Equidistribution and unlikely intersections in arithmetic dynamics

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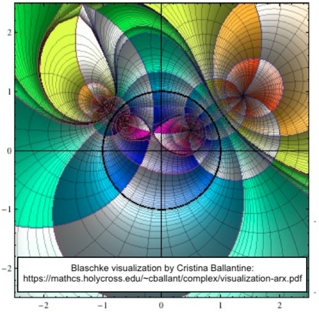
Theorem (Ihara-Serre-Tate)

Let $f(x,y) \in \mathbb{C}[x,y]$ be a non-zero polynomial. Suppose the equation $f(x,y) = 0$ has infinitely many solutions (ζ, η) with ζ, η roots of unity. Then $f(x,y)$ has a factor of the form $x^a - cy^b$, with $a, b \in \mathbb{Z}$ and c a root of unity.

This is NOT a purely geometric phenomenon:

$$y = \prod_{i=1}^{n} \frac{x - a_i}{\overline{a_i}x - 1} \qquad |a_i| < 1.$$

$$f(x,y) := \prod_{i=1}^{n}(x - a_i) - y \prod_{i=1}^{n}(\overline{a_i}x - 1)$$



Blaschke visualization by Cristina Ballantine:
https://mathcs.holycross.edu/~cballant/complex/visualization-arx.pdf

Roots of unity are torsion points of $\mathbb{C}^*$, and polynomials $x^a - cy^b$, with c a root of unity are torsion translates of algebraic subgroups of $\mathbb{C}^* \times \mathbb{C}^*$.

algebra $\longleftrightarrow$ geometry $\longleftrightarrow$ arithmetic

The principle of unlikely intersections: a variety is unlikely to contain a Zariski dense subset of special points unless the variety is itself special.

We will focus on the Manin-Mumford conjecture.

Theorem (Raynaud '83)

Let X be a smooth projective curve of genus $g \geq 2$ defined over a number field K, and let

$$j_p : X \hookrightarrow Jac(X)$$

an Abel-Jacobi embedding of X into its Jacobian based at $p \in X(\overline{K})$. Then $j_p(X) \cap Jac(X)^{\mathrm{tor}}$ is finite.

More generally, Raynaud proved:

> A subvariety Y of an abelian variety A contains a Zariski dense subset of torsion points if and only if Y is a torsion translate of an abelian subvariety of A.

Manin-Mumford, Mordell-Lang, André-Oort, Pink-Zilber...see book of Umberto Zannier: https://www.jstor.org/stable/j.ctt7rndx

special = torsion

special = in division group of fin. gen. subgroup

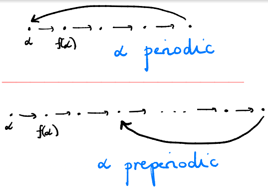
special = CM

Dynamically special subvarieties

Let S be a set, and $f : S \to S$ a map. We will write

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$$

for the nth composition of f.



Definition

Let S be a set, and $f : S \to S$ a map. The *(forward) orbit* of a subset $T \subset S$ is the sequence of iterates $\{f^n(T) : n \in \mathbb{N}\}$.

If T has finite forward orbit, then T is *preperiodic*. If $f^n(T) = T$ for some $n \in \mathbb{N}$, then p is *periodic*.

Dynamical reinterpretation of torsion points:

Lemma

Let G be a group (with operation $+$), and $[m]$ the multiplication-by-m map

$$[m]g := \underbrace{g + g + \cdots + g}_{m \text{ times}}.$$

Then g is torsion in G if and only if g is a preperiodic point for $[m]$.

Dynamically special subvarieties

This simple observation is a hint towards connections between dynamics and arithmetic geometry. Call and Silverman initiated the study of arithmetic dynamics by establishing a dictionary between arithmetic geometry of abelian varieties and of algebraic dynamics.

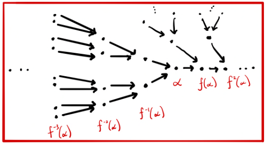
arithmetic of elliptic curves $\longleftrightarrow$ complex dynamics

- torsion $\longleftrightarrow$ preperiodic
- canonical height $\longleftrightarrow$ dynamical height
- $\mathbb{Z}$ and $\mathbb{Q}$ subgroups of points $\longleftrightarrow$ forward and grand orbits
- complex multiplication elliptic curve $\longleftrightarrow$ post-critically finite map

Mordell-Lang

André-Oort.

grand orbit of α



post-critically finite map

$$\mathcal{P}_f := \bigcup_{f'(c)=0} \bigcup_{n\geq 1} f^n(c)$$

is a finite set.

