A conjecture of Serre

Shimura varieties

Main Result

Strategy

Galois representations

The Proof

Primes of ordinary reduction for abelian varieties of type IV and simple signature

Elena Mantovan

California Institute of Technology

April 18, 2023
VaNTAGe Seminar
Shimura varieties in positive characteristic and related topics

Joint w/ V. Cantoral-Farfan, W. Li, R. Pries and Y. Tang.

Application of the geometry of *Shimura varieties in positive characteristic* to

A question of Serre about

the **Galois representations** of abelian varieties defined over number fields.

Today: restrict to abelian varieties defined over $\mathbb{Q}$.

1. A conjecture of Serre
2. Shimura varieties
3. Main Result
4. Strategy
5. Galois representations
6. The Proof

Table of Contents

1 A conjecture of Serre

2 Shimura varieties

3 Main Result

4 Strategy

5 Galois representations

6 The Proof

Galois representations of abelian varieties

Let A be an abelian variety over $\mathbb{Q}$, of dimension g:
for any rational prime ℓ, there is a Galois representation

$$\rho_{A,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_{2g}(\mathbb{Q}_\ell) = \mathrm{End}^0(H^1_{\acute{e}t}(A_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))$$

Let p be a prime of good reduction for A: $\rho_{A,\ell}(Frob_p) \in GL_{2g}(\mathbb{Q}_\ell)$
Consider α an eigenvalue of $\rho_{A,\ell}(Frob_p)$:

- (Weil Conjectures) $|\alpha|_\infty = p^{\frac{1}{2}}$;
- what can we say about $|\alpha|_p$? or equiv. about $val_p(\alpha)$?

Examples

- if A is an elliptic curve ($g = 1$): $\{val_p(\alpha_1), val_p(\alpha_2)\}$ is either $\mathrm{ord} = \{0, 1\}$ or $\mathrm{ss} = \{\frac{1}{2}, \frac{1}{2}\}$
- if A is an abelian surface ($g = 2$): $\{val_p(\alpha_i) \mid 1 \leq i \leq 4\}$ is $\mathrm{ord}^2 = \{0, 0, 1, 1\}$ or $\mathrm{ss}^2 = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\}$ or $\mathrm{ord} \oplus \mathrm{ss} = \{0, \frac{1}{2}, \frac{1}{2}, 1\}$

The **Newton polygon** at p of A is the convex polygon of slopes

$$\nu_p(A) = \{\mathrm{val}_p(\alpha) \mid \alpha \text{ eigenvalues of } Frob_p \text{ on } H^1_{dR}(A/\mathbb{Q}_p)\}$$

A Newton polgyon satisfies

(P1) $\nu_p(A)$ starts as $(0,0)$ and ends at $(2g, g)$;

(P2) $\nu_p(A)$ is symmetric: $\lambda \in \nu_p(X)$ iff $1-\lambda \in \nu_p(X)$;

(P3) all slopes $\lambda \in \mathbb{Q} \cap [0,1]$.

E.g. the **ordinary** polygon $\mathrm{ord}^g = \{0, \dots, 0, 1, \dots, 1\}$.

Theorem [Grothendieck–Manin Conjecutre] (Oort, 2000)

Given a Newton polygon ν and a prime p: there exists $\mathcal{A}/\overline{\mathbb{F}}_p$ such that $\nu_p(\mathcal{A}) = \nu$.

Question: given $A/\mathbb{Q}$ and ν, what can we say about $S_\nu(A/\mathbb{Q}) = \{p \mid \nu_p(A) = \nu\}$?

A conjecture of Serre

Conjecture (Serre)

After passing to a finite extension $L/\mathbb{Q}$: the set $S_\nu(A/L) = \{p \in |L| \mid \nu_p(X) = \nu\}$ has **natural density** 1 if $\nu = \mathrm{ord}^g$ and 0 otherwise.

Hence, $S_{\mathrm{ord}}(A/\mathbb{Q}) = \{p \in |\mathbb{Q}| \mid \nu_p(X) = \mathrm{ord}^g\}$ has **positive density** $\geq \frac{1}{[L:\mathbb{Q}]}$.

Examples

Denote $\delta_L = \delta(S_{\mathrm{ord}}(A/L))$

- if A is an elliptic curve over $\mathbb{Q}$:
  - (Serre, 1977): if A is not CM: $\delta_{\mathbb{Q}} = 1$
  - (Shimura–Tanayama, 1967): if A is CM: $\delta_L = 1$ if $L = \mathrm{End}^0_{\mathbb{C}}(A)$ and $\delta_{\mathbb{Q}} = \frac{1}{2}$
- if A is an abelian surface over $\mathbb{Q}$:
  - (Katz, 1982): $\delta_L = 1$ for some $L/\mathbb{Q}$.
  - (Sawin, 2016): $\delta_{\mathbb{Q}} \in \{1, \frac{1}{2}, \frac{1}{4}\}$ depending on $\mathrm{End}^0_{\mathbb{C}}(A)$.

