Galois groups and the dynamics of Cantor actions

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This talk combines problems/tools from a few areas of mathematics:

- **The object of interest:**

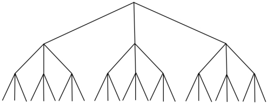
  Arboreal representations of absolute Galois groups of number fields
  - were initially introduced as a tool to compute density of prime divisors in non-linear recurrence relations.
  - are separable profinite groups acting on the boundary of a tree.
  - can be studied by the methods of topological dynamics and geometric group theory.

- **Results:**
  - Use dynamical invariants to classify profinite groups arising from arboreal representations.
  - Give descriptions of certains arboreal representations using the methods of topological dynamics and geometric group theory.

Dynamics on the boundary of a rooted tree

Build a regular tree: Choose a positive integer $d \geq 2$.

Vertex sets:
Let $V_0 = \{*\}$, and $|V_n| = d^n$.



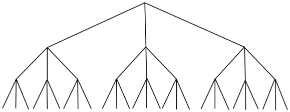
Edge sets:
Join each $v_n \in V_n$ to d vertices in V_{n+1}, so that each $v_{n+1} \in V_{n+1}$ is joined to a single vertex in V_n.

Path space:
The set of all infinite paths in T is the boundary

$$\partial T = \{(v_n)_{n\geq 1} : [v_n, v_{n+1}] \text{ is an edge } \} \subset \prod_{n\geq 1} V_n.$$

∂T is a Cantor set, i.e. a totally disconnected compact metrizable space without isolated points.

Actions by tree automorphisms:



Let G be a discrete countable group.

The group $\mathrm{Aut}(T)$ consists of maps $g : T \to T$ which preserve its tree structure:

(1) Vertices are mapped to vertices, and edges to edges.

(2) For each $n \geq 1$, and $g \in G$, $g|V_n$ is permutation, i.e. there is a homomorphism

$$\Phi_n : G \to \mathrm{Perm}(V_n) : (g, v_n) \mapsto gv_n.$$

(3) Permutations on consecutive levels are compatible, i.e. two vertices joined by an edge are mapped to two vertices joined by an edge.

(4) In addition, we require that $\Phi_n(G)$ is a transitive subgroup of $\mathrm{Perm}(V_n)$, for each $n \geq 1$.

The collection of maps $\Phi_n : G \to \mathrm{Perm}(V_n)$ defines an action of G on the boundary Cantor set ∂T by homeomorphisms

$$\Phi(g) = \varprojlim \Phi_n(g) : \partial T \to \partial T : (v_n)_{n\geq 1} \mapsto (gv_n)_{n\geq 1}.$$

Since G is countable, the orbits of the action are countable subsets of ∂T.

Condition (4) implies that every orbit

$$\mathrm{Orb}((v_n)_{n\geq 1}) = \{(gv_n)_{n\geq 1} : g \in G\}$$

is dense in ∂T, i.e. the action of G on ∂T is **minimal**.

The closure $\mathcal{G} = \overline{\Phi(G)} \subset \mathrm{Homeo}(\partial T)$ is a profinite group (the inverse limit of finite groups), called the **Ellis group** of the action.

$\mathcal{G}$ acts on ∂T transitively, i.e. there is a single orbit of the action of $\mathcal{G}$.

Example

Odometer

An **odometer**, or an **adding machine** is an element $\sigma \in \mathrm{Aut}(\mathrm{T})$ such that the restriction $\sigma|V_n$ a transitive permutation of V_n, for all $n \geq 1$.

Let σ be the adding machine, then $G = \mathbb{Z}$.

Choosing a path $v = (v_n)_{n\geq 1} \in \partial T$, we can use the action of σ to identify

$$V_n \cong \mathbb{Z}/d^n\mathbb{Z},$$

with the action of σ as addition of 1. Then

$$\partial T \cong \mathbb{Z}_d = \varprojlim\{\mathbb{Z}/d^n\mathbb{Z} \to \mathbb{Z}/d^{n-1}\mathbb{Z}\}$$

as topological spaces, and

$$\mathcal{G} \cong \mathbb{Z}_d$$

as groups.

