The Ceresa and Gross–Kudla–Schoen cycles associated to modular curves

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Sep 24, 2024

References

Papers I learned the subject from:

- Galois actions on fundamental groups of curves and the cycle $C - C^-$ (Hain–Matsumoto, 2005)
- Iterated integrals, diagonal cycles, and rational points on elliptic curves (Darmon–Rotger–Sols, 2012)
- On Ceresa cycles of Fermat curves (Eskandari–Murty, 2021)

My papers on the subject:

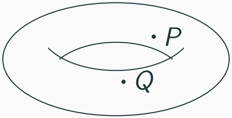
- Group-theoretic Johnson classes and a non-hyperelliptic curve with torsion Ceresa class (Bisogno–L.–Litt– Srinivasan, 2020)
- The Ceresa class: tropical, topological, and algebraic (Corey–Ellenberg–L., 2020)
- The Ceresa class and tropical curves of hyperelliptic type (Corey–L.,2022)
- Non-vanishing of Ceresa and Gross–Kudla–Schoen cycles associated to modular curves (Kerr–L.–Qiu–Yang, 2024)

Algebraic cycles

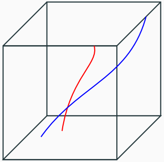
Michelangelo: Every block of stone has a statue inside it and it is the task of the sculptor to discover it.

An **algebraic cycle** on an algebraic variety X defined over K is a $\mathbb{Z}$-linear combination of closed subvarieties over K.

Task of mathematician: discover the algebraic cycles and classify them.



points on an elliptic curve ($\mathbb{C}$)



1-cycles on a 3-fold ($\mathbb{R}$)

As was discussed in Ari and Padma's talks, a main tool in studying algebraic cycles is the Abel–Jacobi map, both Hodge theoretic and ℓ-adic.

Today: ℓ-adic Ceresa class for $C/\mathbb{C}((t))$ and Hodge for modular curves.

From Ari's talk: the cycle class map and Abel–Jacobi maps

Algebraic cycle is the algebraic geometry notion parallel to sub-topological space and sub-manifold in algebraic topology and differential geometry.

There is a **cycle class map** mapping algebraic cycles to singular (Weil) cohomology classes. Any cycle in the kernel is called homologically trivial.

$$\text{Cycle class map } \mathrm{cl} : \mathsf{CH}^r(X) \to H^{2r}(X)$$

The kernel of cl is a subgroup of $\mathsf{CH}^r(X)$ denoted as $\mathsf{CH}^r_{hom}(X)$.

For X defined over $\mathbb{C}$, Griffiths' Abel–Jacobi map

$$\mathrm{AJ} : \mathsf{CH}^r_{hom}(X) \to \frac{\mathsf{H}^{2r-1}(X, \mathbb{C})}{\mathrm{Fil}^r + \mathsf{H}^{2r-1}(X, \mathbb{Z})}.$$

For X/K, K not algebraically closed, there is an ℓ-adic Abel–Jacobi map

$$\mathrm{AJ}_\ell : \mathsf{CH}^r_{hom}(X) \to \mathsf{H}^1(\mathsf{Gal}(\bar{K}/K), \mathsf{H}^{2r-1}_{\acute{e}t}(X_{\bar{K}}, \mathbb{Z}_\ell(r))).$$

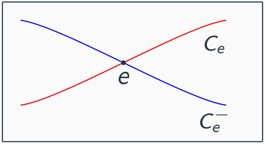
Today: Both of the Abel–Jacobi images/cycle classes can be used to detect non-triviality of algebraic cycles in the Chow groups.

Recall: the Ceresa cycle

Let e be a fixed point of a curve C. The Ceresa cycle is defined as

$$C_e - C_e^- \in \mathrm{CH}^{g-1}_{hom}(\mathrm{Jac}(C))$$

where C_e is the Abel–Jacobi image of $C \hookrightarrow \mathrm{Jac}(C)$ given by $P \mapsto P - e$ and C_e^- is the image of $C \hookrightarrow \mathrm{Jac}(C)$ given by $P \mapsto e - P$.



The Ceresa cycle is homologically trivial because -1 acts trivially on H^2.

Question: What are the cycle classes of the Ceresa cycle?

It turns out the Ceresa cycle classes are related to the Hodge and Galois structures on the geometric fundamental group of the curve.