Table of Contents

1 A conjecture of Serre

2 Shimura varieties

3 Main Result

4 Strategy

5 Galois representations

6 The Proof

Shimura varieties

Consider a **Shimura datum** (G, X) where $G/\mathbb{Q}$ connected reductive group.

(**Shimura**) X is hermitian locally symmetric domain $\circlearrowleft G(\mathbb{R})$

- If $\Gamma \subset G(\mathbb{Q})$ is a *suff. small* discrete subgr. then $\Gamma\backslash X$ is $\mathbb{C}$-manifold;
- $\Gamma\backslash X = \mathrm{Sh}_\Gamma(G, X)(\mathbb{C})^{\mathrm{an}}$ where $\mathrm{Sh}_\Gamma(G, X)$ algebraic variety;
- $\mathrm{Sh}_\Gamma(G, X)$ is defined over a number field $E_\Gamma(G, X)$;
- $E_\Gamma(G, X)$ is determined from the space automorphic forms on G of level Γ.

(**Deligne**) X is a $G(\mathbb{R})$-conjugacy class $\{h : \mathbb{S}^1 \to G(\mathbb{R})\}$ (Hodge structures):

- $\Gamma\backslash X$ moduli space of Hodge structures
- if $G \subseteq \mathrm{GSp}_{2g}$, and $h \in X$ have weights $(-1, 0)$ and $(0, -1)$ (*Hodge type*) $\mathrm{Sh}_\Gamma(G, X)$ is moduli space of abelian varieties with additional structures;
- $E(G, X)$ is determined from (G, X) (the *reflex field*).

Shimura varieties of PEL type

Let $\mathrm{GSp}_{2g} = \mathrm{GSp}(V, \langle\,,\rangle)$.

Examples

Let B semisimple central algebra over F, where F is either CM or totally real field.

- Define a B-module structure on $(V, \langle\,,\rangle)$,

$$G^B = \mathrm{GL}_B \cap \mathrm{GSp}_{2g}$$

- $\mathrm{Sh}_\Gamma(G^B, X)$ is moduli space of polarized ab. var. with multiplication by B,

$$B \hookrightarrow \mathrm{End}^0_{\mathbb{C}}(A)$$

- X prescribes the isom. class of $\mathrm{Lie}_{\mathbb{C}}(A)$ as $B \otimes_{\mathbb{Q}} \mathbb{C}$-module, e.g. if $B = F$

$$\mathfrak{f} : \mathrm{Hom}(F, \mathbb{C}) \to \mathbb{Z}_{\geq 0},\ \tau \mapsto \mathfrak{f}(\tau) = \dim_{\mathbb{C}} \mathrm{Lie}_{\mathbb{C}}(A)(\tau) \quad \text{(the } \textit{signature} \text{ of } A\text{)}$$

Suppose $B \subseteq \mathrm{End}^0_{\mathbb{C}}(A)$ with signature $\mathfrak{f}$: consider $\mathrm{Sh} = \mathrm{Sh}(G^B, X_{\mathfrak{f}})$

- if $\forall p : \mathrm{Sh}_{\overline{\mathbb{F}}_p}[\nu] = \{x \mid \nu_p(x) = \nu\} = \varnothing$ then $S_\nu(A) = \{p \mid \nu_p(A) = \nu\} = \varnothing$

Theorem (Kottwitz, Rapoport–Richartz, Wedhorn, Viehmann-Wedhorn, ...)

Assume p is unramified for (G, X).

- $\mathrm{Sh}_{\overline{\mathbb{F}}_p}[\nu]$ is a locally closed subspace (*Newton stratum*)
- $\mathrm{Sh}_{\overline{\mathbb{F}}_p}[\nu] \neq \varnothing$ iff $\nu \in B_p(G, X)$ (the *Kottwitz* set)
- $\mathrm{Sh}_{\overline{\mathbb{F}}_p}[\mathrm{ord}^g] = \varnothing$ iff p is not *totally split* in the reflex field $E/\mathbb{Q}$
- there is a unique non-empty open Newton stratum (μ-*ordinary* stratum)
- μ-ordinary Newton polygon is ord^g iff p is totally split in $E/\mathbb{Q}$

$$S_{\mathrm{ord}}(A/\mathbb{Q}) = \{p \mid \nu_p(A) = \mathrm{ord}\} \subseteq S(E/\mathbb{Q}) = \{p \mid p \text{ tot. split in } E/\mathbb{Q}\}$$

Note: $\delta(S(E/\mathbb{Q})) = \frac{1}{[E:\mathbb{Q}]} < 1$ if $E \neq \mathbb{Q}$, and $\delta(S(E/L)) = 1$ if $L = E$

The smallest Shimura variety containing A

Let $h_A : \mathbb{S}^1 \to \mathrm{GSp}_{2g}(\mathbb{R})$ be the Hodge structure of A.

