Galois Theory of Local Systems

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jt w/ Landesman + Lam

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I) Introduction

G - group (finite, or algebraic, ...)

$C_1, \dots, C_n \subseteq G$ conjugacy classes

(i) Deligne-Simpson / Hurwitz problem:

<u>Q</u> $\exists\, g_1 \in C_1, g_2 \in C_2, \dots, g_n \in C_n$ s.t.

$$\prod g_i = id \ ?$$

$\Sigma_{0,n} :=$ 

$= \mathbb{P}^1 - n$ pts

$\pi_1(\Sigma_{0,n}) = \langle \gamma_1, \dots, \gamma_n \mid \prod \gamma_i = id \rangle$

<u>Q</u> (Deligne-Simpson)

$\exists \rho : \pi_1(\Sigma_{0,n}) \to G$ s.t. $\rho(\gamma_i) \in C_i$ for all i?

G finite: $\exists$ G-cover $f: Y \to \mathbb{P}^1$ w/ specified branching?

$G = SL_r, GL_r, \ldots$: $\exists$ local system w/ specified local behavior?

Rem (1) Alg. gp case related to finite group case: can look at $G(\mathbb{F}_q)$

(2) For GL_r: sol'n by Simpson, Kostov, ...

(ii) Rigid Tuples: For which $C_1, \ldots, C_n \subseteq G$ is $(g_1, g_2, \ldots, g_n) \in C_1 \times \cdots \times C_n$ s.t. $\prod g_i = id$ unique up to simultaneous conjugation?

G finite: $Y \to \mathbb{P}^1/\mathbb{Q}(\mu_N)$, useful for inverse Galois problem

$G = GL_r, SL_r, \ldots$: rigid local systems, descend to $\mathbb{P}^1_{\mathbb{Q}(\mu_N)} \setminus n$ pts

Rem (1) Again alg. gp & finite gp case related

(2) GL_r: classified by Katz ('96)

All "of geometric origin"

Classification (Katz):

$$(C_1, \ldots, C_n) \subseteq GL_r^n \text{ rigid}$$

$$\Big\downarrow \text{ "middle convolution"}$$

$$(C_1', \ldots, C_n') \subseteq GL_{r'}^n \text{ rigid}, \quad r' < r$$

Iterate! Miracle: middle convolution is invertible

(iii) Dynamics:

$$g_1, \ldots, g_n \in G \text{ s.t. } \prod g_i = \mathrm{id}$$

$$\sigma_i : (g_1, \ldots, g_n) \longmapsto (g_1, \ldots, g_i g_{i+1} g_i^{-1}, g_i, \ldots, g_n)$$

$\langle \sigma_1, \ldots, \sigma_{n-1} \rangle \curvearrowright G^n/G$ where quotient is by conjugation

Q What are the dynamics of this action? (Markoff, BGS, Goldman, ...)

Q $G = GL_r, SL_r, \ldots$: what are finite orbits?

- G finite: Components/topology of Hurwitz spaces
- $G = GL_r, SL_r, \dots$: the rest of this talk.

<u>Rem</u> Strict generalization of study of rigid tuples/rigid local systems.

2) Some history

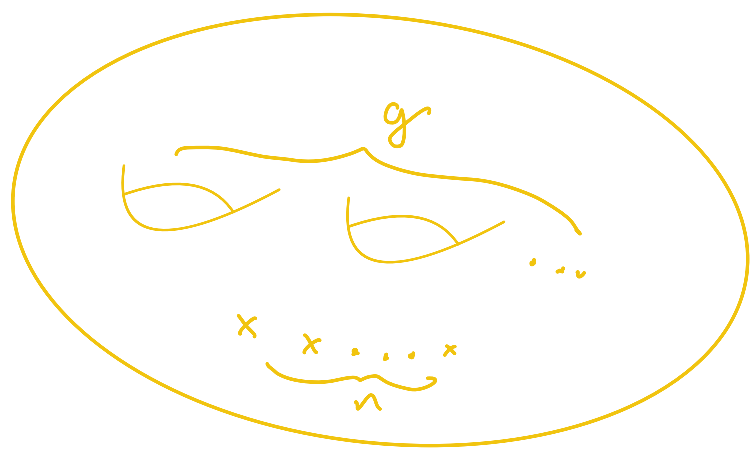
$G = GL_r, SL_r, \dots$

$$\langle \sigma_1, \dots, \sigma_{n-1} \rangle \curvearrowright G^n/\sim_{conj}$$

$$\sigma_i : (g_1, \dots, g_n) \longmapsto (g_1, \dots, g_i g_{i+1} g_i^{-1}, g_i, \dots, g_n)$$

<u>Q</u> What are the finite orbits?

Conceptualization/generalization:

$\Sigma_{g,n} =$ 

$$X_{g,n}(G) = \mathrm{Hom}(\pi_1(\Sigma_{g,n}), G)/\sim_{\mathrm{conj}}$$

$$\begin{aligned}
\mathrm{Mod}_{g,n} &= \pi_0(\mathrm{Homeo}^+(\Sigma_{g,n})) \\
&\simeq \pi_1(\mathcal{M}_{g,n}) \\
&\overset{(g=0)}{\simeq} \langle \sigma_1, \dots, \sigma_{n-1} \rangle = \text{"spherical braid gp on } n \text{ strands"} \\
&\overset{(n=0)}{\simeq} \mathrm{Out}(\pi_1(\Sigma_{g,n}))
\end{aligned}$$

<u>Q</u>) $\mathrm{Mod}_{g,n} \curvearrowright X_{g,n}(G)$

What are the finite orbits?

