# JACOBIANS WITH <br> COMPLEX MULTIPLICATION 

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## COLEMAN'S CONJECTURE


[..]
Finally we would like to state one last conjecture. Let $g$ be an integer $g \geq 4$.

Conjecture 6. There are only finitely many curves over $\mathbb{C}$ of genus $g$ whose Jacobians admit the structure of a CM Abelian variety.

Quoted from: R. Coleman, Torsion points on curves. In: Galois representations and arithmetic algebraic geometry (Kyoto, 1985/Tokyo, 1986), pp. 235-247; Adv. Stud. Pure Math., 12, North-Holland, Amsterdam, 1987.

- Why this conjecture?
- Why $g \geq 4$ ?
- Would Serre call this a conjecture?


## BASIC NOTIONS

Let $A / k$ be an abelian variety, $\operatorname{dim}(A)=g$.

## Definition

We say $A$ is a CM abelian variety (or: $A$ has $C M$ ) if $\operatorname{End}^{0}(A)$ contains a commutative semisimple algebra $S$ with $\operatorname{dim}_{\mathbb{Q}}(S)=2 g$.

- If $A$ is simple, $A$ has CM if $\operatorname{End}^{0}(A)$ contains a field of degree $2 g$ over $\mathbb{Q}$
- In general: let $A \sim A_{1}^{m_{1}} \times \cdots \times A_{r}^{m_{r}}$ be a decomposition of $A$ as product of simple factors; then $A$ has CM if each $A_{i}$ has CM

Caution: If $k \neq \bar{k}$ it may happen that $A / k$ does not have CM but $A_{\bar{k}} / \bar{k}$ does.

Unless stated otherwise, I will work over the complex numbers. "Curve" means: complete nonsingular curve.

The Torelli map

$$
\tau: \mathcal{M}_{g} \rightarrow \mathcal{A}_{g, 1}, \quad[C] \mapsto[\operatorname{Jac}(C)]
$$

is injective on points, immersive outside the hyperelliptic locus. We write

$$
\mathcal{T}_{g}^{\circ} \subset \mathcal{T}_{g} \subset \mathcal{A}_{g, 1}
$$

for the image of $\tau$ and its closure, called the Torelli locus.
The boundary $\mathcal{T}_{g}^{\text {dec }}=\mathcal{T}_{g} \backslash \mathcal{T}_{g}^{\circ}$ is the intersection of $\mathcal{T}_{g}$ with the locus of decomposable ppav.

Does it ever happen that a Jacobian $J$ has $C M$ ?
Example. Take an integer $g \geq 1$ and let $n=2 g+1$. Let $C_{n}$ be the curve given by the equation

$$
y^{n}=x(x-1)
$$

So " $x$ " gives a morphism $C_{n} \rightarrow \mathbb{P}^{1}$ which is a cyclic cover of degree $n$. Therefore, $\mathbb{Q}[\mathbb{Z} / n \mathbb{Z}]$ acts on $J_{n}=\operatorname{Jac}\left(C_{n}\right)$.

Clear: if $d \mid n$ then we have $C_{n} \rightarrow C_{d}$, which on Jacobians gives $J_{n} \leftarrow J_{d}$. Define:

$$
J_{n}^{\text {new }}:=J_{n} / \sum_{d \mid n} J_{d}
$$

- The group algebra

$$
\begin{equation*}
\mathbb{Q}[\mathbb{Z} / n \mathbb{Z}] \cong \prod_{d \mid n} \mathbb{Q}\left(\zeta_{d}\right) \tag{*}
\end{equation*}
$$

acts on $J_{n}$.