The **Mumford–Tate group** of A is the smallest subgroup $M_A \subseteq \mathrm{GSp}_{2g}$ over $\mathbb{Q}$ such that $h_A : \mathbb{S}^1 \to M_A(\mathbb{R}) \subseteq \mathrm{GSp}_{2g}(\mathbb{R})$

- $\mathrm{Sh}(G, X)$ is moduli space of abelian varieties satisfying $M_A \subseteq G$
- $\mathrm{Sh}(M_A, [h_A])$ is the *smallest* Shimura variety containing A.
- If $B = \mathrm{End}^0_{\mathbb{C}}(A)$ then $h_A : \mathbb{S}^1 \to G^B(\mathbb{R})$ for $G^B = \mathrm{GL}_B \cap \mathrm{GSp}_{2g}$,
- $\mathrm{Sh}(G^B, [h_A])$ is the smallest Sh. var. of *PEL type* containing A.

Note: $\exists\ \mathcal{A}$ satisfying $M_{\mathcal{A}} \neq G^B$ for $B = \mathrm{End}^0_{\mathbb{C}}(\mathcal{A})$ ($\mathcal{A}$ has *exceptional cycles*)

Caltech

Table of Contents

1 A conjecture of Serre

2 Shimura varieties

3 Main Result

4 Strategy

5 Galois representations

6 The Proof

Abelian varieties of type IV and simple signature

WLOG: A absolutely simple

Assumptions: type IV and simple signature

(A1) $\mathrm{End}^0_{\mathbb{C}}(A) = F$ is a CM field

(A2) $\mathfrak{f}_A$ is *simple* : $(\mathfrak{f}(\tau), \mathfrak{f}(\tau^*)) = (0, n)$ for all but one pair where is $(1, n-1))$

where $*$ is complex conjugation on F and $n = \frac{2\dim A}{[F:\mathbb{Q}]}$ relative dimenion of A.

Then $M_A \subseteq G^F = \mathrm{GL}_F \cap \mathrm{GSp}_{2g}$

If $(A1-2)$: $M_A = G^F$ (no exceptional cycles)

Proof: all connected reductive algebraic subgroups of G^F are of PEL-type.

Main Theorems

Theorem A (CLMPT) [Serre's conjecture]

Let A be an abelian variety over a number field $\mathbb{Q}$.

Assume (A1) $\mathrm{End}^0_{\mathbb{C}}(A) = F$ is a CM field and (A2) $\mathfrak{f}_A$ is simple.

- After passing to a finite extension $L/\mathbb{Q}$, the set $S_{\mathrm{ord}}(A/L)$ has density 1

Theorem B (CLMPT)

Let A be an abelian variety over a number field $\mathbb{Q}$, satisfying
(A1) $\mathrm{End}^0_{\mathbb{C}}(A) = F$ is a CM field and (A2) $\mathfrak{f}_A$ is simple.

Assume **also** (A3) $F/\mathbb{Q}$ is an abelian Galois extension and (A4) $F = \mathrm{End}^0_F(A)$

- The set $S_{\mu\text{-ord}}(A/\mathbb{Q}) = \{p \in |\mathbb{Q}| \mid \nu_p(A) = \mu\text{-ord}_p(G^F, X_{\mathfrak{f}})\}$ has density 1

Remarks on Theorem B

Ordinary Primes

Elena Mantovan

A conjecture of Serre

Shimura varieties

Main Result

Strategy

Galois representations

The Proof

For any Newton polygon ν: consider $S_\nu(A/L) = \{p \in |L| \mid \nu_p(A) = \nu\}$:

- Thm A computes the density of $S_\nu(A/L)$ for $L/\mathbb{Q}$ sufficiently large.
- Thm B computes the density of $S_\nu(A/L)$ for **any** $L/\mathbb{Q}$.

Assume (A1–4). Denote δ_L the density of $S_\nu(A/L) = \{p \in |L| \mid \nu_p(A) = \nu\}$ is:

- if $\nu = \mathrm{ord}^g$: $\delta_L = \frac{1}{[FL:L]}$
- if $\nu \neq \mathrm{ord}^g$ and $\nu = \mu\text{-ord}_p$ for some p: $S_\nu(A/L)$ is **infinite** and $\delta_L = \frac{a_\nu}{[FL:L]} > 0$ if $L \not\supseteq F$ (explicit $a_\nu \in \mathbb{Z}$)
- if $\nu \neq \mu\text{-ord}_p$ for all p: $\delta_L = 0$.

