A criterion for algebraic degeneracy of integral points

Héctor Pastén

Pontificia Universidad Católica de Chile

Joint work with Natalia García-Fritz.

VaNTAGe

Siegel's theorem

Assumptions: We work in the most classical setting:

- Varieties such as curves, surfaces, n-folds are irreducible.
- Varieties will be considered over $\mathbb{Q}$
- Integrality will be considered with respect to $\mathbb{Z}$.

Theorem (Siegel, 1929)

Let $C/\mathbb{Q}$ be a smooth projective curve of genus g. Let D be a non-zero reduced effective divisor. If

$$\deg(K_C + D) > 0$$

then every set of D-integral points is finite. (Note that $\deg K_C = 2g - 2$.)

About the proof

- Siegel proved this using his diophantine approximation result (before Roth) and abelian varieties.
- Decades before Siegel, Runge (1887) proved several special cases by very elementary means. This was effective but it does not work for general number fields or S-integers.
- Corvaja and Zannier (2002) found a simpler proof using Schmidt's subspace theorem. These ideas led to higher-dimensional results, e.g.:

Theorem (Levin 2009)

Let $X/\mathbb{Q}$ be a smooth projective surface, let D_1, D_2, D_3, D_4 be irreducible reduced effective divisors on X such that no 3 of them meet at the same point. If all the D_j are ample, then every set of $(\sum_j D_j)$-integral points is finite. (Also over other number fields, and for S-integers.)

What to expect?

Conjecture (Bombieri–Lang–Vojta)

Let X be a smooth projective variety over $\mathbb{Q}$ and let D be an effective reduced normal crossings divisor on X. If $K_X + D$ is big, then every set of D-integral points in X is Zariski degenerate.

- **Slogan:** Enough positivity of D should imply algebraic degeneracy of D-integral points.
- When $D = 0$ integrality imposes no condition, so we are talking about algebraic degeneracy of rational points in varieties of general type (K_X big.)
- This includes Siegel's theorem and the results of Faltings on sub-varieties of abelian varieties.

Integrality

Let X be a smooth projective variety and D a reduced effective divisor on X. We need to clarify what does it mean that a set of rational points $\Phi \subseteq (X - D)(\mathbb{Q})$ is D-integral. We'll do this using **heights**.

Attached to every divisor D on a variety X there is a height function

$$h_X(D, -) : (X - \operatorname{sup} D)(\mathbb{Q}) \to \mathbb{R}$$

defined up to bounded error.

Integrality

There is also the Archimedean Weil (proximity) function

$$\lambda_{X,\infty}(D,-) : (X - \operatorname{sup} D)(\mathbb{Q}) \to \mathbb{R}$$

When D is effective and reduced, one can see the proximity function as

$$\lambda_{X,\infty}(D,x) = \log \max \left\{ 1, \frac{1}{d_\infty(D,x)} \right\}$$

where d_∞ is an Archimedean distance.

Integrality

The following abstract definition includes every sensible way to define D-integrality (e.g. coordinates of an embedding when D is ample, etc.)

Definition

Let X be a smooth projective variety over $\mathbb{Q}$ and D a reduced effective divisor on X. A set of rational points $\Phi \subseteq (X - D)(\mathbb{Q})$ is D-**integral** if the following holds as x varies in Φ:

$$h_X(D, x) = \lambda_{X,\infty}(D, x) + O(1).$$

Important notes:

- For D effective we always have (up to adding $O(1)$):

$$0 \leq \lambda_{X,\infty}(D, x) \leq h_X(D, x)$$

- The height and the proximity are linear on D, up to adding $O(1)$.

Integrality: an example.

Let $X = \mathbb{P}^1$ and $D = \infty$. Then $X - D = \mathbb{A}^1$. The set

$$\mathbb{Z} \subseteq \mathbb{A}^1(\mathbb{Q}) = (X - D)(\mathbb{Q})$$

oughts to be D-integral. We can detect this by the condition

$$h_X(D, x) = \lambda_{X,\infty}(D, x).$$

Indeed, this height is the usual one on $\mathbb{Q}$: for a, b coprime integers

$$h_X(D, a/b) = \log \max\{|a|, |b|\}.$$

We also note that $\lambda_{X,\infty}(D, x) = \log \max\{1, |x|\}$.

So, our claim is that integrality is detected by the condition

$$\max\{|a|, |b|\} = \max\{1, |a/b|\}.$$

Runge's theorem

Theorem (Runge, 1887)

Let X be a smooth projective curve over $\mathbb{Q}$ and let D be a reduced effective divisor on X. If D is not irreducible over $\mathbb{Q}$, then every set of D-integral points $\Phi \subseteq (X - D)(\mathbb{Q})$ is finite.