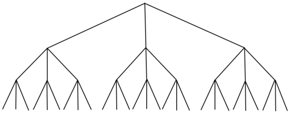
Arboreal representations of absolute Galois groups

Let $f(x)$ be a polynomial of degree d over a number field K.

Fix $\alpha \in K$.

For $n \geq 1$ and $f^n = f \circ f^{n-1}$ consider the equation

$$f^n(x) - \alpha = 0.$$



Let $V_n = f^{-n}(\alpha)$ be the set of its solutions.

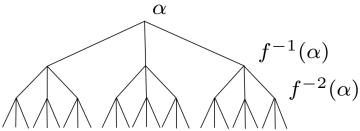
Assume that $f^n(x) - \alpha$ is irredicible for all $n \geq 1$.

Then $|V_n| = d^n$ since $K \supset \mathbb{Q}$.

We have $a \in V_{k+1}$ and $b \in V_k$ joined by an edge if $f(a) = b$.

Thus each V_n defines a field extension $K(V_n)$.

The Galois groups H_n permutes the solutions of $f^n(x) = \alpha$ preserving the tree structure.



Since $f^n(x)$ is irreducible, then H_n acts transitively on V_n.

The **arboreal representation** of the absolute Galois group of K associated to $f(x)$ and α is the inverse limit group

$$\mathrm{Gal}_{f,\alpha} = \varprojlim \{H_n \to H_{n-1} \mid n \geq 1\} \subset \mathrm{Aut}(T).$$

$\mathrm{Gal}_{f,\alpha}$ is a profinite group which acts transitively on the Cantor set ∂T.

Some history: density of primes

Given $\alpha \in K$, consider the set $S = \{f(\alpha), f^2(\alpha), \ldots\}$.

What is the density of prime divisors of elements in S?

Odoni 1985: Obtained an upper bound for this density by counting fixed points of elements of $\rho_{f,\alpha}$.

Jones 2013: A survey of related work on arboreal representations.

Many contributions since 2013, in the works of **Benedetto**, **Ferraguti**, **Jones**, **Juul**, **Looper**, **Pagano**, **Tucker**, and others.

One of the questions asked: for which f, K and α does the group $\mathrm{Gal}_{f,\alpha}$ have finite index in $\mathrm{Aut}(T)$?

From arboreal representations to dynamics of actions

In topological dynamics and geometric group theory, one studies actions of discrete groups, while an arboreal representation is a profinite group.

Construction (Lukina 2019[1])

1. $\mathrm{Gal}_{f,\alpha}$ is the inverse limit of a sequence of finite groups indexed by natural numbers, and so contains a dense countably generated subgroup G_0.
2. Give G_0 discrete topology.
3. The Galois group $\mathrm{Gal}_{f,\alpha}$ is identified with the uniform closure $\overline{\Phi(G_0)} \subset Homeo(X)$.

The choice of G_0 is not unique.

Therefore, one has to look at the dynamical invariants of actions of countable groups which are the **invariants of the profinite closure** of the action.

[1] *O. Lukina, Arboreal Cantor actions, JLMS 2019*

Equivalences in dynamical systems

Isomorphism:

Let G, G' be (discrete or profinite) groups, acting on Cantor sets ∂T, $\partial T'$.

The actions $(\partial T, G)$ and $(\partial T', G')$ are **isomorphic** if there exist:

1. a homeomorphism $\phi : \partial T \to \partial T'$,
2. and a group isomorphism $\psi : G \to G'$ such that for any $g \in G$ and any $v \in \partial T$
$$\phi(gv) = \psi(g)\phi(v).$$

Remark: A slightly stronger notion, for $\psi = \mathrm{id}$, is that of a **conjugacy** of the dynamical systems.

Elements with fixed points (isotropy groups)

Consider the action of a profinite (Ellis) group $\mathcal{G}$ on the boundary ∂T of a rooted tree.

Look at elements of $\mathcal{G}$ which fix a given path $v \in \partial T$.