- We have an isogeny decomposition

$$
J_{n} \sim \prod_{d \mid n} J_{d}^{\text {new }}
$$

which is compatible with $(*)$
$-\operatorname{dim}\left(J_{d}^{\text {new }}\right)=\frac{\varphi(d)}{2}=\frac{\left[\mathbb{Q}\left(\zeta_{d}\right): \mathbb{Q}\right]}{2}$
Conclusion: $J_{n}$ is a CM abelian variety

## Exercise/challenge:

Construct other examples of CM Jacobians

You will find that it is not so obvious how to do this. Fermat curves have CM Jacobians but they occur only in special genera.

## MOTIVATION FOR COLEMAN'S

CONJECTURE: AN ANALOGY

## Why Coleman's Conjecture?

[..]
This is an analogue of the Manin-Mumford conjecture because the CM points on the moduli space of principally polarized Abelian varieties of genus $g$ are analogous to torsion points. In fact the CM liftings to $\mathbb{Q}_{p}$ of an ordinary Abelian variety over $\overline{\mathbb{F}}_{p}$ are the torsion points in the moduli space of all liftings (see [K]). Dwork and Ogus have obtained a partial result in this direction, see [D-O].


## Manin-Mumford Conjecture $=$ Theorem of Raynaud

Let $C / \mathbb{C}$ be a curve of genus $g \geq 2$, let $b \in C$ be a base point and consider the emnbedding

$$
i: C \hookrightarrow J=\operatorname{Jac}(C) \quad \text { given by } P \mapsto[b-P] .
$$

Then $C \cap\{$ torsion points of $J\}$ is finite.

First proven by Raynaud, 1983.

## An analogy-?

abelian variety torsion points

$$
C \hookrightarrow J
$$

Manin-Mumford (Raynaud's thm):
$C \cap\{$ torsion points $\}$ is finite
moduli space $\mathcal{A}_{g, 1}$
CM points
$\mathcal{M}_{g} \hookrightarrow \mathcal{A}_{g, 1}$ (Torelli map)
Coleman's Conjecture:
$\mathcal{T}_{g}^{\circ} \cap\{\mathrm{CM}$ points $\}$ is finite (??)

In fact, Raynaud proved something stronger than Manin-Mumford:

## Definition

Let $A / \mathbb{C}$ be an abelian variety. An irreducible subvariety $S \subset A$ is called a special subvariety if $S$ is the translate of an abelian subvariety over a torsion point.

Theorem (Raynaud)

Let $A / \mathbb{C}$ be an abelian variety, $Z \subset A$ an irreducible subvariety. Then
$Z$ is special $\Longleftrightarrow \quad$ the torsion points in $Z$ are Zariski dense
abelian variety
torsion points

$$
C \hookrightarrow J
$$

Manin-Mumford (Raynaud's thm):
$C \cap\{$ torsion points $\}$ is finite special subvariety

torsion points dense
moduli space $\mathcal{A}_{g, 1}$
CM points
$\mathcal{M}_{g} \hookrightarrow \mathcal{A}_{g, 1}$ (Torelli map)
Coleman's Conjecture:
$\mathcal{T}_{g}^{\circ} \cap\{\mathrm{CM}$ points $\}$ is finite (??)
special subvariety
§?
CM points dense

## Definition

An irreducible subvariety $S \subset \mathcal{A}_{g, 1}$ is called a special subvariety if $S$ is an irreducible component of a Shimura subvariety.

Not an easy definition...

- We can try to explain the definition in more detail or
- we can try to characterize special subvarieties by certain properties.

What the definition means:
The special subvarieties of $\mathcal{A}_{g, 1}$ are the Hodge loci, i.e., the maximal irreducible subvarieties on which certain cohomology classes are Hodge classes.

Special subvarieties are "defined by" Shimura data $(G, X)$, consisting of a reductive $\mathbb{Q}$-group $G$ and a hermitian symmetric domain $X$ with transitive $G(\mathbb{R})$-action, satisfying certain axioms (Deligne). In this language one can try to classify special subvarieties.

Basic example: fix some ring $R$, and consider all (principally polarized) abelian varieties $A$ that admit an action by $R$, i.e., for which we have $R \subset \operatorname{End}(A)$.

