *finite* extension field.

If a group acts on a curve over $\mathbb{C}$, it also acts on a curve over some finite extension of $\mathbb{Q}$.

See Grothendieck’s EGA (s/o to Jeff Achter for the argument).

Motivation

Knowledge of automorphism groups and the corresponding monodromy has important applications:

- inverse Galois theory
- the study of the mapping class group
- Jacobian varieties

Question

Which groups can be the automorphism group of a Riemann surface of a particular genus, and with what signature (or monodromy) and what quotient genus?

A recent AMS Contemporary Mathematics book has a paper with maaaaany open problems in the area (written with coauthors Allen Broughton and Aaron Wootton).

Riemann's Existence Theorem

A finite group G acts on a compact Riemann surface X of genus g >1 if and only if there are elements of the group

$$a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r$$

which generate the group, satisfy the following equation,

$$\prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} c_j = 1_G$$

$$[a_i, b_i] = a_i^{-1} b_i^{-1} a_i b_i$$

h is the genus of X/G

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and so that $m_j = \mathrm{ord}(c_j)$ satisfy the Riemann Hurwitz formula

$$g = 1 + |G|(h-1) + \frac{|G|}{2} \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right).$$

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signature: $[h;\ m_1, \ldots, m_r]$

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generating vector: $(a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r)$

Circa 2000, Thomas Breuer devised an algorithm to determine all automorphism groups of Riemann surfaces for a fixed genus, assuming a complete classification of groups of sufficiently large order.

He coded the algorithm in GAP, and ran it up to genus 48.

> For many more details, see Breuer's book "*Characters and automorphism groups of compact Riemann surfaces*".
>
> This data up to genus 15 (and a lot of additional data) is on the LMFDB. More data and higher genus coming "soon"!

Large automorphism groups up to $g = 101$ on Marston Conder's webpage.

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Tom Tucker

Colgate University

Aaron Wootton

University of Portland

Journal of Pure and Applied Algebra. (2021). Original version on arXiv.

Riemann's Existence Theorem

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signature: $[h;\ m_1, \ldots, m_r]$

generating vector: $(a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r)$

For a *fixed* group G...

Potential signatures are those signatures $[h; m_1, \ldots, m_r]$ which satisfy the Riemann Hurwitz formula:

$$g = 1 + |G|(h-1) + \frac{|G|}{2}\sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right).$$

Actual signatures are those which also have a generating vector associated to them.

These are not always the same.

Example

The signature [0; 3,3,9] is a potential signature for $G = C_9$ since it satisfies Riemann-Hurwitz for a curve of genus 2 and a group of order 9 with elements of order 3 and 9.

But this signature cannot be an actual signature for *abelian* groups. There's an issue with the lcm of the m_i.

Sometimes they are badly not the same for a fixed group.

Example

Take $q = p^n$ for an odd prime p. Then $[0; \underbrace{2,\ 2, \ldots,\ 2}_{r>4}]$

is a potential signature for SL(2,q) but there is only one element of order 2 in this group.

That one element certainly doesn't generate the whole group!

Easy to compute.

Potential signatures are those signatures $[h; m_1, \ldots, m_r]$ which satisfy the Riemann Hurwitz formula:

$$g = 1 + |G|(h-1) + \frac{|G|}{2}\sum_{j=1}^{r}\left(1 - \frac{1}{m_j}\right).$$

Actual signatures are those which also have a generating vector associated to them.

Hard to compute.

Our Question

Which *groups* only have a finite number of potential signatures which fail to be actual signatures?

We say such groups *act with almost all signatures* (or are AAS).

$\mathcal{O}(G) = \{\mathrm{Ord}(g) : g \in G\} - \{1\}$ is the *order set*.

With a very small number of exceptions, any signature of the form

$$[h; \underbrace{n_1, \ldots, n_1}_{t_1}, \underbrace{n_2, \ldots, n_2}_{t_2}, \ldots, \underbrace{n_s, \ldots, n_s}_{t_s}]$$

for $n_i \in \mathcal{O}(G)$ and $t_i \in \mathbb{Z}^+$ is a potential signature.

$\mathcal{O}(G) = \{\mathrm{Ord}(g) : g \in G\} - \{1\}$ is the *order set*.

Theorem

A group G is AAS if and only if:

I. The commutator (or derived) subgroup $[G : G]$ contains an element of order every $n_i \in \mathcal{O}(G)$.

II. G may be generated by elements of order n_i for each $n_i \in \mathcal{O}(G)$.

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If **I.** is false, then potential signatures $[h; n_i]$ for $h > 0$ are never actual signatures.