Proof. Recall if $E = E(G^F, X_{\mathfrak{f}})$ is the reflex field of $\mathrm{Sh}(G^F, X_{\mathfrak{f}})$ then

$$S_{\mathrm{ord}}(A/\mathbb{Q}) \subseteq S(E/\mathbb{Q}) = \{p \mid p \text{ tot. split in } E/\mathbb{Q}\}$$

More precisely, $\quad S_{\mathrm{ord}}(A/\mathbb{Q}) = S_{\mu\text{-ord}}(A/\mathbb{Q}) \cap S(E/\mathbb{Q})$

By $(A1-2)$ $E \simeq F$, and by Cheboratev $S(F/\mathbb{Q})$ has the density $\frac{1}{[F:\mathbb{Q}]}$.

Previous results

Theorem B:

- (Shimura-Tanayama, 1967) if A is CM then all $p \in S_{\mu\text{-ord}}(A/\mathbb{Q})$
- (S-T; Serre, 1977) if $g = 1$
- (S-T; S; Sawin, 2016) if $g = 2$

Theorem A:

- (Katz, 1982) if $g = 2$
- (Pink, 1998) if $\mathrm{End}^0_{\mathbb{C}}(A) = \mathbb{Q}$ and M_A is small
- (Fité, 2018) if $g = 3$,
  if $g = 4$ and $\mathrm{End}^0_{\mathbb{C}}(A) = F$ quad. imag. and $\mathfrak{f}_A = (2, 2)$.

Remark on Assuptions (A1–4)

Ordinary Primes
Elena Mantovan
A conjecture of Serre
Shimura varieties
Main Result
Strategy
Galois representations
The Proof

- $\exists$ infinitely $(F, \mathfrak{f})$:
  (A1) F is a CM field (A2) $\mathfrak{f}$ is simple (A3) $F/\mathbb{Q}$ is an abelian Galois ext.
- (Shimura var. Thy) For any $(F, \mathfrak{f})$ satisfying (A1–3) :
  $\exists$ infinitely many $\mathcal{A}/\overline{\mathbb{Q}}$ with $\mathrm{End}^0_{\mathbb{C}}(\mathcal{A}) = F$ and $\mathfrak{f}_{\mathcal{A}} = \mathfrak{f}$

$\exists$ infinitely many $\mathcal{A}/\mathbb{Q}$ satisfying (A1–3) and (A4) $F = \mathrm{End}^0_F(\mathcal{A})$

An example

Ordinary Primes

Elena Mantovan

A conjecture of Serre

Shimura varieties

Main Result

Strategy

Galois representations

The Proof

Let C the smooth projective curve given by affine equation

$$y^5 = x(x-1)(x-t) \quad t \in \mathbb{Q} - \{0,1\}.$$

Note $\mu_5 \subseteq \mathrm{Aut}_{\mathbb{C}}(C)$ by $(x,y) \mapsto (x, \zeta_5 y)$.

Let J_C be the Jacobian of C. Then $\mathbb{Q}(\zeta_5) \subseteq \mathrm{End}^0_{\mathbb{C}}(J_C)$

- (A3) $\mathbb{Q}(\zeta_5)/\mathbb{Q}$ is a CM and abelian extension
- (A4) $\mathbb{Q}(\zeta_5) = \mathrm{End}^0_{\mathbb{Q}(\zeta_5)}(J_C)$

If J_C is **not CM** (true for general C) then

- (Moonen, 2010) (A1) $\mathrm{End}^0_{\mathbb{C}}(J_C) = \mathbb{Q}(\zeta_5)$.
- (Deligne–Mostow, 1987): (A2) the signature is simple: $\mathfrak{f}_C = (1,0,2,1)$.

Theorem B: a special case

$$\mu\text{-ord}_p(G^{\mathbb{Q}(\zeta_5)}, f_C) = \begin{cases} \mathrm{ord}^4 = \{0,\dots,0,1,\dots,1\} & \text{if } p \equiv 1 \bmod 5 \\ \mathrm{ord}^2 \oplus \mathrm{ss}^2 = \{0,0,\frac{1}{2},\frac{1}{2},1,1\} & \text{if } p \equiv 4 \bmod 5 \\ (1/4,3/4) = \{\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4}\} & \text{if } p \equiv 2,3 \bmod 5 \end{cases}$$

Let C the curve $y^5 = x(x-1)(x-t)$, $t \in \mathbb{Q} - \{0,1\}$. Assume J_C is not CM.