Dynamically special subvarieties

With this analogy, one may replace the 'classical' Manin-Mumford conjecture with a dynamical version.

Conjecture (Zhang '06) [DYNAMICAL MANIN-MUMFORD CONJECTURE]

Let $f : X \to X$ be a polarized endomorphism of a complex projective variety X. A subvariety Y of X contains ~~infinitely many~~ Zariski dense preperiodic points of f if and only if Y is preperiodic for f.

Necessity of condition:

$f(x, y) = (x^2, y^3)$ on $\mathbb{A}^2$

$f^n(x, y) = (x^{2^n}, y^{3^n})$

$(x,y) \in \mathrm{Prep}(f) \iff x, y \in \mu$

$f^n(\Delta) \neq f^m(\Delta)$

$\forall n \neq m$.

Definition

$f : X \to X$ is *polarized* if there exists an ample line bundle $\mathcal{L}$ on X and an integer $d \geq 2$ so that $f^*\mathcal{L} \simeq \mathcal{L}^d$.

Fakhruddin: f polarized $\Rightarrow$ $\exists\, i : X \hookrightarrow \mathbb{P}^N$, $F : \mathbb{P}^N \to \mathbb{P}^N$ such that

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook, "i"'] & X \arrow[d, hook, "i"] \\ \mathbb{P}^N \arrow[r, "F"'] & \mathbb{P}^N \end{tikzcd}$$

The dynamical Manin-Mumford conjecture generalizes the 'classical' Manin-Mumford conjecture.

- X an abelian variety over $\mathbb{C}$
- $f = [m]$ with $m > 1$,

then f is polarized, and a subvariety Y is preperiodic if and only if Y is a torsion translate of an abelian subvariety.

Theorem of the cube $\Rightarrow$

$$[m]^*\mathcal{L} \simeq \mathcal{L}^{\frac{m^2+m}{2}} \otimes [-1]^*\mathcal{L}^{\frac{m^2-m}{2}}.$$

So for ample, symmetric $\mathcal{L}$ on X abelian variety, we have

$$[m]^*\mathcal{L} \simeq \mathcal{L}^{m^2}.$$

For other endomorphisms of abelian varieties,

'Y is preperiodic $\Leftrightarrow$ Y is a torsion translate of an abelian subvariety'

does NOT always hold.

Counterexample of Ghioca-Tucker-Zhang '11, Pazuki '10:

E : CM elliptic curve, $\mathrm{End}(E) \simeq R$ order in imaginary quadratic

- $\alpha, \beta \in R$, $|\alpha| = |\beta|$.

(ex: $\alpha = 5$, $\beta = 3+4i$, E square lattice)

Then Δ_E is preperiodic for $([\alpha],[\beta]) : E \times E \to E \times E$

$\Leftrightarrow$ α/β is a root of unity.

BUT! $\mathrm{Prep}([\alpha],[\beta]) = (E \times E)^{\mathrm{tor}}$ is Zariski dense in Δ_E.

Conjecture (modification by Ghioca-Tucker-Zhang)

Let X be a projective variety, $\phi : X \to X$ a polarized endomorphism, $Y \subset X$ a subvariety with no component in the singular locus of X. Then Y is preperiodic for ϕ if and only if there exists a Zariski dense subset of smooth points $x \in Y \cap \mathrm{Prep}(\phi)$ so that the tangent subspace of Y at x ~~be~~ is preperiodic under the induced action of ϕ on the Grassmanian $\mathrm{Gr}_{\dim Y}(T_{X,x})$.

They proved that their modification holds for:

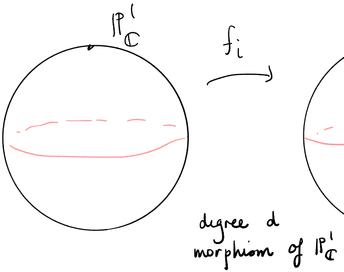
- endomorphisms of abelian varieties
- lines in $\mathbb{P}^1 \times \mathbb{P}^1$

Idea of condition: forbid bad behaviour arising from maps which "come from" algebraic groups.

Question

For which pairs (X, f) of varieties with polarized endomorphisms $f : X \to X$ does the Zhang dynamical Manin-Mumford conjecture hold?

