Abhyankar's Conjectures and Fundamental Groups

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Outline

(Very) short bibliography

S. S. Abhyankar, "Galois theory on the line in nonzero characteristic", Bull. Amer. Math. Soc. (1992), 68–133.

S. S. Abhyankar, "Resolution of singularities and modular Galois theory", Bull. Amer. Math. Soc. (2001), 131–169.

D. Harbater, A. Obus, R. Pries, K. Stevenson, "Abhyankar's conjectures in Galois theory: Current status and future directions", Bull. Amer. Math. Soc. (2018), 239–287.

Topological fundamental groups

Starting point: Fundamental group $\pi_1(X)$, where X is a (punctured) Riemann surface of genus g with r holes ("of type (g, r)"):

$$\pi_1(X) = \frac{\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r \rangle}{\prod_{i=1}^{g} [a_i, b_i] c_1 \cdots c_r} =: \Pi_{g,r}.$$

- Observe that if $r \geq 1$, then this is a *free* group on $2g + r - 1$ generators.
- Recall that there is a universal cover $\widetilde{X} \to X$ on which $\pi_1(X)$ acts freely, and $X \cong \widetilde{X}/\pi_1(X)$.
- In particular, each finite index normal subgroup H of $\pi_1(X)$ corresponds to a finite topological cover of X with deck transformation group $\pi_1(X)/H$.

Algebraic fundamental groups

- Hard to talk about homotopy classes of loops in algebraic geometry.
- Instead, one builds the theory of the fundamental group from the idea of covering spaces.
- These are not Zariski-topological covers (topology is too coarse), but rather finite *étale* covers.
- Explicitly, $Y \to X$ is étale if, locally, Y is cut out over X by n polynomials $f_1, \ldots, f_n$ in n variables such that the Jacobian matrix of the f_i is invertible. Think: On $\mathbb{C}$-points, gives topological cover
- A finite étale cover $f \colon Y \to X$ is $|Aut(Y/X)|$-*Galois* if $|\mathrm{Aut}(Y/X)|$ acts transitively on the fibers.
- One can then define $\pi_1^{\mathrm{alg}}(X)$ to be the inverse limit of $\mathrm{Aut}(Y/X)$ over all Galois covers $Y \to X$. So G is the finite Galois group of a cover *if and only if* G is a quotient of $\pi_1^{\mathrm{alg}}(X)$.

Branched covers and inertia groups

- If X is a smooth, connected affine curve with smooth projective completion $\overline{X}$, then any finite étale cover $Y \to X$ extends uniquely to a finite morphism $\overline{Y} \to \overline{X}$, where $\overline{Y}$ is the smooth projective completion of Y.
- Such a finite morphism is called a *branched* cover.
- If $Y \to X$ is G-Galois, then G acts on $\overline{Y}$, and the stabilizer of a point of $\overline{Y}$ is called the *inertia group* at that point.
- For short, we refer to the inertia groups of $\overline{Y} \to \overline{X}$ as those of $Y \to X$. cyclic
- Over $\mathbb{C}$, the inertia groups of a cover are always *cyclic*. $\to \mathbb{C}((t))$
- If $\overline{X} \cong \mathbb{P}^1$ over any algebraically closed field and $Y \to X$ is Galois, then the inertia groups generate the Galois group of the cover.

Fundamental groups and inertia groups in characteristic zero

- The Riemann existence theorem shows that if X is a smooth algebraic curve over $\mathbb{C}$ and $f^{an}\colon Y^{an} \to X(\mathbb{C})$ is a finite topological cover, then Y^{an} is actually $Y(\mathbb{C})$ for some algebraic curve Y, and the map f^{an} comes from an *algebraic* morphism from $Y \to X$.
- One derives from this that $\pi_1^{\text{alg}}(X)$ is the profinite completion of $\pi_1^{\text{top}}(X(\mathbb{C}))$.
- In fact, as long as X is defined over an algebraically closed field of characteristic zero, the isomorphism class of $\pi_1^{\text{alg}}(X)$ depends only on the type (g, r) of X, not on the base field. It is $\hat{\Pi}_{g,r}$.
- The inertia groups do not depend on the base field either (they are cyclic).