The Hodge and Galois structures on fundamental groups

Anabelian Geometry: The arithmetic fundamental group $\pi_{1,\text{ét}}(C)$ determines C when C is a hyperbolic curve defined over a number field.

$$1 \to \pi_{1,\text{ét}}(C_{\bar{K}}) \to \pi_{1,\text{ét}}(C) \to \mathsf{Gal}(\bar{K}/K) \to 1.$$

Question: How to extract information from the fundamental group?

Let $\pi = \pi_1(C_{\bar{K}})$ be the geometric fundamental group. When $\mathrm{char} K = 0$, $\pi = \pi_1(\Sigma_g)$ is the fundamental group of a genus g Riemann surface.

Let J be the augmentation ideal of the group ring $\mathbb{Z}[\pi]$ or $\mathbb{Z}_\ell[\pi^\ell]$.

Then $J/J^2 \simeq \pi_{ab} = \mathsf{H}_1(C)$ and $J^2/J^3 \simeq \mathsf{H}_1(C)^{\otimes 2}/\langle\theta\rangle$.

We have the following exact sequence of free $\mathbb{Z}$ or $\mathbb{Z}_\ell$ modules.

$$0 \to J^2/J^3 \to J/J^3 \to J/J^2 \to 0. \qquad (*)$$

Sequence (*) splits as $\mathbb{Z}$ or $\mathbb{Z}_\ell$ modules, but π has extra structure.

The Hodge and Galois structures on fundamental groups

Question: What structures does J/J^k carry?

$$0 \to J^2/J^3 \to J/J^3 \to J/J^2 \to 0. \qquad (*)$$

Hodge: The free $\mathbb{Z}$-mod J/J^k carries a mixed Hodge structure for $k \geq 3$, given by Hain in terms of iterated integrals.

Harris–Pulte: The splitting of (*) respecting the mixed Hodge structures is measured by a class in $Ext_{MHS}(J/J^2, J^2/J^3)$ "=" the Ceresa cycle class.

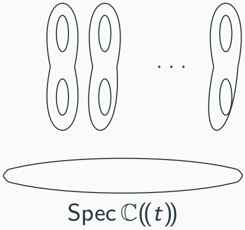
The Hodge Ceresa class inspired the construction of Chow–Heegner points by Darmon–Rotger–Sols which we will discuss in details.

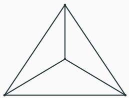
Galois: With $e \in C(K)$, there is a section $\mathrm{Gal}(\bar{K}/K) \to \pi_{1,\text{ét}}(C)$ which induces a Galois action on π^ℓ and thus all J^k.

Hain–Matsumoto: The splitting of (*) respecting the Galois action is measured by a class in $H^1(\mathrm{Gal}(\bar{K}/K), \mathrm{Hom}(J/J^2, J^2/J^3))$ "=" the ℓ-adic Ceresa cycle class.

Idea: Galois action on the geometric fundamental group $\Rightarrow$ Ceresa class.

Curves over $\mathbb{C}(\!(t)\!)$





A curve $C/\mathbb{C}(\!(t)\!)$ can be viewed as a family of Riemann surfaces over a punctured disk such that the fiber over the puncture is singular.

The (weighted) dual graph of the singular fiber (a tropical curve) contains the data for the $\mathrm{Gal}(\overline{\mathbb{C}(\!(t)\!)}/\mathbb{C}(\!(t)\!))$ action on the fiber.

The ℓ-adic Ceresa class can be computed using this data and one can further define an invariant for tropical curves inspired by this.

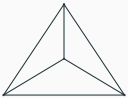
Corey–Ellenberg–L.: For $C/\mathbb{C}(\!(t)\!)$, its ℓ-adic Ceresa class is torsion.

The Ceresa class of curves over $\mathbb{C}(\!(t)\!)$

Question: Ceresa trivial $\iff$ hyperelliptic? From Ari's talk, we know trivial in $\mathrm{CH}^{g-1}(\mathrm{Jac}(C)) \otimes \mathbb{Q}$ is not enough, but integrally?

From computation: when the dual graph of the special fiber (viewed as a tropical curve) is not hyperelliptic, the Ceresa class is always nonzero.

Example: Given $C/\mathbb{C}(\!(t)\!)$ with special fiber the following dual graph with edge length 1, its ℓ-adic Ceresa class is non-trivial of order 16.



If instead of assigning edge length, define a "Ceresa" invariant for a graph over a polynomial ring where the variables are the edges. This invariant asserts when the tropical curve is of hyperelliptic type.

Corey–L.: A connected graph G of genus $g \geq 2$ is Ceresa-Zharkov trivial if and only if G is of hyperelliptic type, meaning any tropical curve with underlying graph G has a hyperelliptic Jacobian.