History: $(g=0,\ n=4,\ G=SL_2)$

Painlevé VI: (Fuchs)

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left\{\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right\}$$

finite orbits $\rightsquigarrow$ alg. sol'ns to Painlevé VI

- Painlevé: "new" transcendents (~1912)
- Umemura: rigorous proof, conditional on

classification of finite orbits

- Classification ($g=0, n=4, G=SL_2$)
  - Fuchs, Picard, ..., Hitchin, Boalch, Andreev-Kitaev, Doran, ...
  - Dubrovin-Mazzocco:

    $$C_1 = C_2 = C_3 = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right], \quad C_4 = \left[\begin{pmatrix} \mu & \\ & \mu^{-1} \end{pmatrix}\right]$$

  - Lisovyy-Tykhyy (2014): ← effective Manin-Mumford for Tori
    - computer-aided
    - 4 cts families
    - 1 ∞ discrete family
    - 45 exceptional sol'ns
- ($g=0, n>4$): sporadic examples (Calligaris-Mazzocco, ...)
  - $G = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ (Cousin-Moussard)
  - ($n=5$) computations by Tykhyy
- ($g \geq 1, G = SL_2$) Classification by Biswas-Gupta-Mj-Whang

<u>Conjectures</u> • $G = GL_r$

<u>Conj</u> (Kisin, Whang) If $g >> r$, all finite orbits correspond to rep'ns

$$\pi_1(\Sigma_{g,n}) \to GL_r$$

w/ finite image.

Conj (Esnault-Kerz, Budur-Wang)
Finite orbits are Zariski-dense in
$X_{g,n}(GL_r)$.

Rem · These conjectures contradict each other.
· Motivation: relative Fontaine-Mazur,
p-curvature conjecture...
· Expectation: ($g \geq 3$?) Finite orbits all of geometric origin

3) Theorems

Thm 1 (Landesman-L-)
Suppose $g \geq r^2$. Then
$$\rho : \pi_1(\Sigma_{g,n}) \to GL_r(\mathbb{C})$$
has finite $\mathrm{Mod}_{g,n}$-orbit $\Longleftrightarrow$
ρ has finite image.

Cor Kisin-Whang's conjecture ✓
Esnault-Kerz / Budur-Wang's conjecture ✗

Pf Input from NAHT and Langlands

Thm 2 (Lam-Landesman-L-)
Complete classification of

$$(A_1, \ldots, A_n) \in (GL_2)^n/\sim \quad \text{s.t. } \prod A_i = id$$

w/ finite $\mathrm{Mod}_{0,n}$-orbit, when some A_i has infinite order.

Precise statement:

Say $(A_1, \ldots, A_n) \in (SL_2)^n$ w/ $\prod A_i = id$ is interesting if

- finite $\mathrm{Mod}_{0,n}$-orbit
- $\langle A_1, \ldots, A_n \rangle \subseteq SL_2$ Zariski-dense (otherwise classified by Cousin-Heussard, Tykhyy)
- doesn't move in continuous family of finite orbits (these were classified by Diarra)
- no $A_i = \pm id$

Thm 2' (Lam-Landesman-L–)

Suppose $(A_1, \dots, A_n)$ is interesting, and some A_i has infinite order. Then $\exists \lambda_1, \dots, \lambda_n \in \mathbb{C}^*$ s.t.

$$(\lambda_1 A_1, \dots \lambda_n A_n) = MC_\chi(B_1, \dots, B_n)$$

"middle convolution"

where $\langle B_1, \dots, B_n \rangle \subseteq GL_{n-2}(\mathbb{C})$ is a finite complex reflection group.

Defn • $B \in GL_r(\mathbb{C})$ is a pseudoreflection if

$$rk(B - id) = 1$$

• $G \subseteq GL_r(\mathbb{C})$ is a finite complex reflection group (FCRG) if it is finite and gen'd by pseudoreflections

In what sense is this a classification?

FCRGs classified by Shephard & Todd (1956)

- 1 infinite family $G(m,p,n) \subseteq GL_n(\mathbb{C})$
  $(e_i \longmapsto \zeta_m^{a_i} e_{\sigma(i)})$
- 34 exceptional gps (Aut(icosahedron), $W(E_8)$, ...)

Cor (LLL-)

If $(A_1, \dots, A_n)$ is interesting & some A_i has infinite order, then $n \leq 6$. (Sharp.)

Middle convolution (MC_χ):

$$X = \mathbb{P}^1 \setminus D, \quad \infty \in D$$

$$\begin{tikzcd} X \times X \setminus \Delta \arrow[r, hook, "j"] \arrow[d, "\pi_1"'] \arrow[dr, "\alpha"] & \mathbb{P}^1 \times X \arrow[dr, "\pi_2"] & \\ X & \mathbb{A}^1 - \{0\} & X \end{tikzcd}$$

$(x,y) \mapsto x - y$

$\mathbb{V} \in LocSys(X)$, χ - rk 1 local system on $\mathbb{A}^1 - \{0\}$

$$MC_\chi(\mathbb{V}) := R^1\pi_{2*} j_* (\pi_1^* \mathbb{V} \otimes \alpha^* \chi)$$

Special case: $\mathbb{V}$ finite monodromy gp G (think FCRG)

$\chi^a = \text{triv}$

$\exists$ $\begin{array}{c} \mathcal{Y} \\ \downarrow \pi \\ x \in X \end{array}$ family of curves s.t. $\mathcal{Y}_x$ is a cover of $\mathbb{P}^1$ w/ Galois gp $G \times \mathbb{Z}/a\mathbb{Z}$ branched over $D \cup \{x\}$

s.t. $\mathrm{IC}_X(\mathbb{V}) \subseteq R\pi_* \underline{\mathbb{C}}$.

Upshot: Finite orbits have a good geometric explanation.