Fact: in the moduli space, these points $[A] \in \mathcal{A}_{g, 1}$ form a countable union of irreducible algebraic subvarieties. These components are special subvarieties of $\mathcal{A}_{g, 1}$. In this particular case, they are called special subvarieties "of PEL type".

In this example, we use that an endomorphism of $A$ corresponds to a Hodge class in

$$
\underline{\operatorname{End}}\left(H^{1}(A, \mathbb{Z})\right)=H^{1}(A, \mathbb{Z})^{\vee} \otimes H^{1}(A, \mathbb{Z})
$$

So if we ask that $R \hookrightarrow \operatorname{End}(A)$, this means that certain classes in End $\left(H^{1}(A, \mathbb{Z})\right)$ should be Hodge classes.

In general, the geometric "meaning" of the Hodge classes is less clear; according to the Hodge conjecture they should be related to algebraic cycles, but even so the interpretation is not so direct as in the case of endomorphisms.

A characterization: special subvarieties are "linear", in some sense.

## Theorem (BM)

Let $Z \subset \mathcal{A}_{g, 1}$ be an irreducible algebraic subvariety. Then
$Z$ is special $\Longleftrightarrow Z$ is totally geodesic and contains a CM point
(There is another characterization of special subvarieties through a "linearity property" in mixed characteristic.)


Yves André


Frans Oort

Basic fact: (first proven by Mumford, 1969) If $S \subset \mathcal{A}_{g}$ is a special subvariety, the CM points in $S$ lie dense, even for the analytic topology.

## André-Oort Conjecture $=$ Theorem of K-Y-U-P-T

Let $Z \subset \mathcal{A}_{g, 1}$ be an irreducible subvariety. Then
$Z$ is special $\Longleftrightarrow$ the CM points in $Z$ are Zariski dense

First proven conditionally (under GRH) by Klingler and Yafaev, using work of Ullmo and Yafaev. First proven unconditionally (for $\mathcal{A}_{g, 1}$ ) by Tsimerman, using his work with Pila (+ input from work of Andreatta-Goren-Howard-Madapusi Pera)

abelian variety
torsion points

$$
C \hookrightarrow J
$$

Manin-Mumford (Raynaud's thm):
$C \cap\{$ torsion points $\}$ is finite special subvariety

torsion points dense
moduli space $\mathcal{A}_{g, 1}$
CM points
$\mathcal{M}_{g} \hookrightarrow \mathcal{A}_{g, 1}$ (Torelli map)
Coleman's Conjecture:
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special subvariety
I
CM points dense

Reformulation of Coleman's conjecture:

## Coleman-Oort Conjecture

For $g \geq$ ?? there are no special subvarieties $S \subset \mathcal{A}_{g, 1}$ such that

- $\operatorname{dim}(S)>0$,
- $S \subset \mathcal{T}_{g}$,
- $S \not \subset \mathcal{T}_{g}^{\text {dec }}$

Note: if for some $g$ there are infinitely many Jacobians of CM type, the Zariski closure of these CM points contains an irreducible component $S$ with $\operatorname{dim}(S)>0$.

By André-Oort, $S$ is a special subvariety.
By construction, $S \subset \mathcal{T}_{g}$ and $S$ is not fully contained in the boundary of $\mathcal{T}_{g}$.

Remark: It is easy to construct special subvarieties $S \subset \mathcal{T}_{g}$ of positive dimension that are fully contained in the boundary $\mathcal{T}_{g}^{\text {dec }}$ of $\mathcal{T}_{g}$.

## Coleman-Oort Conjecture

For $g \geq$ ?? there are no special subvarieties $S \subset \mathcal{A}_{g, 1}$ such that

- $\operatorname{dim}(S)>0$,
- $S \subset \mathcal{T}_{g}$,
- $S \not \subset \mathcal{T}_{g}^{\text {dec }}$


## WHAT IS KNOWN?

EXAMPLES

## What is known?

For all genera $g \leq 7$, there do exist special subvarieties $S \subset \mathcal{T}_{g}$ of positive dimension that are not contained in the boundary of $\mathcal{T}_{g}$.