$$\delta(\{p \in |\mathbb{Q}| \mid \nu_p(C) = \nu\}) = \begin{cases} \frac{1}{4} \text{ if } \nu = \mathrm{ord}^4 \\ \frac{1}{4} \text{ if } \nu = \mathrm{ord}^2 \oplus \mathrm{ss} \\ \frac{1}{2} \text{ if } \nu = (1/4,3/4) \\ 0 \text{ otherwise} \end{cases}$$

$$\delta(\{\mathfrak{p} \in |\mathbb{Q}(\zeta_5)| \mid \nu_{\mathfrak{p}}(C) = \nu\}) = \begin{cases} 1 \text{ if } \nu = \mathrm{ord}^4 \\ 0 \text{ otherwise} \end{cases}$$

Table of Contents

1 A conjecture of Serre

2 Shimura varieties

3 Main Result

4 Strategy

5 Galois representations

6 The Proof

Strategy for $g = 1$ following Serre

Ordinary Primes
Elena Mantovan
A conjecture of Serre
Shimura varieties
Main Result
Strategy
Galois representations
The Proof

Assume $A/\mathbb{Q}$ is not CM. **Write** $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Assume $g = 1$: $\rho_{A,\ell} : \Gamma \to \mathrm{GL}_2(\mathbb{Q}_\ell)$

- $\nu_p(A) \neq \mathrm{ord}$ iff $\nu_p(A) = \mathrm{ss}$ iff $\mathrm{tr}(Frob_p \mid H^1_{\mathrm{dR}}(A/\mathbb{Q}_p)) = 0$
- $\nu_p(A) \neq \mathrm{ord}$ iff $\mathrm{tr}(\rho_{A,\ell}(Frob_p)) = 0$

Proposition (Serre)

Let $G/\mathbb{Q}_\ell$ conn'd alg. group, and $Z \subset G(\mathbb{Q}_\ell)$ closed, stable under conjugation.
Assume $\rho : \Gamma \to G(\mathbb{Q}_\ell)$ has **dense** image,
If $Z \subset G(\mathbb{Q}_\ell)$ of Haar measure 0 then $\{p \mid \rho(Frob_p) \in Z\}$ has density 0

- $Z = \{g \in \mathrm{GL}_2(\mathbb{Q}_\ell) \mid tr(g) = 0\}$ is closed, conjug. stable, of Haar measure 0.
- if A is not CM then $\overline{\rho_{A,\ell}(\Gamma)}^{\mathrm{Zar}} = \mathrm{GL}_2$

If A is CM: $\overline{\rho_{A,\ell}(\Gamma)}^{\mathrm{Zar}} \neq \mathrm{GL}_2$ and is not connected

Strategy for $g = 2$ following Deligne and Katz

Ordinary Primes
Elena Mantovan
A conjecture of Serre
Shimura varieties
Main Result
Strategy
Galois representations
The Proof

Assume $A/\mathbb{Q}$ is not CM. Write $\Gamma = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

Assume $g = 2$: $\rho_{A,\ell} : \Gamma \to \mathrm{GSp}_4(\mathbb{Q}_\ell)$

- $\nu_p(A) = \mathrm{ss}^2$ iff $\mathrm{tr}(Frob_p \mid H^1_{\mathrm{dR}}(A/\mathbb{Q}_p)) = 0$
- $\nu_p(A) \neq \mathrm{ord}^2$ iff either $\nu_p(A) = \mathrm{ss}^2$ or $\nu_p(A) = \mathrm{ord} \oplus \mathrm{ss}$

(Deligne): $\nu_p(A) \neq \mathrm{ord}^2$ iff $p \mid \mathrm{tr}(Frob_p \mid H^2_{\mathrm{dR}}(A/\mathbb{Q}_p))$

- iff $p \mid \mathrm{tr}\left((\wedge^2 \rho_{A,\ell})(Frob_p)\right)$ for $\wedge^2 : \mathrm{GSp}_4 \subset \mathrm{GL}_4(\mathbb{Q}_\ell) \to \mathrm{GL}_6(\mathbb{Q}_\ell)$,
- iff $\mathrm{tr}\left((\chi \otimes \wedge^2) \circ \rho_{A,\ell}(Frob_p)\right) \in \mathbb{Z}$ for $\chi : \mathrm{GSp}_4(\mathbb{Q}_\ell) \to \mathbb{Q}_\ell$ similitude char.

(Weil Conjectures): $\left|\mathrm{tr}\left((\chi \otimes \wedge^2) \circ \rho_{A,\ell}(Frob_p)\right)\right| \leq 6$

- iff $\mathrm{tr}\left((\chi \otimes \wedge^2) \circ \rho_{A,\ell}(Frob_p)\right) = c$ for $c \in \{-6, -5, \dots, 0, \dots, 5, 6\}$

(Katz; Fité-Kedlaya-Rotger-Surtherland): Describe $\overline{\rho_{A,\ell}(\Gamma)}^{\mathrm{Zar}} \subseteq \mathrm{GSp}_4(\mathbb{Q}_\ell)$

Table of Contents

1 A conjecture of Serre

2 Shimura varieties

3 Main Result

4 Strategy

5 Galois representations

6 The Proof

The ℓ-**adic monodromy group** of $A/\mathbb{Q}$ is $G_{A,\ell} = \overline{\mathrm{Im}(\rho_{A,\ell})}^{\mathrm{Zar}} \subseteq \mathrm{GL}_{2g}(\mathbb{Q}_\ell)$

$G_{A,\ell}$ might be **not** connected. Denote $\pi_0(G_{A,\ell})$ group of connected components.