- *split* endomorphisms: maps of the form $(f_1, \ldots, f_m) : (\mathbb{P}^1)^m \to (\mathbb{P}^1)^m$



$$f_i(z) = \frac{a_d z^d + \cdots + a_0}{b_d z^d + \cdots + b_0}$$

$\left(d \geq 2 \quad a_d, b_d \quad \text{not both zero} \right)$

THE DYNAMICAL MANIN-MUMFORD CONJECTURE AND THE DYNAMICAL BOGOMOLOV CONJECTURE FOR ENDOMORPHISMS OF $(\mathbb{P}^1)^n$

D. GHIOCA, K. D. NGUYEN, AND H. YE

Work of :

Baker-Hsia, DeMarco-K.-Ye, Ghioca-Nguyen-Ye, Ghioca-Tucker, Ghioca-Tucker-Zhang...

The dynamical Manin-Mumford conjecture

Question

For which pairs (X, f) of varieties with polarized endomorphisms $f : X \to X$ does the dynamical Manin-Mumford conjecture hold?

- *split* endomorphisms: maps of the form $(f_1, \ldots, f_m) : (\mathbb{P}^1)^m \to (\mathbb{P}^1)^m$
- polynomial automorphisms of $\mathbb{A}^2$ (Dujardin-Favre)
- monomial maps of toric varieties (Lin)
- lifts of the Frobenius (Medvedev-Scanlon, Xie)

Dujardin-Favre:

maps of Hénon type

\+

condition on $\mathrm{Jac}(f)$

Medvedev-Scanlon, Xie:

In this paper, we write $\mathbb{C}_p$ for the completion of the algebraically closure of $\mathbb{Q}_p$ with the induced norm. Denote by $\mathbb{C}_p^{\circ}$ its valuation ring and $\mathbb{C}_p^{\circ\circ}$ the maximal ideal of $\mathbb{C}_p^{\circ}$. Let $F : \mathbb{P}^N_{\mathbb{C}_p} \to \mathbb{P}^N_{\mathbb{C}_p}$ be an endomorphism taking form

$$F : [x_0 : \cdots : x_N] \mapsto [x_0^q + p' P_0(x_0, \cdots, x_N) : \cdots : x_N^q + p' P_N(x_0, \cdots, x_N)]$$

where q is a power of p, $p' \in \mathbb{C}_p^{\circ\circ}$, and $P_0, \cdots, P_N$ are homogeneous polynomials of degree q in $\mathbb{C}_p^{\circ}[x_0, \cdots, x_N]$. We say that F is *a lift of Frobenius on* $\mathbb{P}^N_{\mathbb{C}_p}$.

K a number field, $f : X \to X$ a dominant endomorphism of a projective variety, polarized by $f^*\mathcal{L} \simeq \mathcal{L}^d$, $d \geq 2$, all defined over K.

Theorem (Néron-Tate, Call-Silverman)

There is a well-defined function $h_f : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ given by

$$h_f(P) := \lim_{n\to\infty} \frac{h(f^n(P))}{d^n},$$

known as the canonical dynamical height associated to f. This height has the following properties:

- $h_f(f(P)) = dh_f(P)$,
- $h_f(P) = h(P) + \mathcal{O}(1)$,
- *h_f has a local decomposition. If $P \in L/K$ is not in the support of $\mathcal{L}$,*

$$\left(\hat{h}_f(P) =\right) \quad h_f(P) = \frac{1}{[L:K]} \sum_{v \in M_L} [L_v : K_v] \lambda_{f,\mathcal{L},v}(P),$$

where the $\lambda_{f,\mathcal{L},v}$ are local canonical height functions.

and $h_f = h + O(1)$!!

By Northcott's theorem and $h_f(f(P)) = dh_f(P)$, a point $P \in X(\overline{K})$ is preperiodic if and only if $h_f(P) = 0$.

Example. Let $X = \mathbb{P}^1$, $f(z) = z^d$. By definition,

$$h_f(P) = \lim_{n\to\infty} \frac{h(P^{d^n})}{d^n} = h(P).$$

The preperiodic points are those of Weil height zero: roots of unity, 0, and ∞.

Example. Let $X = A$ an abelian variety, with $f = [m]$ for some $m \geq 2$. Then h_f agrees with the Néron-Tate height on A associated to $\mathcal{L}$, as do the local heights.