If **II.** is false, then potential signatures $[0; \underbrace{n_i, n_i, \ldots, n_i}_{\geq 4}]$ are never actual signatures.

I. The commutator subgroup $[G : G]$ contains an element of order every $n_i \in \mathcal{O}(G)$.

II. G may be generated by elements of order n_i for each $n_i \in \mathcal{O}(G)$.

If **I.** is true, we exhibit generating vectors for any signature $[h; m_1, \ldots, m_r]$ with h beyond a certain bound.

If **II.** is true, then we exhibit generating vectors for any signature $[h; m_1, \ldots, m_r]$ for r beyond a certain bound.

Theorem

Any non-abelian finite simple group is AAS.

I. Non-abelian simple groups all have commutator subgroup the full group.

II. Take an element of order n_i. The set of conjugates of that element is a set of elements of order n_i and which generate a normal subgroup. Since the group is simple, this is all of G.

Proposition

If a group G is AAS, then it is either a non-abelian p-group, or a perfect group.

A *perfect group* is one where the commutator subgroup is the whole group.

Since the commutator subgroup must contain elements of every order in $\mathcal{O}(G)$, any AAS group must be non-abelian.

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Theorem

Any non-abelian finite simple group is AAS.

Question

For each non-abelian simple group up to order 10 000, what is the largest genus g so that there exists a potential signature for a curve of that genus which is, in fact, not an actual signature?

Question

many

For ~~each~~ ^ non-abelian simple group up to order 10 000, what is the largest genus g so that there exists a potential signature for a curve of that genus which is, in fact, not an actual signature?

Question

For ~~each~~ [many] non-abelian simple group up to order 10 000, what is the largest genus g so that there exists a potential signature for a curve of that genus which is, in fact, not an actual signature?

Well, at least for covers of $\mathbb{P}^1$?

group	g	potential, but not actual signatures
PSL(2,7)	210	[0; 2, 2, 2, 3] [0; 2, 2, 2, 4] [0; 2, 2, 2, 2, 2]
$A_6 \cong$ PSL(2,9)	31	[0; 2, 2, 2, 3] [0; 3, 4, 4]
PSL(2,11)	56	[0; 2, 2, 2, 3]
PSL(2,16)	817	[0; 3, 3, 5] [0; 2, 5, 5] [0; 5, 5, 5] [0; 3, 5, 5]
PSL(2,25)	1821	[0; 2, 4, 5] [0; 2, 4, 6] [0; 2, 5, 5] [0; 2, 5, 6] [0; 2, 6, 6] [0; 3, 3, 5] [0; 3, 4, 4] [0; 3, 4, 6] [0; 3, 5, 5] [0; 3, 6, 6] [0; 4, 4, 5] [0; 4, 5, 6] [0; 5, 5, 5] [0; 5, 6, 6]

group	g	potential, but not actual signatures
A_7	3150	[0; 2, 3, 7] [0; 2, 4, 5] [0; 2, 4, 6] [0; 2, 5, 5] [0; 2, 5, 6] [0; 2, 6, 6] [0; 3, 3, 4] [0; 2, 2, 2, n_i] [0; 2, 2, 2, 2, 2]
M_{11}	9900	[0; 2, 3, 8] [0; 2, 3,11] [0; 2, 4, 6] [0; 2, 4, 5] [0; 2, 4, 8] [0; 2, 5, 5] [0; 2, 5, 6] [0; 2, 6, 6] [0; 3, 3, 4] [0; 3, 3, 5] [0; 3, 3, 6] [0; 3, 3, 11] [0; 3, 4, 4] [0; 3, 5, 5] [0; 4, 4, 4] [0; 2, 2, 3, 3] [0; 2, 2, 2, 2, 2]
A_5, and PSL(2,q) for $q = 8, 13, 17, 19, 23, 27$		Every potential signature is an actual signature. (Potential signature with $h = 0$.)

Aaron Wootton
University of Portland

Question

Classify all actions for the alternating group

with quotient genus > 0.

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Ruben Hidalgo
Universidad de la Frontera

Sebastian Reyes-Carocca & Anita Rojas
Universidad de Chile

Submitted article. Original version on arXiv.