Theorem (Silverberg, 1992; Larsen–Pink, 1997)

- $\pi_0(G_{A,\ell})$ is independent on ℓ
- $\exists\ \mathbb{Q}^{\mathrm{conn}}/\mathbb{Q}$ finite Galois field s.t. $\rho_{A,\ell} : \mathrm{Gal}(\mathbb{Q}^{\mathrm{conn}}/\mathbb{Q}) \simeq \pi_0(G_{A,\ell})$ for all ℓ
- if $A/\mathbb{Q}$ has no exceptional cycles: $\mathbb{Q}^{\mathrm{conn}}$ is the field of definition of $\mathrm{End}^0_{\mathbb{C}}(A)$

- After passing to $L/\mathbb{Q}$, may assume $G_{A,\ell}$ is connected ($L = \mathbb{Q}^{\mathrm{conn}}$)
- If (A1-2): (A4) $F = \mathrm{End}^0_F(A)$ is equivalent to $\mathbb{Q}^{\mathrm{conn}} \subseteq F$;
- If (A1-4): $\exists$ epimorphism of abelian grp. $\mathrm{Gal}(F/\mathbb{Q}) \longrightarrow \pi_0(G_{A,\ell})$

Mumford–Tate conjecture

Let $G^0_{A,\ell}$ the identity component of the ℓ**-adic monodromy group** $G_{A,\ell}$ of A.

Mumford–Tate Conjecture

$$G^0_{A,\ell} = M_A(\mathbb{Q}_\ell).$$

- (Faltings, 1987) $G^0_{A,\ell} = \mathcal{G}_A(\mathbb{Q}_\ell)$ for a conn'd reductive group $\mathcal{G}_A$ over $\mathbb{Q}$
- (Vasiu, 2008): the Mumford-Tate conjecture holds if (A1-2) .

By Assumptions (A1-2): $G^0_{A,\ell} = G^F(\mathbb{Q}_\ell)$.

After passing to $\mathbb{Q}^{\mathrm{conn}}/\mathbb{Q}$, we may assume $G_{A,\ell} = \overline{\mathrm{Im}(\rho_{A,\ell})}^{\mathrm{Zar}} = G^F(\mathbb{Q}_\ell)$

Table of Contents

1 A conjecture of Serre

2 Shimura varieties

3 Main Result

4 Strategy

5 Galois representations

6 The Proof

Idea of Proof: Thm A

Step 1: Find suitable invariant $a_p = a(Frob_p)$:

- it detects ordinariness

$$\text{if } \nu_p(A) \neq \mathrm{ord}^g \text{ then } a_p = c \text{ for finitely many values}$$

  - $|a_p| \leq C$ bounded independently of p (use Weil Conjectures)
  - if $\nu_p(A) \neq \mathrm{ord}^g$ then $a_p \in \mathbb{Z}$ (use Newton stratification of Shimura variety)
- it is the trace of an algebraic representation of GSp_{2g}.

$$a(Frob_p) = \mathrm{tr}(\rho_{A,\ell}(Frob_p) \mid W_\ell) \text{ where } \mathrm{GSp}_{2g} \to \mathrm{GL}(W)$$

Step 2: Compute $\mathrm{tr}(- \mid W_\ell)$ on $G_{A,\ell} = G^F(\mathbb{Q}_\ell) \subset \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$

- $Z = \coprod_c \{g \in G^F(\mathbb{Q}_\ell) \mid \mathrm{tr}(g \mid W_\ell) = c\}$ is closed, stable under conjugation
- Z has Haar measure 0 $\qquad \Longrightarrow \delta(S_{\mathrm{ord}}(A/F\mathbb{Q}^{\mathrm{conn}})) = 1$

Detecting ordinariness

Recall if $\nu_p(A) = \mathrm{ord}^g$ then $p \in S(F/\mathbb{Q}) = \{p \mid p \text{ tot. split in } F/\mathbb{Q}\}$.

WLOG assume $p \in S(F/\mathbb{Q})$.