The arithmetic

Example. Let $E_t : y^2 = x(x-1)(x-t), t \neq 0, 1$ be a Legendre elliptic curve, and $\pi : E \to \mathbb{P}^1$ the x-coordinate projection. The following diagram commutes:

unramified →

2-to-1 →

degree 4 self-map of $\mathbb{P}^1$ →

$$\begin{tikzcd} E_t \arrow[r, "{[2]}"] \arrow[d, "\pi"'] & E_t \arrow[d, "\pi"] \\ \mathbb{P}^1 \arrow[r, "f_t"'] & \mathbb{P}^1 \end{tikzcd}$$

$$f_t(z) = \frac{(z^2 - t)^2}{4z(z-1)(z-t)}$$

The height h_f on $\mathbb{P}^1$ is $h_{f_t}(\pi(P)) = \hat{h}_t(\pi(P))$, where $\hat{h}_t$ is Néron-Tate on E_t.

This type of map is known as a (flexible) Lattès map, and the preperiodic points are the images under π of torsion points of E_t.

$\Rightarrow$ $\mathrm{Prep}(f_t)$ dense in $\mathbb{P}^1_{\mathbb{C}}$.

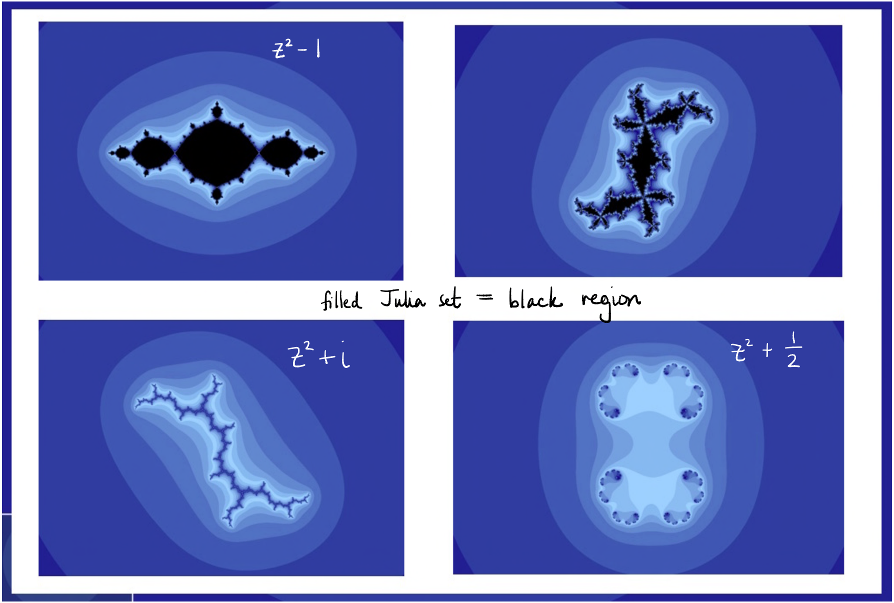
For self-maps of $\mathbb{P}^1$, these local heights are not mysterious! They are most easily understood for polynomials, where they are Green's functions for the dynamical object known as a *filled Julia set*.

Let's look at the archimedean setting, viewing f as a dynamical system $f : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ by composition.

- **Fatou set** of f: the collection of $\alpha \in \mathbb{C}$ for which the iterates $\{f^n\}_{n\in\mathbb{N}}$ form a normal family on a neighborhood of α (orbits near α behave like the orbit of α).
- **Julia set** $\mathcal{J}(f)$: the complement of the Fatou set; no prediction possible for orbits near α.
- if f is a polynomial **filled Julia set** $K(f)$ of f: the collection of $\alpha \in \mathbb{C}$ which remain bounded under iteration:

$$K(f) := \{\alpha \in \mathbb{C} : |f^n(\alpha)| \not\to \infty\}.$$

For polynomials, the Julia set is alternately characterized as the topological boundary of the filled Julia set $\mathcal{K}(f)$.

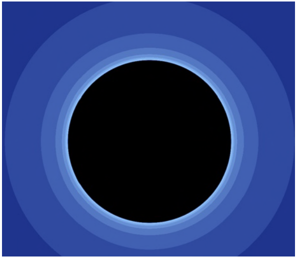

$z^2 - 1$
filled Julia set = black region
$z^2 + i$
$z^2 + \frac{1}{2}$

Example: $f(z) = z^2$.

We can actually compute $f^{\circ n}(z) = z^{2^n}$.

- $|\alpha| < 1 \Rightarrow f^n(\alpha) \to 0$
- $|\alpha| > 1 \Rightarrow f^n(\alpha) \to \infty$
- $|\alpha| = 1$ behavior depends on whether α is a root of unity or not.

Immediate consequence: $\mathcal{J}(f) = S^1$.