Flavors of fundamental group in characteristic p

Now, suppose X is a curve of type (g, r) over an algebraically closed field of characteristic p, with smooth projective completion $\overline{X}$. From now on, we write $\pi_1(X) = \pi_1^{\text{alg}}(X)$.

- As we know,

$$\pi_1(X) \cong \varprojlim_{Y \to X \text{ Galois finite étale}} \operatorname{Aut}(Y/X)$$

- If we instead take the inverse limit over Galois covers whose inertia groups are prime to p, we obtain $\pi_1^{\text{tame}}(X)$.
- If we only take the inverse limit over Galois covers whose degrees are prime to p, we obtain $\pi_1^{p'}(X)$.

Comparison Theorem

Let $\Pi_{g,r}$ be the topological fundamental group of a ~~curve~~ Riemann surface of type (g, r) over $\mathbb{C}$.

Theorem (Grothendieck, SGA 1)

If X is a curve of type (g, r) over an algebraically closed field of characteristic p, then there exists a surjective homomorphism $\hat{\Pi}_{g,r} \to \pi_1^{tame}(X)$ which is an isomorphism on maximal prime-to-p quotients.

So, e.g. a finite prime-to-p group is a quotient of $\pi_1(X)$ iff it is a quotient of $\Pi_{g,r}$ (equiv. $\hat{\Pi}_{g,r}$)

→ If G has finite prime-to-p quotient H, then G is a quotient of $\pi_1(X)$ only if H is a quotient of $\Pi_{g,r}$.

Wild cover examples

Let $X = \mathbb{A}^1_k$, with k algebraically closed of characteristic p. Note that if k had characteristic zero, the affine line would have trivial fundamental group!

$$\begin{array}{c} y \\ \mathbb{Z}/p \big\downarrow \\ X = \mathbb{A}^1_k \text{ (x-line)} \end{array} \qquad y^p - y = x^n \qquad y \mapsto y+1. \quad \mathbb{Z}/p\text{-action.}$$

$\frac{\partial}{\partial y}: 1 \rightsquigarrow$ does not vanish $\to$ étale.

Inertia group at ∞ is $\mathbb{Z}/p$ (not prime to p).

As n runs through $\mathbb{N} \setminus p\mathbb{N}$, these covers are all linearly disjoint.

$\rightsquigarrow (\mathbb{Z}/p)^a$ is a quotient of $\pi_1(\mathbb{A}_k)$ for all $a \in \mathbb{N}$.

Quotients of $\pi_1(\mathbb{A}^1_k)$

- We have seen that arbitrary finite powers of $\mathbb{Z}/p$ appear as quotients of $\pi_1(\mathbb{A}^1_k)$.
- In fact, there are interesting non-abelian examples as well!

$$\begin{array}{ll} y & y^n - ax^s y^t + 1 = 0 \\ \downarrow & \\ x & (x\text{-line}) \end{array}$$

$(t \not\equiv 0 \pmod p)$, $n = p + t$).
is étale (fun exercise)
If we take Galois closure, Galois group can be $PSL_2(p)$, $PSL_2(8)$, S_n, A_n depending on t, p.

Question

What finite groups G can appear as quotients of $\pi_1(\mathbb{A}^1_k)$? That is, for what finite groups G do there exist G-Galois covers of $\mathbb{A}^1_k$?

$\pi_1(\mathbb{A}^1_k)$ is not topologically finitely generated!

- Note that the examples above show that $(\mathbb{Z}/p)^n$ is a quotient of $\pi_1(\mathbb{A}^1_k)$ for all $n \in \mathbb{N}$, so $\pi_1(\mathbb{A}^1_k)$ is *not* topologically finitely generated!
- This means that even if we can answer the question of which finite groups appear as quotients of $\pi_1(\mathbb{A}^1_k)$, we will still not have determined the full structure of $\pi_1(\mathbb{A}^1_k)$.

Abhyankar's philosophy regarding finite Galois covers (informally)

- Groups that shouldn't be the Galois group of a cover aren't.
- Groups shouldn't *not* be the Galois group of a cover are.

Or, slightly more formally:

- A finite group should appear as a quotient of a fundamental group in characteristic p if and only if its maximal prime-to-p quotient appears as a quotient of the "corresponding" fundamental group in characteristic zero.