Let $V_p = H^1_{\mathrm{dR}}(A/\mathbb{Q}_p)$ a $F \otimes \mathbb{Q}_p$-vector space

- If $p \in S(F/\mathbb{Q})$: the action of $Frob_p$ on V_p is $F \otimes \mathbb{Q}_p$-**linear**.
- (Kisin) $T_p = \mathrm{tr}_{F\otimes\mathbb{Q}_p}(Frob_p \mid V_p) \in \mathcal{O}_F \subseteq F \otimes \mathbb{Q}_p$
- $a_p = \mathrm{Nm}_{F/\mathbb{Q}}(T_p) \in \mathbb{Z}$
- (Weil Conjectures) $|a_p| \leq Cp^d$ for $d = \frac{[F:\mathbb{Q}]}{2}$
- $b_p = p^{-d} a_p \in \mathbb{Q}$ satisfies $|b_p| \leq C$

For $p \in S(F/\mathbb{Q})$: if $\nu_p(A) \neq \mathrm{ord}^g$ then $b_p \in \mathbb{Z}$ (use Shimura variety)

- $b_p = \mathrm{tr}(\rho_{A,\ell}(Frob_p)|W_\ell)$ where $G^F \to GL(W)$ not $\mathrm{GSp}_{2g} \to GL(W)$

Idea of Proof: Thm B

Step 1: Find suitable invariant $a_p = a(Frob_p)$:

- it detects μ-ordinariness

$$\text{if } \nu_p(A) \neq \mu\text{-ord}_p \text{ then } a_p = c \text{ for finitely many values}$$

- it is the trace of an algebraic representation on a subgroup $G \subseteq \mathrm{GSp}_{2g}$.

$$a(Frob_p) = \mathrm{tr}(\rho_{A,\ell}(Frob_p) \mid W_\ell) \text{ where } G \to \mathrm{GL}(W)$$

Step 2: Compute $\mathrm{tr}(- \mid W_\ell)$ on $G_{A,\ell} \subseteq G(\mathbb{Q}_\ell)$ (and $G_{A,\ell}$ might be not conn'd).

For $Z = \coprod_c \{g \in G(\mathbb{Q}_\ell) \mid \mathrm{tr}(g \mid W_\ell) = c\}$ and **each** conn'd comp. $G^{(\sigma)}_{A,\ell}$ of $G_{A,\ell}$:

- $Z \cap G^{(\sigma)}_{A,\ell}$ closed, *stable under conjugation*, of Haar measure 0

By (A3-4): All conn'd comp. $G^{(\sigma)}_{A,\ell}$ are stable under conjugation.

Idea of Proof (via the geometry of Shimura variety)

A conjecture of Serre

Shimura varieties

Main Result

Strategy

Galois representations

The Proof

By Assumption (A1-2): Sh is a *simple* Shimura variety (á la Harris–Taylor)

- the Kottwitz's set $B_p(\mathrm{Sh})$ can be computed **explicitely**.
- the Newton stratification of $\mathrm{Sh}_{\mathbb{F}_p}$ is **totally** ordered, for all p good.
- the set $B_p(\mathrm{Sh})$ depends **only** on $Frob_p|_F \in \mathrm{Gal}(F/\mathbb{Q})$
- for each $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ $\exists!$ a polygon μ_σ: $\mu\text{-}\mathrm{ord}_p = \mu_\sigma$ if $Frob_p|_F = \sigma$

- $S_{\mu\text{-ord}}(A/\mathbb{Q}) = \coprod_\sigma S_{\mu_\sigma}(A/\mathbb{Q})$ where for each σ we have

$$S_{\mu_\sigma}(A/\mathbb{Q}) \subseteq S_\sigma(F/\mathbb{Q}) = \{p \mid Frob_p|_F = \sigma\}$$

- (Chebotarev) $S_\sigma(F/\mathbb{Q})$ has density $\frac{1}{[F:\mathbb{Q}]}$

For each σ: the sets $S_{\mu_\sigma}(A/\mathbb{Q})$ and $S_\sigma(F/\mathbb{Q})$ have the same density

The complement of $S_{\mu_\sigma}(A/\mathbb{Q}) \subseteq S_\sigma(F/\mathbb{Q})$ has density 0 (use Serre's Proposition)

Step 1: detecting μ-ordinariness

Fix $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ and assume $p \in S_\sigma(F/\mathbb{Q})$.
Goal: an invariant $a_\sigma(Frob_p)$ detecting $\nu_p(A) = \mu_\sigma$.

Let $V_p = H^1_{\mathrm{dR}}(A/\mathbb{Q}_p)$ a $F \otimes \mathbb{Q}_p$-vect. sp.