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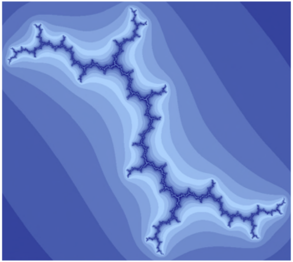
General phenomena illustrated:

- All but finitely many preperiodic points are contained in the Julia set.
- There is an invariant probability measure μ_f supported on the Julia set: $f^*\mu_f = d\mu_f, f_*\mu_f = \mu_f$. This is known as the equilibrium measure for f.

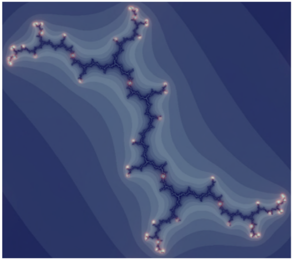
Theorem (Friere-Lopes-Mañe, Lyubich '83)

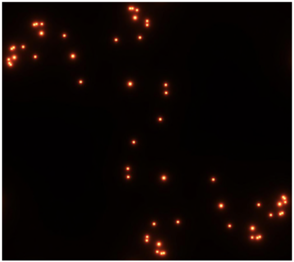
Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a degree d morphism. There exists a unique f-invariant probability measure μ_f supported on $J(f)$ such that for all but at most two points $\alpha \in \mathbb{P}^1$,

$$\frac{1}{d^n} \sum_{f^n(x)=\alpha} \delta_x \longrightarrow \mu_f \quad \textit{in the weak-star topology.}$$



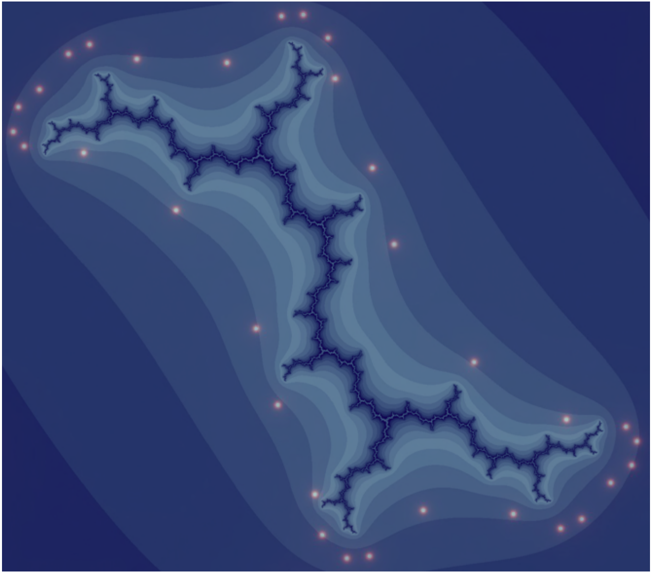
Julia set of z^2+i





$\le 6^{th}$ preimages under z^2+i of $\alpha = i$.

Archimedean equidistribution: a first result

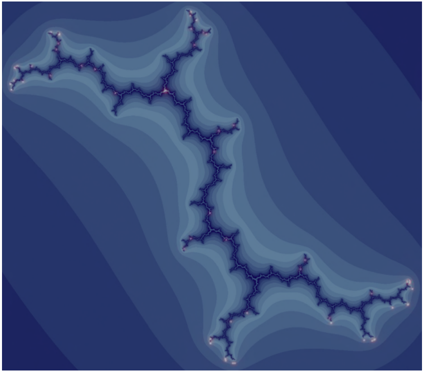


$\leq 5^{th}$ preimages of $\alpha = 9$ under $z^2 + i$.

Theorem (Lyubich '83)

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a degree d morphism, and for $n \geq 1$ let $\mathrm{Per}_n(f)$ denote the set of points of period n for f. Then

$$\frac{1}{|\mathrm{Per}_n(f)|} \sum_{x \in \mathrm{Per}_n(f)} \delta_x \longrightarrow \mu_f \quad \textit{in the weak-star topology.}$$



points of period $n \in \{1,2,3,4,5\}$
for $f(z) = z^2 + i$.

Arithmetic equidistribution: the hammer

Note that the relation $h_f(f(\alpha)) = dh_f(\alpha)$ implies that if α is a point and x_n an nth preimage of α (all algebraic), then

$$h_f(x_n) = \frac{1}{d^n} h_f(\alpha) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus in the arithmetic setting, the previous two equidistribution results are unified if we view $\{x : f^n(x) = \alpha\}$ and $\mathrm{Per}_n(f)$ as Galois orbits of points of small height, as done by Szpiro, Ullmo, and Zhang.

for abelian varieties.