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Fulbright Scholars Program

Riemann's Existence Theorem ($h = 0$)

A finite group G acts on a compact Riemann surface X of genus g >1 if and only if there are elements of the group

$$c_1, \ldots, c_r$$

which generate the group, satisfy the following equation,

$$\prod_{j=1}^{r} c_j = 1_G$$

and so that $m_j = \mathrm{ord}(c_j)$ satisfy the Riemann Hurwitz formula

$$g = 1 - |G| + \frac{|G|}{2} \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right).$$

generating vector: $(c_1, \ldots, c_r)$

Equivalent Actions

Some generating vectors represent the “same” action

If $(c_1, \ldots, c_r)$ is one generating vector, so is $(c_1^g, \ldots, c_r^g)$ for any $g \in G$.

conjugation by g

This defines an equivalence relation on generating vectors up to inner automorphisms of G.

Equivalent Actions

What other "sameness" conditions are there?

Two generating vectors are *topologically equivalent* if they are in the same orbit under the action of $\mathrm{Aut}(G) \times \mathscr{B}_r$.

Artin braid group

eg. $(c_1, \ldots, c_r)$ to $(c_1, \ldots c_{i-1}, c_{i+1}, c_{i+1}^{-1} c_i c_{i+1}, \ldots, c_r)$

Example

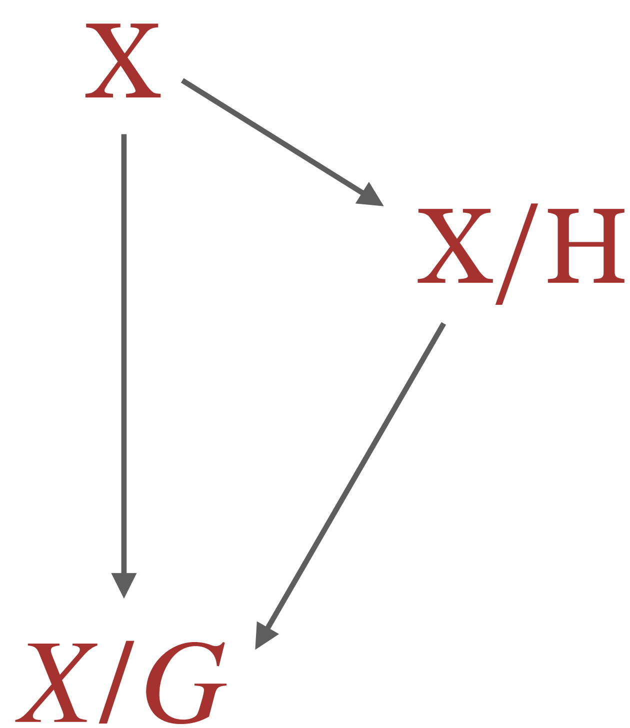
In genus 3, $H = C_2 \times C_4$ acts with signature $[0; 2,\ 2,\ 4,\ 4]$ in 3 topologically inequivalent ways.

(1 2)(3 4)(5 6)(7 8)	(1 3)(2 4)(5 7)(6 8)	(1 3)(2 4)(5 7)(6 8)
(1 3)(2 4)(5 7)(6 8)	(1 3)(2 4)(5 7)(6 8)	(1 4)(2 3)(5 8)(6 7)
(1 5 2 6)(3 7 4 8)	(1 5 2 6)(3 7 4 8)	(1 5 2 6)(3 7 4 8)
(1 7 2 8)(3 5 4 6)	(1 6 2 5)(3 8 4 7)	(1 5 2 6)(3 7 4 8)

generating vectors

Equivalent Actions

Any subgroup $H \leq G$ will also act on X with a quotient structure X/H . This gives us many intermediate actions.



But *which* action does a subgroup H correspond to for a particular action of G? (Assume X/H is genus 0 too.)

Example

In genus 3, $H = C_2 \times C_4$ acts with signature $[0; 2, 2, 4, 4]$ in 3 topologically inequivalent ways.

(1 2)(3 4)(5 6)(7 8)	(1 3)(2 4)(5 7)(6 8)	(1 3)(2 4)(5 7)(6 8)
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(1 5 2 6)(3 7 4 8)	(1 5 2 6)(3 7 4 8)	(1 5 2 6)(3 7 4 8)
(1 7 2 8)(3 5 4 6)	(1 6 2 5)(3 8 4 7)	(1 5 2 6)(3 7 4 8)

↑

$G = C_2 \times D_4$ in genus 3 with signature [0; 2, 2, 2, 4].

Example

In genus 3, $H = C_2 \times C_4$ acts with signature $[0; 2, 2, 4, 4]$ in 3 topologically inequivalent ways.