This philosophy was informed by a great deal of examples that Abhyankar originally encountered while studying resolution of singularities on surfaces in characteristic p.

Abhyankar's conjecture for affine curves

Let k be an algebraically closed field of characteristic p, let G be a finite group, and let $p(G)$ be the subgroup of G generated by its p-Sylow subgroups. Suppose X is a curve of type (g, r) over k with $r \geq 1$.

Conjecture (Abhyankar — Proof by Harbater and Raynaud)

The group G is a quotient of $\pi_1(X)$ if and only if $G/p(G)$ can be generated by $2g + r - 1$ elements.

Case: $X = \mathbb{A}^1_k$, then G is a quotient of $\pi_1(\mathbb{A}^1_k)$ iff

$$G = p(G)$$

"quasi-p groups"

Ex: p-groups, simple groups G where $p \mid |G|$.

Proof of Abhyankar's conjecture: "only if" (easy) direction

G quotient of $\pi_1(X) \Rightarrow G/p(G)$ is a quotient of $\pi_1(X)$

$\Rightarrow G/p(G)$ is a quotient of $\Pi_{g,r}$

$\Rightarrow G/p(G)$ is generated by $2g+r-1$ elements.

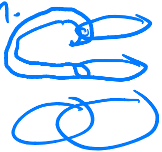
Proof of Abhyankar's conjecture: Techniques for "if" (hard) direction

- Group cohomology and embedding problems

$$\begin{tikzcd} & G \arrow[d] \\ \pi_1(X) \arrow[ur, dashed, "?"] \arrow[r, two heads] & G/H \end{tikzcd}$$

- Formal/Rigid patching

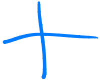
"Gluing" together covers over non-archimedean fields where overlap is not on a Zariski open.



- Semi-stable models

Controlled bad reduction to char. p.

Ex: $xy = p$ (char. 0) $\xrightarrow{\text{reduction}}$ $xy = 0$



reduced, singularities are ODP's.

Solvable case: Serre's idea

Theorem (Serre)

Let $1 \to H \to G \to G/H \to 1$ be exact, with H solvable and G quasi-p. If G/H is a quotient of $\pi_1(\mathbb{A}^1_k)$, then so is G.

Idea of proof:

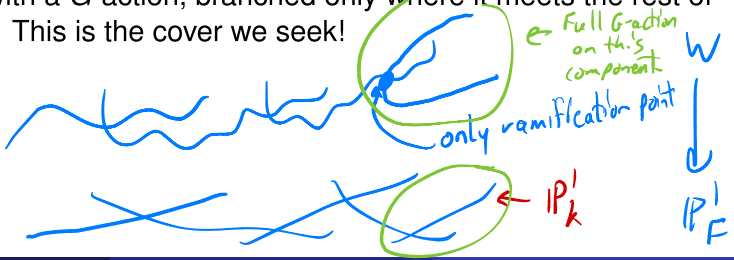
- By étale cohomology, $\pi_1(\mathbb{A}^1_k)$ has cohomological dimension 1, and is thus projective.
- So $\pi_1(\mathbb{A}^1_k) \twoheadrightarrow G/H$ lifts to $\pi_1(\mathbb{A}^1_k) \to G$.
- Need to make sure this map is surjective. Can assume H elementary abelian, irreducible under action of G/H.
- It might not be surjective, but there exists a twist by an element of $H^1(\pi_1(\mathbb{A}^1_k), H)$, provided this latter cohomology group strictly contains $H^1(G/H, H)$
- Can ensure this by pulling back the G/H-cover by an mth power map.

Affine line case: Raynaud's first idea

- Let S be a p-Sylow group of G, and let $G(S) \subseteq G$ be generated by all proper quasi-p-subgroups of G whose p-Sylow subgroups are contained in S.
- Assume $G(S) = G$.
- Let G_i be the proper subgroups of G whose p-Sylows are contained in S. By induction, assume there are G_i-Galois covers $f_i \colon Y_i \to \mathbb{A}^1_k$.
- By Abhyankar's Lemma, can assume the inertia groups are p-groups P_i, all contained in S.
- Now use rigid patching over $k((t))$ to patch these covers together to a G-cover with inertia group S.
- Lastly, take an appropriate specialization to get a cover over k.