Theorem (Baker-Rumely, Chambert-Loir, Favre-Rivera-Letelier)

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a degree d morphism defined over a number field K. Let $x_n \in \mathbb{P}^1(\overline{K})$ be a set of points satisfying $h_f(x_n) \to 0$ as $n \to \infty$, and write $G_n = \mathrm{Gal}(\bar{K}/K)x_n$ for the Galois orbit of x_n. Then for any $v \in M_K$,

$$\frac{1}{|G_n|} \sum_{x \in G_n} \delta_x \longrightarrow \mu_{f,v} \quad \textit{in the weak-star topology},$$

where $\mu_{f,v}$ is the v-adic equilibrium measure for f on $\mathbb{P}^{1,an}_v$.

Arithmetic equidistribution: the hammer

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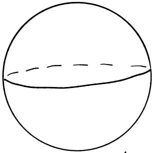
where $\mu_{f,v}$ is the v-adic equilibrium measure for f on $\mathbb{P}^{1,an}_v$.

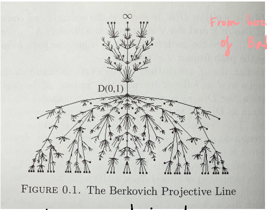
$\mu_{f,v}$ is related to the local height $\lambda_{f,\mathcal{L},v}$ by the Laplacian (in the sense of distributions): a local height can be recovered from the local measure.

$\mathbb{P}^{1,an}_v$ is the v-adic Berkovich projective line:

$\mathbb{P}^{1,an}_v$:



v archimedean



From book of Baker-Rumely

v nonarchimedean

Yuan '12: generalization of equidistribution of points of small height to $f : \mathbb{P}^n \to \mathbb{P}^n$ and thus to polarized endomorphisms.

Theorem (Baker-Rumely, Chambert-Loir, Favre-Rivera-Letelier)

Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a degree d morphism defined over a number field K. Let $x_n \in \mathbb{P}^1(\overline{K})$ be a set of points satisfying $h(x_n) \to 0$ as $n \to \infty$, and write $G_n = \mathrm{Gal}(\bar{K}/K)x_n$ for the Galois orbit of x_n. Then for any $v \in M_K$,

$$\frac{1}{|G_n|} \sum_{x \in G_n} \delta_x \longrightarrow \mu_{f,v} \quad \textit{in the weak-star topology,}$$

where $\mu_{f,v}$ is the v-adic equilibrium measure for f on $\mathbb{P}^{1,an}_v$.

Takeaway: for maps defined over a number field, any Zariski dense and sufficiently generic subset of preperiodic points determines an adelic equilibrium measure associated to the map.

Arithmetic equidistribution allows us to translate questions on the geometry of preperiodic points to questions of measure classification.

Let's return to the split dynamical Manin-Mumford question.

General plan of attack over $\overline{\mathbb{Q}}$, based on ideas of Szpiro-Ullmo-Zhang and Baker-DeMarco.

Let $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ be degree $d \geq 2$ morphisms defined over a number field K. Suppose the diagonal subvariety $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ contains infinitely many preperiodic points for (f, g). Then f and g have infinitely many common preperiodic points.

Step 1: equidistribution. By arithmetic equidistribution, infinitely many common preperiodic points ensures that f and g have the same:

- adelic equilibrium measures,
- adelic dynamical height functions,
- set of preperiodic points,
- Julia sets.

Step 2: measure classification.

Theorem (Levin '90)

If two rational maps f, g of degree $d \geq 2$ of $\mathbb{P}^1_{\mathbb{C}}$ share the same equilibrium measures and the same set of preperiodic points, then either f, g are exceptional, or $f^k \circ g^k = f^{2k}$ for some $k \geq 1$.

From this we see that $(f^k, g^k)(\Delta) \subset (f^{2k}, g^{2k})(\Delta)$, so by irreducibility Δ is preperiodic as conjectured. QED

The classification theorem really must be a *global* statement: there are plenty of rational maps with Julia set $\mathbb{P}^1$, for example.

Generalities: algebraic correspondence instead of equality, working in higher dimensions, results over $\mathbb{C}$, ...

Example: quadratic polynomials

Let $f(z) = z^2 - 2$ and $g(z) = z^2 - 6$. $\mu_{f,v} = \mu_{g,v}$ for all non-archimedean places $v \in M_{\mathbb{Q}}$, the Julia sets at the archimedean place have infinite overlap, and each has all finite preperiodic points contained in the interval $[-3, 3]$.