Chow–Heegner divisor from Hodge theoretic perspective

Next discuss a non-vanishing criteria using Griffiths' Abel–Jacobi image. Recall the Hodge Ceresa class measures the splitting of sequence (*) which we could dualize and rewrite as

$$0 \to \mathrm{H}^1(C) \to (J/J^3)^\vee \to \mathrm{H}^1(C)^{\otimes 2}_{\cup} \to 0.$$

Chow–Heegner divisor (Darmon–Rotger–Sols): Use a Hodge class $\xi : \mathbb{Z}(-1) \to \mathrm{H}^1(C)^{\otimes 2}_{\cup}$ to pullback sequence (*) and obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^1(C) \arrow[r] \arrow[d, "="] & E \arrow[r] \arrow[d] & \mathbb{Z}(-1) \arrow[r] \arrow[d, "\xi"] & 0 \\
0 \arrow[r] & \mathrm{H}^1(C) \arrow[r] & (J/J^3)^\vee \arrow[r] & \mathrm{H}^1(C)^{\otimes 2}_{\cup} \arrow[r] & 0.
\end{tikzcd}$$

The non-splitting of the top sequence implies the non-splitting of the bottom. The extension class of the top sequence corresponds to a point on $\mathrm{Jac}(C)$ which is called the Chow–Heegner divisor.

Next: Visualize the Chow–Heegner divisor.

The Gross–Kudla–Schoen modified diagonal cycles

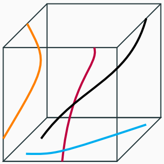
The Gross–Kudla–Schoen modified diagonal cycle:

Let C^3 denote the product $C \times C \times C$. The Δ_{GKS} cycle is defined as

$$\Delta_{GKS}(C, e) := C_{123} - C_{12} - C_{23} - C_{13} + C_1 + C_2 + C_3 \in \mathrm{CH}^2_{hom}(C^3)$$

where e is a fixed point of C and C_I denotes the image of $C \hookrightarrow C^3$ by $P \mapsto P$ for $i \in I$ and $P \mapsto e$ for $i \notin I$ with $i \in \{1, 2, 3\}$.

The Δ_{GKS} cycle is homologically trivial.



The Δ_{GKS} cycle and Ceresa cycle has "the same" cycle class.

Theorem (S-W Zhang, 2010): $C_e - C_e^- \equiv 0 \iff \Delta_{GKS}(C, e) \equiv 0$.

The Chow–Heegner divisor

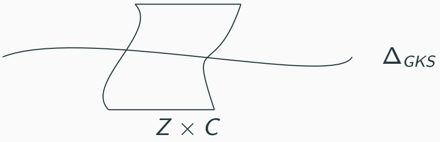
Chow–Heegner divisor: Let $Z \in \mathrm{CH}^1(C \times C)$ be a correspondence. Define

$$\Pi_Z(\Delta_{GKS}) = \pi_{3,*}((Z \times C) \cdot \Delta_{GKS}) \in \mathrm{CH}^1_{hom}(C) \simeq \mathrm{Jac}(C)$$

where $\pi_3 : C^3 \to C$ is the projection via the third coordinate.

$$\Pi_Z(\Delta_{GKS}) \neq 0 \Rightarrow \Delta_{GKS} \neq 0.$$

Key: Intersect a 1-cycle in a 3-fold with a 2-dim subvariety and project down to a curve to get a divisor on a curve which detects non-triviality.



Next: Use this construction to show Ceresa/Δ_{GKS} non-triviality for modular curves, inspired by work of Eskandari–Murty on Fermat curves.

Main Result

Theorem (Kerr–L.–Qiu–Yang, 2024)

Let N be a positive integer such that there exists a weight 2 normalized newform $f \in \mathcal{S}_2^{new}(\Gamma(N))^-$ satisfying $L'(f,1) \neq 0$.

Let Γ be a congruence subgroup satisfying

$$\Gamma \subset \Gamma_1(2N) \cap \Gamma(2) \subset \mathrm{SL}_2(\mathbb{Z})$$

and let $X = \Gamma \backslash \overline{\mathbb{H}}$ be the modular curve associated to Γ.

Then the Ceresa and Δ_{GKS} cycles associated to X are nontrivial in the the Chow groups $\mathrm{CH}^{g-1}(\mathrm{Jac}(X)) \otimes \mathbb{Q}$ and $\mathrm{CH}^2(X^3) \otimes \mathbb{Q}$.