Affine line case: Raynaud's second idea

- Assume G has no non-trivial normal p-subgroup (Serre).
- Assume $G(S) \neq G$ (!)
- Build a G-Galois branched cover $W \to \mathbb{P}^1_{\mathbb{C}}$ in characteristic zero with all inertia groups p-groups.
- Can view this cover over a finite extension of $Frac(W(k))$.
- If we take a *semi-stable* model of this cover and look at the special fiber (which lives over k!), there will be an irreducible component upstairs with a G-action, branched only where it meets the rest of the curve. This is the cover we seek!



General curves: Harbater's idea

- The key case is the of type $(0, 2)$, that is, $\mathbb{A}^1_k \setminus \{0\}$.
- Reduce to the case $G \cong p(G) \rtimes \overline{G}$, where $\overline{G}$ is cyclic of prime-to-p order.
- If P is a p-Sylow group of G, can use a souped-up version of Serre's result to build a $P \rtimes \overline{G}$-cover $g\colon V \to \mathbb{A}^1_{k((t))} \setminus \{0\}$ with inertia groups conjugate to P.
- Using Abhyankar's conjecture, build a $p(G)$-cover $h\colon W \to \mathbb{A}^1_k$. Can force inertia groups to be conjugate to P as well.
- Thickening h, we can glue copies of h and g together using formal patching to get a G-cover over $k((t))$. Then specialize as before.
- General case comes from patching an appropriate cover of $\mathbb{A}^1_k \setminus \{0\}$ to copies of a prime-to-p cover of a type (g, r)-curve.

Open generalizations/extensions

- Abhyankar's inertia conjecture
- Abhyankar's affine arithmetical conjecture ⇝ analog of Abhyankar Conjecture when $k = \mathbb{F}_p$.
- Abhyankar's conjectures for higher-dimensional schemes

↳ Complements of hyperplane arrangements.

Abhyankar's inertia conjecture

- Let $Y \to \mathbb{A}^1_k$ ~~is~~ be a G-Galois cover, and let I be an inertia group. Then I and its conjugates generate G.
- Furthermore, $I \cong P \rtimes \mathbb{Z}/m$, where P is a non-trivial p-group and $p \nmid m$. Galois groups of extensions of $k((t))$.

Conjecture (Abhyankar's inertia conjecture)

Let G be a quasi-p-group, and let $I \subseteq G$ be a subgroup of the form $P \rtimes \mathbb{Z}/m$ such that I and its conjugates generate G. Then there exists a G-cover of the affine line with an inertia group equal to I.

For short, we say we can realize (G, I).

Abhyankar's inertia conjecture: Results

General results:

- If we can realize (G, I) and $I' \supseteq I$ with $|I'/I|$ a power of p, then we can realize (G, I') (Harbater).
- If we can realize (G, I) and $I' \subseteq I$ with $p \nmid |I/I'|$, then we can realize (G, I'). (Abhyankar's lemma)

Specific results

- Lots of individual groups G (abelian p-groups, $PSL_2(p), A_p, \underline{A_{p+2}, A_{p+1}, A_{p+3}, A_{p+4}, A_{p+5}})$ $p \equiv 2 \pmod 3$
- Partial results (e.g., $PSL_2(\ell)$ with $p \mid \ell^2 - 1$, products of alternating groups A_{n_i} with $p \leq n_i < 2p$)
- Techniques are similar to original Abhyankar conjecture (explicit equations, reduction, patching).

No real "program" to solve the entire conjecture.

Abhyankar's affine arithmetical conjecture

- Suppose k is a *finite* field.
- Observe that $\mathrm{Gal}(\overline{k}/k) \cong \hat{\mathbb{Z}}$.
- If X is an affine curve over k, we have the exact sequence

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \mathrm{Gal}(\overline{k}/k) \cong \hat{\mathbb{Z}} \to 1.$$

- Less ambitious conjecture: If G is a quotient of $\pi_1(\mathbb{A}^1_k)$, then it is a quotient of $\pi_1(\mathbb{A}^1_{\mathbb{F}_p})$.
- More ambitious conjecture: If G is a quotient of $\pi_1(\mathbb{A}^1_k \setminus \{0\})$, then it is a quotient of $\pi_1(\mathbb{A}^1_{\mathbb{F}_p})$.