So: Coleman's original conjecture is false for $g \leq 7$.
Note: For $g<4$ this is trivial because then $\mathcal{T}_{g}^{\circ} \subset \mathcal{A}_{g, 1}$ is dense. (This is why Coleman required $g \geq 4$.)

A basic example (de Jong-Noot): Consider the family of curves $C_{\lambda}$ of genus 4 given by

$$
y^{5}=x(x-1)(x-\lambda) \quad(\lambda \text { a parameter })
$$

This family gives a 1-dimensional subvariety $Z \subset \mathcal{M}_{4}$.
Clear: the curves $C_{\lambda}$ have an automorphism of order 5 , given by

$$
(x, y) \mapsto\left(x, \zeta_{5} \cdot y\right)
$$

So the Jacobians $J_{\lambda}$ have $\mathbb{Q}\left[\zeta_{5}\right] \hookrightarrow \operatorname{End}^{0}\left(J_{\lambda}\right)$. This is already "half of CM".

General question: Suppose we have

- an abelian variety $A$, say with $\operatorname{dim}(A)=g$
- a CM field $K \subset \operatorname{End}^{0}(A)$

Let $S \subset \mathcal{A}_{g}$ be the PEL type special subvariety passing through [ $A$ ] that is defined by the condition that $K$ acts by endomorphisms.

What is the dimension of $S$ ?

This dimension does not only depend on $K$ but also on the "CM type".

The Lie algebra

$$
T=T_{0}(A) \cong H^{0}\left(A, \Omega_{A}^{1}\right)^{\vee}
$$

is a module under

$$
K \otimes_{\mathbb{Q}} \mathbb{C} \cong \prod_{\sigma: K \rightarrow \mathbb{C}} \mathbb{C}
$$

So we have a natural decomposition

$$
T=\bigoplus_{\sigma: K \rightarrow \mathbb{C}} T_{\sigma}
$$

Define

$$
n_{\sigma}:=\operatorname{dim}_{\mathbb{C}}\left(T_{\sigma}\right)
$$

By using the polarization:

$$
n_{\sigma}+n_{\bar{\sigma}}=g
$$

(Think of the complex embeddings of $K$ as a collection of pairs $(\sigma, \bar{\sigma})$.)
Basic fact: (theory of Shimura varieties)

$$
\operatorname{dim}(S)=\sum_{\text {all pairs }(\sigma, \bar{\sigma})} n_{\sigma} \cdot n_{\bar{\sigma}}
$$

Back to the example:

$$
C_{\lambda}: \quad y^{5}=x(x-1)(x-\lambda)
$$

Basis for $H^{0}\left(C_{\lambda}, \Omega^{1}\right)$ is given by

$$
\frac{d x}{y^{2}}, \quad \frac{d x}{y^{3}}, \quad \frac{x d x}{y^{4}}, \quad \frac{x d x}{y^{4}} .
$$

We see: the pairs $\left(n_{\sigma}, n_{\bar{\sigma}}\right)$ that occur are $(0,2)$ plus $(1,1)$. Conclusion: in this case

$$
\operatorname{dim}(S)=0 \cdot 2+1 \cdot 1=1
$$

To summarize:

- the curves $C_{\lambda}$ give rise to a 1-dimensional $Z \subset \mathcal{M}_{4}$
- the image of $Z$ is contained in a PEL type special subvariety $S \subset \mathcal{A}_{4}$
- by the above calculation $\operatorname{dim}(S)=1$

Conclusion: $Z$ is dense in $S$, and hence there are infinitely many values of $\lambda$ such that $J_{\lambda}$ has $C M$.