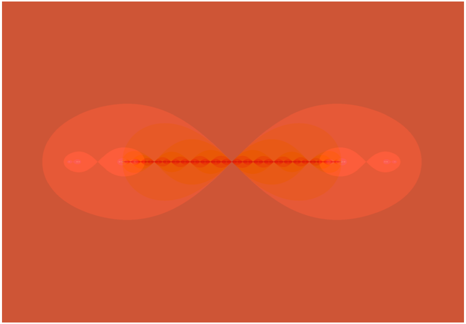


Figure: archimedean Julia overlap for $z^2 - 2$ and $z^2 - 6$.

Nonetheless, $z^2 - 2$ and $z^2 - 6$ have only finitely many common preperiodic points, and the diagonal is not preperiodic under (f, g) on $\mathbb{P}^1 \times \mathbb{P}^1$.

Let X be the smooth genus 2 hyperelliptic curve with affine model

$$C : y^2 = x^6 - rx^4 + sx^2 - 1 = (x^2 - \alpha_1)(x^2 - \alpha_2)(x^2 - \alpha_3),$$

where the α_i are distinct.

$(x, y) \mapsto (x^2, y)$ provides a double cover $X \to E_1$, with

$$E_1 : y^2 = x^3 - rx^2 + sx - 1$$

and $(x, y) \mapsto (1/x^2, iy/x^3)$ provides a double cover $X \to E_2$ with

$$E_2 : y^2 = x^3 - sx^2 + rx - 1.$$

$\pi_x : E_1[2] \longrightarrow \{0, \alpha_1, \alpha_2, \alpha_3\}$

$1/\pi_x : E_2[2] \longrightarrow \{0, \alpha_1, \alpha_2, \alpha_3\}$

$\Longrightarrow$ $\exists t_1, t_2$ and π_1, π_2 double covers

$\pi_i : C \longrightarrow E_{t_i}$

$E_{t_i} : y^2 = x(x-1)(x-t_i)$

Let $P = (\pm\sqrt{\alpha_i}, 0)$. We have the following diagram relating torsion points of X in its Jacobian via j_P to preperiodic points on $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{tikzcd}
X \arrow[r, hook] \arrow[d] & \mathrm{Jac}(X) \arrow[d] & \text{torsion pts} \arrow[l] \arrow[dd, mapsto] \\
C \arrow[r] \arrow[d] & E_{t_1} \times E_{t_2} \arrow[d] & \\
\Delta \arrow[r, hook] & \mathbb{P}^1 \times \mathbb{P}^1 & \text{preperiodic pts for } (f_{t_1}, f_{t_2}). \arrow[l]
\end{tikzcd}$$

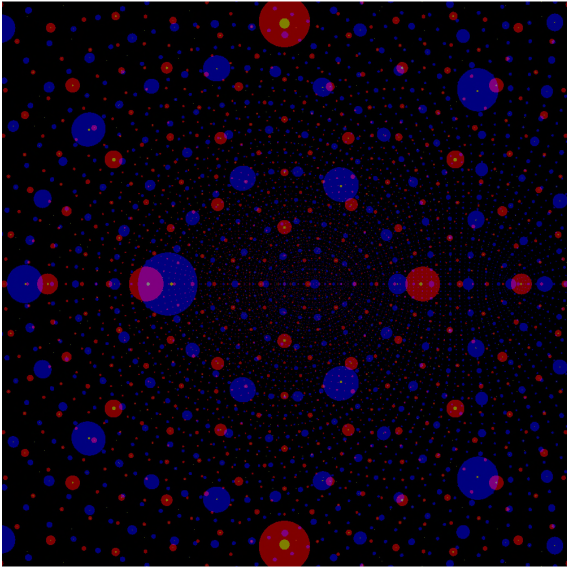
If $t_1, t_2 \in \overline{\mathbb{Q}}$ and $t_1 \neq t_2$, then the diagonal cannot be preperiodic for (f_{t_1}, f_{t_2}), so $j_P(X) \cap J(X)^{\mathrm{tor}}$ is finite.

Question (Quantitative dynamical Manin-Mumford)

Suppose $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ with $h_f \neq h_g$. How large can

$$|\mathrm{Preper}(f) \cap \mathrm{Preper}(g)|$$

be?



Common torsion images for $t_1 = -1, t_2 = 4$

blue: close to torsion for $t_2 = 4$

red: close to torsion for $t_1 = -1$

dense sets with finite overlap.