(1 2)(3 4)(5 6)(7 8)	(1 3)(2 4)(5 7)(6 8)	(1 3)(2 4)(5 7)(6 8)
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(1 5 2 6)(3 7 4 8)	(1 5 2 6)(3 7 4 8)	(1 5 2 6)(3 7 4 8)
(1 7 2 8)(3 5 4 6)	(1 6 2 5)(3 8 4 7)	(1 5 2 6)(3 7 4 8)

$G = D_4 : C_2$ in genus 3 with signature [0; 2, 2, 2, 4].

Another Equivalence

Two subgroups $H_1, H_2 \leq \mathrm{Aut}(X)$ are *conformally equivalent* if there is an automorphism $\tau : X \to X$ so that $\tau H_1 \tau^{-1} = H_2$.

Note: conformally equivalent means the subgroups are conjugate in the full automorphism group of the surface.

Another Equivalence

Two subgroups $H_1, H_2 \leq \mathrm{Aut}(X)$ are conformally (*topologically*) *equivalent* if there is an automorphism (homeomorphism) $\tau : X \to X$ so that $\tau H_1 \tau^{-1} = H_2$.

Note: it is clear that if an action is conformally equivalent then it is topologically equivalent, but the converse is not always true.

- 1991/1994 González-Diez: for H_1, H_2 cyclic subgroup of prime order then topological $\Rightarrow$ conformal.
- 1997 González-Diez and Hidalgo: 2 subgroups isomorphic to C_8 in genus 9 where topological $\not\Rightarrow$ conformal
- 2004 Cirre: gave an example where action was for a non-cyclic group (genus 3)
- 2013 Carvacho: gave one dimensional families of groups of order 2^n generalizing González-Diez/Hidalgo.

Motivation

Fix genus g, $H \leq G$, signature s, and generating vector σ.

The set of all isomorphism classes of Riemann surfaces topologically equivalent to the one defined by (H, s, σ) is a closed, irreducible (non-smooth) subvariety of $\mathscr{M}_g$ denoted by $\mathscr{M}_g(H, s, \sigma)$.

The set of all classes of Riemann surfaces in $\mathscr{M}_g(H, s, \sigma)$ up to conformal equivalence is the normalization of $\mathscr{M}_g(H, s, \sigma)$. (González-Diez, Harvey 1992)

The Problem

We search for $H_1, H_2 \leq \mathrm{Aut}(X)$ which are topologically but **not** conformally equivalent.

- Using already known complete lists of groups which act on Riemann surfaces for a particular genus, we search for isomorphic but not conjugate subgroups $H_1, H_2 \leq G$.

- Compare the generating vectors of H_1 and H_2, and then determine if they are topologically equivalent actions.

- If they are, we know $\mathscr{M}_g(H_1, s, \sigma)$ is a non-normal subvariety in the singular locus of $\mathscr{M}_g$.

Example

The group $C_2 \times D_{g+1}$ acts in each **odd** genus with signature $[0; 2, 2, 2, g+1]$ and C_2^2 is isomorphic to two non-conjugate subgroups of this group.

For every genus $g \geq 3$ the group C_2^2 acts with signature $s = [0; \underbrace{2,\ldots,2}_{g+3}]$ in one hyperelliptic way.

e.g. Accola-Maclachlan curve is a non-normal point

Example

The group D_g acts in each **even** genus with signature $[0; 2, 2, 2, 2, g]$ and C_2^2 is isomorphic to two non-conjugate subgroups of this group.

For every genus $g \geq 3$ the group C_2^2 acts with signature $s = [0; \underbrace{2,\ldots,2}_{g+3}]$ in one hyperelliptic way.

e.g. Wiman curve of type II is a non-normal point

Example

The group D_g acts in each **even** genus with signature $[0; 2, 2, 2, 2, g]$ and C_2^2 is isomorphic to two non-conjugate subgroups of this group.

For every genus $g \geq 3$ the group C_2^2 acts with signature $s = [0; \underbrace{2,\ldots,2}_{g+3}]$ in one hyperelliptic way.

So, for each g > 2, the moduli space $\mathscr{M}_g$ contains a non-normal subvariety of type $\mathscr{M}_g(C_2^2, s, \sigma_h)$.

Example

The group $C_{2(g-1)} \rtimes C_2$ acts with signature $[0; 2, 2, 2, 2]$ and $D_{2(g-1)}$ is isomorphic to two non-conjugate subgroups of this group.

For each odd integer $g > 3$, $D_{2(g-1)}$ acts with signature $[0; 2, 2, 2, 2, 2]$ in one way up to topological equivalence.

This gives us first examples of action of **non-abelian** groups giving non-normal subvarieties.

The End