This is a nice game! Can we do more such examples?
Let's try:

$$
C_{\lambda}: \quad y^{7}=x(x-1)(x-\lambda)
$$

1-parameter family of genus 6 curves.
Basis for $H^{0}\left(C_{\lambda}, \Omega^{1}\right)$ :

$$
\frac{d x}{y^{3}}, \quad \frac{d x}{y^{4}}, \quad \frac{d x}{y^{5}}, \quad \frac{d x}{y^{6}}, \quad \frac{x d x}{y^{5}}, \quad \frac{x d x}{y^{6}} .
$$

We see: the pairs $\left(n_{\sigma}, n_{\bar{\sigma}}\right)$ that occur are $(1,1)$ plus twice $(0,2)$. Conclusion:

$$
\operatorname{dim}(S)=0 \cdot 2+0 \cdot 2+1 \cdot 1=1
$$

and again we get an example with infinitely many CM Jacobians.

If we try

$$
C_{\lambda}: \quad y^{11}=x(x-1)(x-\lambda)
$$

we get a 1-parameter family of genus 10 curves. This time a basis for $H^{0}\left(C_{\lambda}, \Omega^{1}\right)$ is

$$
\frac{d x}{y^{4}}, \quad \ldots \quad, \frac{d x}{y^{10}}, \quad \frac{x d x}{y^{8}}, \quad \frac{x d x}{y^{9}}, \quad \frac{x d x}{y^{10}} .
$$

We find

$$
\operatorname{dim}(S)=3 \times(0 \cdot 2)+2 \times(1 \cdot 1)=2
$$

So now $S$ has bigger dimension than $Z$ and we cannot conclude that $Z$ is a special subvariety.

We can do this more systematically: fix

- an integer $m \geq 2$
- an integer $N \geq 4$
- nonzero elements $a_{1}, \ldots, a_{N} \in \mathbb{Z} / m \mathbb{Z}$ that generate the whole group and satisfy $\sum a_{i}=0$
Then consider the family of curves given by

$$
y^{m}=\left(x-\lambda_{1}\right)^{a_{1}}\left(x-\lambda_{2}\right)^{a_{2}} \cdots\left(x-\lambda_{N}\right)^{a_{N}}
$$

This gives an $(N-3)$-dimensional subvariety of $\mathcal{M}_{g}$ with

$$
g=1+\frac{(N-2) m-\sum_{i=1}^{N} \operatorname{gcd}\left(a_{i}, m\right)}{2}
$$

The Jacobians $J_{\lambda}$ have an action of $\mathbb{Q}[\mathbb{Z} / m \mathbb{Z}]$ and we can consider the PEL type special subvariety $S \subset \mathcal{A}_{g}$ parametrizing abelian varieties with an action of this ring. Then calculate the dimension of $S$.

If $\operatorname{dim}(S)=N-3$ we get an example where we have infinitely many CM Jacobians.

This can be done on a computer. This was done independently by Rohde and myself. Here are the examples that are found:

| genus | $m$ | $N$ | $\left(a_{1}, \ldots, a_{N}\right)$ |
| :---: | :---: | :---: | :---: |
| 1 | 2 | 4 | $(1,1,1,1)$ |
| 2 | 2 | 6 | $(1,1,1,1,1,1)$ |
| 2 | 3 | 4 | $(1,1,2,2)$ |
| 2 | 4 | 4 | $(1,2,2,3)$ |
| 2 | 6 | 4 | $(2,3,3,4)$ |
| 3 | 3 | 5 | $(1,1,1,1,2)$ |
| 3 | 4 | 4 | $(1,1,1,1)$ |
| 3 | 4 | 5 | $(1,1,2,2,2)$ |
| 3 | 6 | 4 | $(1,3,4,4)$ |
| 4 | 3 | 6 | $(1,1,1,1,1,1)$ |