Example

Let $f(x) \in \mathbb{Z}[x]$ be squarefree, monic, of degree $n \geq 2$. Consider the superelliptic curve

$$y^n = f(x)$$

Runge's theorem gives finiteness of integral points with respect to the divisor at infinity (thus, finiteness of solutions (u, v) with u, v integers.)

Indeed, the divisor at infinity has equation $y^n - x^n = 0$ which is reducible over $\mathbb{Q}$.

Runge's theorem

Proof.

Write $D = D_1 + D_2$ with D_j non-zero (hence, ample.) For $x \in \Phi$ we have

$$h_X(D_1, x) + h_X(D_2, x) = \lambda_{X,\infty}(D_1, x) + \lambda_{X,\infty}(D_2, x) + O(1)$$

The point x cannot be close to both D_1 and D_2.
So, if Φ is infinite we may assume that $\lambda_{X,\infty}(D_2, x) = O(1)$ for infinitely many x's. Thus

$$h_X(D_1, x) + h_X(D_2, x) = \lambda_{X,\infty}(D_1, x) + O(1) \leq h_X(D_1, x) + O(1).$$

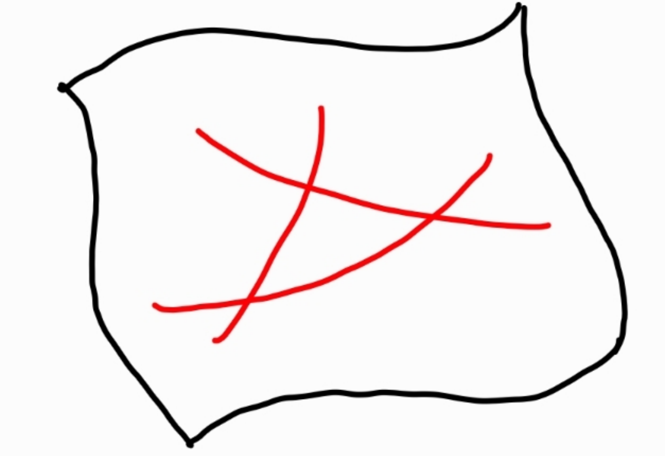
Hence, $h_X(D_2, x) = O(1)$. This contradicts the Northcott property. □

Recall: The Northcott property of $h_X(D, -)$ for D ample says that if we bound this height, we are left with only finitely many rational points.

After Runge's theorem

- The key observation was:
  The point x cannot be close to both D_1 and D_2.
  Other than this, it was just formal properties of heights.
- Levin (2008) extended these ideas to higher dimensions, under the assumption that $D = D_1 + \dots + D_{n+1}$ on a variety X of dimension n, not all the D_j meeting at the same point —plus some necessary positivity assumptions on D_j such as bigness or ampleness to have Northcott.

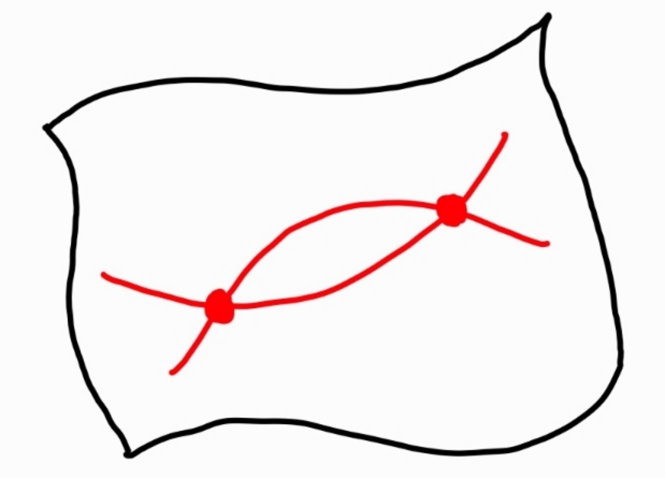
Levin–Runge setting



Our results

- What Natalia Garcia-Fritz and I did is to obtain algebraic degeneracy of integral points for D with $n = \dim X$ components (not $n+1$), under suitable assumptions that are often satisfied.
- **Key complication:** If the components of $D = D_1 + ... + D_n$ are big or ample, then all of these n components can meet at some points, so the key observation no longer holds: a rational point can indeed be close to all the D_j at the same time.

New setting



The main idea

- Recall the key observation in Runge's theorem for curves: **The point** x **cannot be close to both** D_1 **and** D_2**.**
- In the general case $\dim X = n$, a rational point x can very well be close to each $D_1, ..., D_n$ at the same time. Note that under the normal crossings assumption, $\cap_{j=1}^{n} D_j$ is a finite set of algebraic points.
- We define an invariant τ_∞ to control this phenomenon, and it turns out that this invariant satisfies $\tau_\infty < 1$ (enough for us) quite often!