These elements are of two types:

1. $g \in \mathcal{G}$ such that $g \cdot v = v$ and g fixes every point in an open neighborhood $U \subset \partial T$ of v.
2. $g \in \mathcal{G}$ such that $g \cdot v = v$ and for any open neighborhood $U \subset \partial T_d$ of v the restriction $g|U$ is not the identity map.

Approach (developed in a series of joint works with Hurder):

We consider the behavior of such elements on smaller and smaller neighborhoods of $v \in \partial T$.

We see this as an analogue to studying infinitesimal dynamical properties of the system in the absence of derivatives.

Elements which fix a given path $v = (v_1, v_2, \ldots) \in \partial T$ form the isotropy subgroup (= the stabilizer) of the action at v

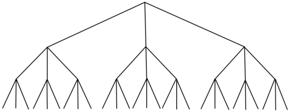
$$\mathcal{D}_v = \{\widehat{h} \in \mathcal{G} \mid \widehat{h} \cdot v = v\} \subset G_\infty,$$

called the *discriminant group* (not the same notion as the discriminant group of a polynomial!).

The discriminant group may be trivial, finite or a profinite (Cantor) group.

Then ∂T is a homogeneous space

$$\partial T \cong \mathcal{G}/\mathcal{D}_v.$$



Remark: $\mathcal{D}_v$ is computable using *group chains*, see

J. Dyer, S. Hurder and O. Lukina, The discriminant invariant of Cantor group actions, Topology Appl., 2016.

For $v_n \in V_n$, the subtree T_{v_n} contains all paths in

$$U_{n,v_n} = \{(w_1, w_2, \ldots) \mid w_n = v_n\},$$

which is also a Cantor set.

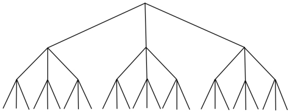
Elements which preserve U_{n,v_n} form a subgroup $\widehat{U}_{n,v_n} \subset \mathcal{G}$ and

$$U_{n,v_n} \cong \widehat{U}_{n,v_n}/\mathcal{D}_v.$$

The adjoint action of $\mathcal{D}_v$ on $\widehat{U}_{n,v_n}$ is well-defined

$$Ad_n : \mathcal{D}_v \times \widehat{U}_{n,v_n} \to \widehat{U}_{n,v_n} : (\widehat{h}, \widehat{g}) \mapsto \widehat{h}\widehat{g}\widehat{h}^{-1}.$$

Remark: If $Ad_n(\widehat{h}) = id$, then $\widehat{h}$ fixes every point in U_{n,v_n}.



For $n \geq 0$, define the groups

$$K_n = \{\,\widehat{h} \in \mathcal{D}_v : \widehat{h}|U_{n,v_n} = id\} \qquad (\widehat{h} \text{ fixes every point in } U_{n,v_n})$$

$$Z_n = \{\widehat{h} \in \mathcal{D}_v : Ad(\widehat{h})|\widehat{U}_{n,v_n} = id\} \qquad (Ad(\widehat{h}) \text{ fixes every point in } \widehat{U}_{n,v_n}).$$

Then $Z_n \subset K_n$, for all $n \geq 0$.

Form direct limit groups with respect to inclusions $\iota^n_{n+1} : K_n \to K_{n+1}$

$$\mathcal{K} = \sqcup_{n\geq 0} K_n / \{a \sim b \,\text{iff}\, \iota^k_s(a) = \iota^m_s(b),\ s \geq k, m\},$$

$$\mathcal{Z} = \sqcup_{n\geq 0} Z_n / \{a \sim b \,\text{iff}\, \iota^k_s(a) = \iota^m_s(b),\ s \geq k, m\}.$$

Theorem (Hurder and Lukina 2021[2])

The isomorphism classes of the direct limit groups $\mathcal{K}$ and $\mathcal{Z}$ are invariants of the conjugacy class of $(\partial T, \mathcal{G})$.