- If $p \in S_\sigma(F/\mathbb{Q})$: the action of $Frob_p$ on V_p is not $F \otimes \mathbb{Q}_p$-linear
- $Frob_p$ is $K \otimes \mathbb{Q}_p$-linear for $K = F^{\langle\sigma\rangle}$ by (A3-4)
- (Kisin) $T_p = \mathrm{tr}_{K\otimes\mathbb{Q}_p}(Frob_p \mid \wedge^e_{K\otimes\mathbb{Q}_p} V_p) \in \mathcal{O}_K \subseteq K \otimes \mathbb{Q}_p$ for $e = [F : K]$
- (Weil Conjectures) $a_p = \mathrm{Nm}_{K/\mathbb{Q}}(T_p) \in \mathbb{Z}$ satisfies $|a_p| \leq Cp^d$
- $b_p = p^{-d} a_p \in \mathbb{Q}$ satisfies $|b_p| \leq C$

For $p \in S_\sigma(F/\mathbb{Q})$: if $\nu_p(A) \neq \mu_\sigma$ then $b_p \in \mathbb{Z}$

- $b_p = \mathrm{tr}(\rho_{A,\ell}(Frob_p)|W_\ell)$ where $G^K \to GL(W)$ for $G^K = \mathrm{GL}_K \cap \mathrm{GSp}_{2g}$

Step 2: Evaluating traces on $G_{A,\ell}$

Ordinary Primes
Elena Mantovan
A conjecture of Serre
Shimura varieties
Main Result
Strategy
Galois representations
The Proof

For $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$: $\mathrm{tr}(-, W)$ on G^K (depending on $K = F^{\langle\sigma\rangle}$)

Need: Match to the conn'd component of $G_{A,\ell}$.

(A4): $\mathrm{Gal}(F/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathbb{Q}^{\mathrm{conn}}/\mathbb{Q}) \simeq \pi_0(G_{A,\ell}) = G_{A,\ell}/G^0_{A,\ell}$

For each $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, there is a conn'd component $G^{(\sigma)}_{A,\ell}$ of $G_{A,\ell}$

- if $p \in S_\sigma(F/\mathbb{Q})$ then $\rho_{A,\ell}(Frob_p) \in G^{(\sigma)}_{A,\ell}$
- $G^{(\sigma)}_{A,\ell} \subseteq G^K(\mathbb{Q}_\ell)$ for $K = F^{\langle\sigma\rangle}$

Goal: $Z \cap G^{(\sigma)}_{A,\ell} = \coprod_c \{g \in G^{(\sigma)}_{A,\ell} \mid \mathrm{tr}(g \mid W_\ell) = c\}$ has Haar measure 0

Enough: $\mathrm{tr}(- \mid W_\ell)$ is *not constant* on $G^{(\sigma)}_{A,\ell}$ (b/c if it is it takes integral value)

- for $\sigma = id$: $G^{(id)}_{A,\ell} = G^0_{A,\ell} = G^F(\mathbb{Q}_\ell)$
- for $\sigma \neq id$: $G^{(\sigma)}_{A,\ell} = B^{(\sigma)}_{A,\ell} \cdot G^F(\mathbb{Q}_\ell) \subset G^K(\mathbb{Q}_\ell)$ for some $B^{(\sigma)}_{A,\ell} \in G^K(\mathbb{Q}_\ell)$

Bounding $\pi_0(G_{A,\ell})$

Need: Compute cosets representative $B_{A,\ell}^{(\sigma)} \in \mathrm{GSp}_{2g}$ for $\pi_0(G_{A,\ell})$
Caveat: These depend on $A/\mathbb{Q}$
Strategy: Remove dependence on $A/\mathbb{Q}$

- Compute a list of **potential** cosets representatives that is independent of $A/\mathbb{Q}$

Idea: Bound $\pi_0(G_{A,\ell}) = G_{A,\ell}/G_{A,\ell}^0 \subseteq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)/G^F(\mathbb{Q}_\ell)$ independently on $A/\mathbb{Q}$

For each coset $[\alpha] \in \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)/G^F(\mathbb{Q}_\ell)$:
• choose $B_\alpha \in \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)$ • show $\mathrm{tr}(- \mid W_\ell)$ is not constant on $B_\alpha \cdot G^F(\mathbb{Q}_\ell)$
Careful: need to match $[\alpha]$ and $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ s.t. $B_\alpha \cdot G^F(\mathbb{Q}_\ell) \subset G^K(\mathbb{Q}_\ell)$

There is a monomorphism $\phi : \pi_0(G_{A,\ell}) \hookrightarrow H/H_1$
where $H \subseteq \mathit{Weyl}(\mathrm{GSp}_{2g}, T)$ and $H_1 = \mathit{Weyl}(\mathrm{G}^F, T)$ *independent* of $A/\mathbb{Q}$.

For each $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$: identify $I_\sigma \subset H/H_1$ such that

• $\phi(G_{A,\ell}^{(\sigma)}) \in I_\sigma$ and • $B_\alpha \in G^K(\mathbb{Q}_\ell)$ for all $[\alpha] \in I_\sigma$

A conjecture of Serre

Shimura varieties

Main Result

Strategy

Galois representations

The Proof

Thank you!