| genus | $m$ | $N$ | $\left(a_{1}, \ldots, a_{N}\right)$ |
| :---: | :---: | :---: | :---: |
| 4 | 5 | 4 | $(1,3,3,3)$ |
| 4 | 6 | 4 | $(1,1,1,3)$ |
| 4 | 6 | 4 | $(1,1,2,2)$ |
| 4 | 6 | 5 | $(2,2,2,3,3)$ |
| 5 | 8 | 4 | $(2,4,5,5)$ |
| 6 | 5 | 5 | $(2,2,2,2,2)$ |
| 6 | 7 | 4 | $(2,4,4,4)$ |
| 6 | 10 | 4 | $(3,5,6,6)$ |
| 7 | 9 | 4 | $(3,5,5,5)$ |
| 7 | 12 | 4 | $(4,6,7,7)$ |

Note: Suppose that in this game, we find that $N-3=\operatorname{dim}(Z)<\operatorname{dim}(S)$. Then we cannot conclude that $\bar{Z}$ is a special subvariety of $\mathcal{A}_{g}$.

But a priori we also cannot conclude that $\bar{Z}$ is not a special subvariety...

## Theorem (BM)

Among all families of cyclic covers of $\mathbb{P}^{1}$ with varying branch points, the above list of examples is complete. I.e., in all other cases the closure of " $Z$ " is not a special subvariety.

The proof relies on methods in mixed characteristic due to Dwork and Ogus.

## 5

## COLEMAN'S CONJECTURE:

VARIOUS APPROACHES

From now on we say that an algebraic subvariety $S \subset \mathcal{A}_{g}$ satisfies condition (ST)—for "special subvariety in the Torelli locus"-if:

- $S$ is a special subvariety of $\mathcal{A}_{g}$
- $\operatorname{dim}(S)>0$
- $S \subset \mathcal{T}_{g}$
- $S \cap \mathcal{T}_{g}^{\circ}$ is nonempty

So with this terminology, the goal is to show that for $g$ large enough, there are no $S \subset \mathcal{A}_{g}$ that satisfy (ST).
"The Italian team:"
Colombo, Frediani, Ghigi, Penegini, Pirola, Porru, Torelli, ....
(lots of papers, with different groups of authors):

- Further examples of subvarieties satisfying (ST), e.g. via the study of families of ramified covers with non-abelian group, or with a curve of genus $>0$ as basis. (All examples have $g \leq 7$.)
- Extensive study of the differential-geometric properties of the Torelli map, through the second fundamental form.

This has led to dimension estimates. The best estimate I'm aware of (improving earlier results of Ghigi-Pirola-Torelli, improving earlier results of ...) is due to Frediani and Pirola:

Theorem (Frediani-Pirola, improving earlier results)
Suppose $S \subset \mathcal{A}_{g}$ satisfies $(\mathrm{ST})$. Then $\operatorname{dim}(S) \leq 2 g-1$ if $g$ is even, $\operatorname{dim}(S) \leq 2 g$ is $g$ is odd.

Another important contribution is due to Hain, with later improvements by de Jong and Zhang. Hain's approach is based on the study of mapping class groups.

## Theorem (Hain, de Jong-Zhang)

Suppose $S \subset \mathcal{A}_{g}$ satisfies $(\mathrm{ST})$ and the $\mathbb{Q}$-group $G$ that defines $S$ is simple. Then one of the following is true:

- $S$ is a ball quotient
- $S \cap \mathcal{T}_{g}^{\text {dec }}$ has codimension $\leq 2$ in $S$
- the Baily-Borel compactification of $S$ has a nonempty boundary of codimension $\leq 2$.

Note: every special subvariety $S$ is locally symmetric: $S=\Gamma \backslash X^{+}$, with $X^{+}$a hermitian symmetric domain and $\Gamma \subset G(\mathbb{Q})$ an arithmetic group. We say that $S$ is a ball quotient if $X^{+}$is the complex $n$-ball for some $n$.