[2] *S. Hurder and O. Lukina, Limit group invariants for non-free Cantor actions, Ergod. Theory Dynam. Systems, 2021.*

We say that the direct limit group $\mathcal{K}$ is *bounded*, if there exists $m \geq 0$ such that for all $n \geq m$ the inclusions $\iota : K_n \to K_{n+1}$ are isomorphisms.

The following classification was introduced in **Hurder and Lukina 2021**:

Stable and wild actions

A group action $(\partial T, \mathcal{G})$ is *stable* if the group chain $\mathcal{K}$ is bounded, and the action is *wild* otherwise.

Remark: If $\mathcal{K}$ is bounded with $m = 0$, then the space ∂T contains a dense subset of points which are fixed only by the identity element, i.e. the action is *topologically free*.

Remark: The invariants above are motivated by questions about the nature of attractors and exceptional minimal sets of foliations in dynamical systems and foliation theory.

Remark: A finer, more detailed classification using the properties of $\mathcal{K}$ and $\mathcal{Z}$ is introduced in **Hurder and Lukina 2021**.
More examples can be found in **Álvarez López, Barral Lijó, Lukina and Nozawa 2022**.

Examples

Theorem (Lukina JLMS 2019)

Let p and d be distinct odd primes, let K be a finite unramified extension of the p-adic numbers $\mathbb{Q}_p$, and let

$$f(x) = (x+p)^d - p.$$

Then the action of $\mathrm{Gal}_{f,0}$ on ∂T is stable.

Theorem (Lukina JLMS 2019)

Suppose the arboreal representation $\mathrm{Gal}_{f,\alpha}$ is a subgroup of finite index in $Aut(T)$. Then the action $(\partial T, \mathrm{Gal}_{f,\alpha})$ is wild.

Profinite IMGs

Let t be a transcendental element, and let $K(t)$ be a field of functions.

Let $f(x)$ be a polynomial of degree $d \geq 2$ with coefficients in K.

Repeating the arboreal representation construction as before for the solutions of $f^n(x) = t$, we obtain

- the **arithmetic iterated monodromy group** $\mathrm{Gal}_{\mathrm{arith}}(f) \subset Aut(T)$

and

- the **geometric iterated monodromy group** $\mathrm{Gal}_{\mathrm{geom}}(f) \trianglelefteq \mathrm{Gal}_{\mathrm{arith}}(f)$.

Specializing to $t = \alpha$, $\alpha \in K$, one gets an arboreal representation $\mathrm{Gal}_{f,\alpha}$.

Generically, $\mathrm{Gal}_{f,\alpha} \cong \mathrm{Gal}_{\mathrm{arith}}(f)$.

"Small" profinite IMGs

Observation (Jones 2013):

If a polynomial $f(x)$ of degree d is **post-critically finite (PCF)**, then the image of its arboreal representation has infinite index in $\mathrm{Aut}(T)$.

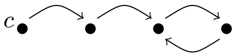
Let C be the set of critical points of $f(x)$. On the Riemann sphere this includes ∞, which is a fixed point of $f(x)$.

Recall: A polynomial $f(x)$ is **(PCF)**, if the set of images

$$P_C = \{f^i(c) : c \in C, i \geq 1\}$$

of its critical set C under the forward iterations of f is finite.

Example: If $f(x)$ is a quadratic PCF polynomial, and $c \neq \infty$, then the orbit of c may be periodic or strictly pre-periodic; ∞ is a fixed point.



Remark (Pink 2013): If K is a number field, $\mathrm{Gal}_{\mathrm{geom}}(f)$ can be computed over $\mathbb{C}(t)$.

On the other hand, f^n define a sequence of coverings of a punctured Riemann sphere, which are used to construct the discrete iterated monodromy group $\mathrm{IMG}(f)$.

Proposition (Nekrashevych 2005, attributed to Pink)

Suppose $f(x)$ has a finite post-critical set P_C. Then the closure $\overline{\mathrm{IMG}(f)}$ in $Aut(T)$ is isomorphic to $\mathrm{Gal}_{\mathrm{geom}}(f)$ over $\mathbb{C}(t)$.