Theorem (Guralnick, Stevenson)

If G is generated by $p(G)$ and g with $g \notin p(G)$, then there exists a finite field $\mathbb{F}_q/\mathbb{F}_p$ such that G is a quotient of $\pi_1(\mathbb{A}^1_{\mathbb{F}_q})$.

Abhyankar-style covers from characteristic zero

- We have already seen that, in order to prove Abhyankar's conjecture for affine curves over an algebraically closed field k of characteristic p, a key step involves realizing a G-cover as an irreducible component of the reduction of a cover of curves from characteristic zero.
- In fact, many G-covers can be realized as reductions of covers in characteristic zero, full stop (no need to take an irreducible component).
- In fact, *every* branched G-cover $Y \to \mathbb{P}^1_k$ can be realized in this way, so long as the inertia groups are prime to p (i.e., it is a tame cover).
- This is more or less equivalent to the fact that the tame fundamental group of an affine curve in characteristic p is a quotient of the corresponding fundamental group in characteristic zero.

Example

Let k be an algebraically closed field of characteristic p. Consider the $\mathbb{Z}/p$-cover of $\mathbb{P}^1_k$, étale over $\mathbb{A}^1_k$, given birationally by

$$y^p - y = x.$$

Sketch: ~~Let~~ Work over $R = W(k)[\zeta_p]$. Let $\lambda = \zeta_p - 1$.

(branched) $\mathbb{Z}/p$-cover over R given by

$$Z^p = 1 + \lambda^p X \qquad Z = 1 + \lambda Y$$

$$\hookrightarrow (1+\lambda Y)^p = 1 + \lambda^p X$$

$$\hookrightarrow \cancel{1} + \cancel{\lambda^p} Y^p + \cancel{p}\tilde{\lambda}^{-1} \cancel{\lambda} Y + \text{stuff} = \cancel{1} + \cancel{\lambda^p} X$$

reduce mod λ

$$\hookrightarrow Y^p - Y = X$$

The lifting problem and Oort groups

We consider the following “lifting problem”:

- Suppose we are given a branched G-Galois cover of smooth curves $f\colon\ Y \to \mathbb{P}^1_k$. When is there a discrete valuation ring R in characteristic zero with residue field k and a G-Galois cover $f_R\colon\ Y_R \to \mathbb{P}^1_R$ such that $f_R \times_R k \cong f$ as G-Galois covers?

If the question above has a positive answer for every cover for a given group G, then G is said to be a *(global) Oort group*. One says that “all G-Galois covers lift to characteristic zero”.

A conjecture and recent counterexample

Conjecture

In characteristic $p > 2$, the group D_{p^n} is an Oort group for all n.

every D_{p^n}-cover in char p comes as reduction from char. 0.

- Known for $n = 1$ (Bouw–Wewers).
- Known for D_9, D_{25}, D_{27} (Dang, Das, Karagiannis, Obus, Thatte)

Counterexample: D_{125} → branched at 2 points (inertia groups $\mathbb{Z}/2$, D_{125})

Kontogeorgis–Terezakis

→ Cyclic part $\mathbb{Z}/125$ upper jumps 9, 45, 225

Idea behind the counterexample

- If $f\colon Y \to \mathbb{P}^1_k$ is a G-cover, then there is a G-action on the vector space V of holomorphic differentials of Y.
- Outside of some exceptional cases, Petri's theorem shows that the canonical ideal I of Y is generated by *quadratic* differentials.
- Kontogeorgis and Terezakis show that f lifts to characteristic zero if and only if the G-module V lifts to characteristic zero in a way that leaves the canonical ideal invariant.
    - That is, letting $I_2 \subseteq \mathrm{Sym}^2(V)$ be the quadratic part of the canonical ideal, the canonical action of G acting on $\mathrm{Sym}^2(V)$ preserves I_2.
- This uses an earlier result of theirs that any lift of the curve Y to characteristic zero can be explicitly given by lifting the quadratic polynomials that cut Y out in its canonical embedding.
- They then show that a certain D_{125}-cover does not satisfy this criterion. The proof is more or less linear algebra!

Question on D_{p^n}-covers

Can one obtain any reasonably nice criterion for when a D_{p^n}-cover does or does not lift to characteristic zero?

Thank you for your attention!