The invariant $\tau_\infty(Y, D)$

Let X be a smooth projective variety over $\mathbb{Q}$, let Y be a 0-dimensional reduced sub-scheme of X over $\mathbb{Q}$, and let $D \geq 0$ be a divisor on X. We define

$$\tau_\infty(Y, D)$$

as the infimum of all (extended) real numbers $\tau \geq 0$ such that the inequality

$$\lambda_{X,\infty}(Y, x) < \tau \cdot h_X(D, x) + O(1)$$

holds for all rational points x outside a proper Zariski closed set $Z_\tau \subsetneq X$.

Remarks:

- If Y is a single rational point and D is ample, then $\tau_\infty(Y, D) \leq 1$.
- A small value of $\tau_\infty(Y, D)$ indicates that the algebraic points in Y are poorly approximable by rational points.
- For our applications we just need $\tau_\infty(Y, D) < 1$.

Our main result

Theorem (GF–P 2023)

Let X be a smooth projective variety over $\mathbb{Q}$ with $n = \dim X$. Let $D_1, ..., D_n$ be reduced effective big divisors on X such that $D = D_1 + ... + D_n$ is normal crossings. Assume that for every component Y of $\cap_{j=1}^{n} D_j$ defined over $\mathbb{Q}$ we have

$$\tau_\infty(Y, D_j) < 1 \qquad \textit{for each } j = 1, ..., n.$$

Then there is a proper Zariski closed subset $Z \subsetneq X$ such that every set $\Phi \subseteq (X - D)(\mathbb{Q})$ of D-integral points is contained in Z, up to finitely many points.

Some applications

The method of proof follows the same ideas as in Runge's theorem; the point is that $\tau_\infty(Y, D_j) < 1$ is good enough to allow us to play the same game —it is not necessary that $\cap_{j=1}^n D_j$ be empty.

We'll discuss some applications. For this we fix the following notation:

- X smooth projective variety over $\mathbb{Q}$ of dimension n
- $D_1, \ldots, D_n$ are reduced effective big divisors on X such that $D = D_1 + \ldots + D_n$ is normal crossings.

Small numerical rank

Theorem (GF–P)

Let r be the rank of $\langle D_1, \dots, D_n \rangle$ in $NS(X)$ and recall $n = \dim X$. If $r < n$ then every set of D-integral points is Zariski degenerate.

Idea of proof:

- Let $q = \dim H^0(X, \Omega^1_X)$ be the irregularity of X.
- If $q = 0$ then algebraic and linear equivalence coincide. Hence, there is a non-constant rational function $f \in K(X)$ whose divisor is supported on the D_j. We note that D-integral sets are mapped to $(0 + \infty)$-integral sets of $\mathbb{P}^1$; these are finite.
- If $q > 0$ the Albanese map $a : X \to A$ is non-trivial, with A an abelian variety. We use Diophantine approximation on abelian varieties to show that $\tau_\infty(Y, D_j) = 0 < 1$. $\square$

Using the volume

For a divisor D one defines its **volume** as

$$vol(D) = \limsup_{s} \frac{n! \dim H^0(X, sD)}{s^n}.$$

Essentially by definition of big divisor, B is big if and only if $vol(B) > 0$. When B is ample, $vol(B) = B^n$.

Theorem (GF–P; general bound for $\tau_\infty(Y, B)$)

Let Y be a zero dimensional subscheme of X defined over $\mathbb{Q}$ with d geometric points. Let B be a big divisor on X. Then

$$\tau_\infty(Y, B) \leq \sqrt[n]{\frac{d}{vol(B)}}.$$

This gives at once an algebraic degeneracy theorem for integral points provided that $vol(D_j)$ are large enough (omitted here.)

Sufficiently positive divisors

Theorem (GF–P)

Assume that $n = \dim X \geq 2$. Let $D = D_1 + ... + D_n$ be a normal crossings divisor. Assume that for each j, we have $\mathcal{O}_X(D) \simeq \mathcal{L}_j^{\otimes m_j}$ with some $m_j \geq 2$ and $\mathcal{L}_j$ ample globally generated. Then there is a proper Zariski closed set $Z \subsetneq X$ such that every set of D-integral points is contained in Z up to finitely many points.

- Roth–McKinnon theorem: *the larger a Seshadri contant is, the worse the approximation rate is.*
- The positivity assumptions provide a large enough Seshadri constant to conclude $\tau_\infty(Y, D_j) < 1$.
- Then we can apply our main result. □

Large fundamental group

Theorem (GF–P)

Assume that $X_{\mathbb{C}}$ has large algebraic fundamental group. If each D_j is ample, then every set of D-integral points is Zariski degenerate.

- By Cerbo–Cerbo the étale Seshadri constant of X is ∞.
- Using this, from Roth–McKinnon we deduce that algebraic points of X are very poorly approximable, obtaining $\tau_\infty(Y, D_j) = 0 < 1$.
- Finally, we conclude by our main result. $\square$

Thanks for your attention.