Remark: The condition on N is satisfied if $N \gg 0$.

Heegner divisors: 0-cycles on modular curves

Consider the modular curve $X_0(N)$ on which a point x represents a cyclic degree N isogeny $E_1 \to E_2$ between generalized elliptic curves.

Heegner divisor: fix $D < 0$, $(D, N) = 1$, the discriminant of an order $\mathcal{O}_D \subset \mathbb{Q}(\sqrt{D})$, and $r \in \mathbb{Z}/2N\mathbb{Z}$ satisfying $r^2 \equiv D \bmod 4N$, there exists $x \in X_0(N)$ representing $\pi : E_1 \to E_2$ making the diagram commute

$$\begin{tikzcd} E_1 \arrow[r, "\pi"] \arrow[d, "\mathcal{O}_D"] & E_2 \arrow[d, "\mathcal{O}_D"] \\ E_1 \arrow[r, "\pi"] & E_2 \end{tikzcd}$$

where π is annihilated by $N\mathbb{Z} + \frac{r+\sqrt{D}}{2}\mathbb{Z}$.

Let point $P_{D,r} \in \mathrm{Jac}(X_0(N))$ denote the degree 0 divisor given by the sum of such x subtracting a multiple of the rational cusp ∞.

Then $P_{D,r} + P_{D,-r}$ is defined over $\mathbb{Q}$.

Heegner divisor is a linear combination of CM points on a modular curve.

Existence of non-torsion Heegner divisor

Gross–Zagier: Let $f \in \mathcal{S}_2^{new}(\Gamma_0(N))^-$ be a new form, $K = \mathbb{Q}(\sqrt{D})$, and

$$L_K(f,s) = L(f,s)L(f \otimes \chi_D, s).$$

Then the height of the f-isotypical component of $P_{D,r}$ is given by

$$L_K'(f,1) = \frac{8\pi^2(f,f)}{h\omega^2|D|^{1/2}} h(P_f)$$

and $h(P_f) \neq 0$ implies the Heegner divisor $P_{D,r} + P_{D,-r}$ is non-torsion.

Consequences: 1. When there exists $f \in \mathcal{S}_2^{new}(\Gamma_0(N))^-$ satisfying $L'(f,1) \neq 0$, there exists non-torsion Heegner divisors on $\mathrm{Jac}(X_0(N))$.

2. For $N \gg 0$, there exists $f \in \mathcal{S}_2^{new}(\Gamma_0(N))^-$ satisfying $L'(f,1) \neq 0$. Thus, we have non-torsion Heegner divisors.

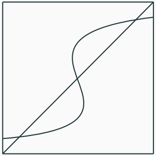
Now we use this tool on the Δ_{GKS} cycle associated to a modular curve.

Main strategy

By direct computation,

$$\Pi_Z(\Delta_{GKS}) = Z \cap \delta - Z \cap (e \times C) - Z \cap (C \times e) - *e$$

where $\delta \subset C \times C$ is the diagonal and $*e$ is to make $\Pi_Z(\Delta_{GKS})$ degree 0.



$\delta : C \hookrightarrow C \times C$

Note that when C is a modular curve and Z a Hecke correspondence, the divisor $\Pi_Z(\Delta_{GKS})$ is supported on CM points, a linear combination of Heegner divisors.

It would be nice to prove its nontriviality but unfortunately unclear.

Main strategy

Our strategy: Fix a non-torsion Heegner divisor $P_{D,r} + P_{D,-r}$ and we show there exists a linear combination of Hecke correspondences Z satisfying $\Pi_Z(\Delta_{GKS}) = P_{D,r} + P_{D,-r}$.

Theorem (Kerr–L.–Qiu–Yang, 2024)

Let $X_N = \Gamma\backslash\overline{\mathbb{H}}$ be the modular curve associated to the congruence subgroup $\Gamma_N := \Gamma_1(2N) \cap \Gamma(2)$.

A Heegner divisor $P_{D,r} + P_{D,-r}$ on $X_0(N)$ can be obtained as the composition of pullback δ^* and pushforward π_*.

Namely, there exists a rational linear combination of special divisors Z on $X_N \times X_N$ such that $P_{D,r} + P_{D,-r} = \pi_* \circ \delta^*(Z)$.

$$\begin{tikzcd} X_N \arrow[r, hook, "\delta"] \arrow[d, "\pi"] & X_N \times X_N \arrow[d] \\ X_0(N) \arrow[r, hook] & X_0(N) \times X_0(N) \end{tikzcd}$$

Thanks