A geometric group theory method:

Elements of $\mathrm{IMG}(f)$ can be written recursively using the presentation of $\mathrm{Aut}(T)$ as the iterated wreath product of the groups $\mathrm{Perm}(\{1, \ldots, d\})$, with a generator g_p for every point $p \in P_C$.

Let $d = 2$, i.e. $f(x)$ is a quadratic polynomial over a number field K.

Pink 2013 provided a detailed description of $\mathrm{Gal}_{\mathrm{geom}}(f)$ and $\mathrm{Gal}_{\mathrm{arith}}(f)$ in this case.

Let $P_c = P_C \backslash \{\infty\}$.

Theorem 1 (profinite geometric IMGs)[3]

Let $f(x)$ be a quadratic PCF polynomial over a number field K. Then the following holds.

1. If $\#P_c = 1$, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$ is stable with trivial discriminant group.
2. If $\#P_c = 2$ and the the post-critical orbit is pre-periodic, then the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$ is stable with finite discriminant group.
3. In all other cases when $\#P_c$ is finite, the action of $\mathrm{Gal}_{\mathrm{geom}}(f)$ is wild.

[3] *O. Lukina, Galois groups and Cantor actions, Trans. Amer. Math. Soc., 2021.*

Theorem 2 (profinite arithmetic IMGs)[3]

Let $f(x)$ be a quadratic PCF polynomial over a number field K. Then the following holds.

1. If $\#P_c = 1$, then the action of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is stable with profinite discriminant group.
2. If $\#P_c = 2$ and the the post-critical orbit is pre-periodic, then the action of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is stable with profinite discriminant group.
3. In all other cases when $\#P_c$ is finite, the action of $\mathrm{Gal}_{\mathrm{arith}}(f)$ is wild.

[3] *O. Lukina, Galois groups and Cantor actions, Trans. Amer. Math. Soc., 2021.*

Higher degrees: invariable generation

Pink 2013:

Let $f(x)$ be quadratic PCF with $\#P_c = r \geq 1$ (P_c does not include ∞), and let $\mathrm{Gal}_{geom}(f)$ be the associated geometric IMG.

Then there exist a set of elements $\{g_1, \ldots, g_r\} \subset \mathrm{Aut}(T)$, depending only on the length of P_c, and the length of a periodic cycle in P_c, such that, for some $w \in \mathrm{Aut}(T)$

$$wGw^{-1} = \mathrm{Gal}_{geom}(f),$$

where $G = \overline{\langle g_1, \ldots, g_r \rangle}$. Here $g_1 g_2 \cdots g_r = g_\infty$, where g_∞ is an odometer, and T is a binary tree.

Remark: g_∞ is the generator corresponding to ∞.

We call the generators $\{g_1, \ldots, g_r, g_\infty\}$ the **model generators**.

Issues:

1. $\mathrm{Gal}_{geom}(f)$ has a countable dense subgroup generated by r elements, but each generator h_i is only determined up to a conjugation by $w_i \in \mathrm{Aut}(T)$. Even if each $w_i \in \mathrm{Gal}_{geom}(f)$, the set
$$\{w_i h_i w_i^{-1} : 1 \leq i \leq r\}$$
need not be a generating set for $\mathrm{Gal}_{geom}(f)$.
2. For $w \in \mathrm{Aut}(T)$, the sets $\{w h_i w^{-1} : 1 \leq i \leq r\}$ and $\{g_i : 1 \leq i \leq r\}$ may topologically generate the same profinite group, but different countable groups.

Issue 1 leads to the notion of an invariable generating set of a group.

Invariable generating set

A set S **invariably generates** a group G, if, for any choice of $g_s \in G$, the set $\{g_s s g_s^{-1} : s \in S\}$ generates G.

p-adic case, p prime

If $d = 2$, then the restriction $G|_{T_n}$ to the first n levels of T is a 2-adic group, and so nilpotent.

The fact that the set of model generators $S = \{g_1, \ldots, g_r\}$ is an invariable generating set follows from that.