Example: the only 1-dimensional hermitian symmetric domain (up to equivalence) is the usual upper half plane. This can also be realized as the complex 1-ball (Möbius transformation). Therefore: every 1-dimensional special subvariety is a ball quotient.

Remarks:

- all known examples of special subvarieties satisfying (ST) are ball quotients
- even in the 1-dimensional case, such varieties are in general not of PEL type


## Definition

Say that a special subvariety $S \subset \mathcal{A}_{g}$ with $\operatorname{dim}(S)>0$ is minimal if there is no special $S^{\prime} \subsetneq S$ with $\operatorname{dim}\left(S^{\prime}\right)>0$.

Note: minimal special subvarieties can have arbitrarily large dimension (for $g$ large enough).
(Almost) obvious: for the Coleman-Oort conjecture, it suffices to show that there are no minimal special subvarieties satifying (ST). This simplifies the classification. If $S$ is minimal, it comes from a $\mathbb{Q}$-simple algebraic group $G$; in that case:

$$
G^{\mathrm{ad}}=\operatorname{Res}_{F / \mathbb{Q}} H
$$

with $F$ a totally real field, $H$ an absolutely simple adjoint group over $F$.

Based on Hain's results, the only cases we need to consider are:

- special subvarieties of dimension $\leq 2$
- ball quotients
- the case where $H$ is of Lie type $A_{1}$ or $B_{2}$

The assumption that $S$ is minimal then still gives some finer information about the algebraic group that defines $S$.

Work of
Zuo, Viehweg, Möller, Chen, Lu, ...

At the origin of their work lies the study of families of abelian varieties $f: A \rightarrow C$ over (open) curves, through the corresponding variation of Hodge structure. Basic idea:

- $\bar{C}=$ the complete nonsingular model of $C$
- $S=\bar{C} \backslash C$
- Higgs bundle $\mathscr{E}=\left(\mathscr{E}^{1,0} \oplus \mathscr{E}^{0,1}, \theta\right)$ with $\mathscr{E}^{1,0}=f_{*} \Omega_{A / C}^{1}$
- $\mathscr{E}$ extends to $\overline{\mathscr{E}}$ over $\bar{C}$
- $\overline{\mathscr{E}}=\mathscr{F} \oplus \mathscr{G}$ with $\mathscr{F}$ flat and $\mathscr{G}^{1,0}=\mathscr{G} \cap \overline{\mathscr{E}}^{1,0}$ ample

With this notation, there is the Arakelov inequality:

$$
2 \cdot \operatorname{deg}\left(\mathscr{G}^{1,0}\right) \leq g(\bar{C}) \cdot \operatorname{deg}\left(\Omega_{\bar{C}}^{1}(\log S)\right)
$$

## Theorem (Viehweg-Zuo)

Let $f: A \rightarrow C$ be a family of $g$-dimensional abelian varieties. Assume the largest unitary local subsystem of $R^{1} f_{*} \mathbb{C}$ is defined over $\mathbb{Q}$ and that the Arakelov inequality is an equality. Then the closure of the image of $C$ in $\mathcal{A}_{g}$ is a special subvariety.

This is only the start of a whole series of results. Zuo and collaborators have used such techniques to prove results about the non-existence of particular types of special subvarieties satisfying (ST). Here is just one sample:

Theorem (Lu-Zuo)
For $g>7$ there are no 1-dimensional special subvarieties $S \subset \mathcal{A}_{g}$ satisfying (ST) that are contained in the hyperelliptic locus.

By combining the results of Lu-Zuo and the work of Hain-de Jong-Zhang, and then analysing the possible Shimura data, I can prove something stronger:

## Theorem

For $g>7$ there are no special subvarieties $S \subset \mathcal{A}_{g}$ satisfying (ST) that are contained in the hyperelliptic locus.

By now, it's maybe not unreasonable to call it a conjecture...!

THANK YOU FOR YOUR ATTENTION!