Corollary

If $f(x)$ is a quadratic PCF polynomial, then every subsets of $|S|$ elements from the set $S \cup \{a\}$, where $a = g_1 \cdots g_r$ is an odometer, is a generating set for $G = \overline{\langle S \rangle}$.

For $d \geq 3$, the restrictions $G|T_n$ need not be nilpotent, even for d prime.

Unicritical case

A polynomial $f(x)$ is **unicritical** if it has a single critical point $c \in \mathbb{C}$.

Then $C = \{c, \infty\}$, and we denote $P_c = P_C \backslash \{\infty\}$.

Theorem (Adams and Hyde 2025)

Let $f(x)$ be a unicritical PCF polynomial of degree $d \geq 2$, and let $\mathrm{Gal}_{geom}(f)$ be the associated geometric monodromy group.

Then there is a set of model generators S of cardinality $|P_c|$, depending only on $|P_c|$ and the length of the periodic cycle in P_c, such that:

1. S is an invariable generating set for $G = \overline{\langle S \rangle}$.
2. There exists $w \in \mathrm{Aut}(T)$, such that

$$wGw^{-1} = \mathrm{Gal}_{geom}(f).$$

Remark: thus G has an invariable generating set without an odometer $g_\infty = g_1 \cdots g_r$.

Non-unicritical cubic case

Let $f(x)$ be a PCF cubic polynomial over a number field K.

Let P_C denote the postcritical set (including ∞).

Joint work with Hlushchanka and Wardell:
profinite geometric IMGs for cubic polynomials satisfying the assumption below.

Assumption (Y): for each $p \in P_C \backslash \{\infty\}$, $f^{-1}(p)$ contains a point that is neither critical nor postcritical.

New phenomenon

For cubic polynomials satisfying Assumption (Y), an invariable generating set **must include an odometer**.

Theorem (Hlushchanka, Lukina, Wardell 2025[4])

Let $f(x)$ be a cubic PCF polynomial satisfying the Assumption (Y), with post-critical set P_C (P_C includes ∞).

Then there exist a set of model generators $S = \{g_1, \dots, g_r, g_\infty\}$ such that:

1. An invariable generating set for $G = \overline{\langle S \rangle}$ must contain an odometer; in particular, S is an invariable generating set for G.
2. There exists $w \in \mathrm{Aut}(T)$, such that

$$wGw^{-1} = \mathrm{Gal}_{geom}(f).$$

Remark: Most likely, the cardinality of a minimal generating set for G is $|S| - 1$, and such a set must contain g_∞.

[4] *M. Hlushchanka, O. Lukina and D. Wardell, Profinite geometric iterated monodromy groups of postcritically finite polynomials in degree* 3*, arXiv: 2507.05033.*

Stable or wild?

Theorem (Hlushchanka, Lukina, Wardell 2025[4])

Let $f(x)$ be a cubic PCF polynomial satisfying the Assumption (Y).

Let $\mathrm{Gal}_{geom}(f)$ be the profinite iterated monodromy group.

Then $\mathrm{Gal}_{geom}(f)$ is regular branch.

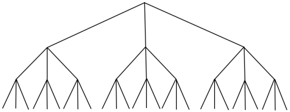
Remarks: A regular branch group is weakly branch.

The action of a weakly branch group is wild.

For $H \subset G$, if the action of H is wild, then the action of G is wild.

Corollary: Let $f(x)$ be a cubic PCF polynomial satisfying Assumption (Y).

Then the actions of $\mathrm{Gal}_{geom}(f)$ and of $\mathrm{Gal}_{arith}(f)$ are both wild.



[4] *M. Hlushchanka, O. Lukina and D. Wardell, Profinite geometric iterated monodromy groups of postcritically finite polynomials in degree* 3*, arXiv: 2507.05033.*

Further work and open problems

The works cited above generated a range of concrete doable open problems and trends for further research.

The answer to the following problem will make the connection between Cantor dynamics and number theory more remarkable:

Main open question

- What difficult problems in number theory be solved using Cantor dynamics?
- What difficult problems in dynamics or foliation theory can be solved using arboreal representations of Galois groups?

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Thank you for your attention!