Answer: as large as we want, if we allow the degrees of f, g to grow.

$$f(z) = z(z-1)(z-2)\cdots(z-n) \quad g(z) = z^2(z-1)(z-2)\cdots(z-n)$$

* $0 \in J(f), \quad 0 \notin J(g)$.

Conjecture (DeMarco-K.-Ye '20)

Fix $d \geq 2$. There exists a uniform constant B_d so that for all $f, g : \mathbb{P}^1 \to \mathbb{P}^1$ of degree d,

$$|\mathrm{Prep}(f) \cap \mathrm{Prep}(g)| \leq B_d$$

whenever $h_f \neq h_g$.

By [DeMarco-K.-Ye '19, '20], this conjecture holds for:

- $f(z) = z^2 + c_1, g(z) = z^2 + c_2$ when $c_1 \neq c_2$.
- $f(z) = f_{t_1}(z), g(z) = f_{t_2}(z)$, where f_{t_i} is the Legendre Lattès map associated to E_{t_i}. $t_1 \neq t_2$

Key ingredients: quantitative equidistribution, arithmetic intersection pairing, degeneration of dynamical measures in non-compact moduli, ...

Question (geometric uniform Manin-Mumford question)

Fix $g \geq 2$. Does there exist a uniform constant $B = B(g)$ so that for all curves X of genus g,

$$|j(X) \cap J(X)^{\mathrm{tor}}| \leq B$$

for any Abel-Jacobi embedding j?

Coleman, Hrushovski, Katz-Rabinoff-Zureick-Brown, Dimitrov-Gao-Habegger... depend on degree of field of definition.

In the 2-dimensional family $\mathcal{L}_2 := \{y^2 = x^6 - rx^4 + sx^2 - 1\}$ of genus 2 curves, the answer by D-K-Y is YES.

Question (uniform dynamical Manin-Mumford conjecture)

Fix $N \geq 1, d \geq 2, e \geq 1$. Does there exist a uniform bound $B = B(N, d, e)$ so that whenever $f : \mathbb{P}^N \to \mathbb{P}^N$ is a degree d morphism and $X \subset \mathbb{P}^N$ an algebraic subvariety of degree e,

$$\deg(\overline{\mathrm{Prep}(f) \cap X}) \leq B?$$

Lower bounds on the intersection pairing

Let f, g be degree d rational maps of $\mathbb{P}^1$ defined over a number field K with adelic measures μ_f, μ_g, respectively. Assume for simplicity that $\infty \in \mathrm{Prep}(f) \cap \mathrm{Prep}(g)$. We define the *height pairing* of f and g to be

$$h_f \cdot h_g := \frac{1}{2} \sum_{v \in M_K} n_v \left(\int_{\mathbb{P}_v^{1,an}} (\lambda_{f,v} - \lambda_{g,v}) \; d\mu_{g,v} + \int_{\mathbb{P}_v^{1,an}} (\lambda_{g,v} - \lambda_{f,v}) \; d\mu_{f,v} \right).$$

$h_f \cdot h_g = 0$ iff $h_f = h_g$; more generally, the pairing gives a quantitative measure of the difference between the two adelic height functions.

Theorem (DKY)

There exists a positive constant $\delta > 0$ so that for all $t_1 \neq t_2 \in \overline{\mathbb{Q}} \setminus \{0, 1\}$, the Legendre Lattès maps f_{t_1} and f_{t_2} satisfy

$$h_{f_{t_1}} \cdot h_{f_{t_2}} \geq \delta.$$

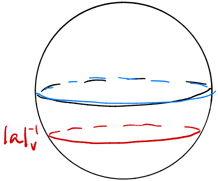
We cannot generally have a uniform lower bound on the pairing, even with fixed degree.

Example. Let $f(z) = z^2, g(z) = az^2$, $a \in K^*$. Then h_f is the standard height, and h_g scaled at all places with $|a|_v \neq 1$. One easily computes

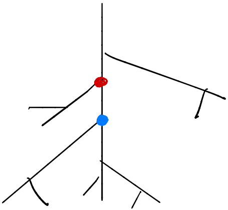
$$h_f \cdot h_g = h(a);$$

in particular, $h_f \cdot h_g$ can be arbitrarily close to 0 with $h_f \neq h_g$.

$\mu_{f,v}$ vs $\mu_{g,v}$



v archimedean : Lebesgue on radius $|a|_v^{-1}$ circle



v non-archimedean : different point mass.

Proposition (DeMarco-K.-Nguyen-Tucker-Ye)

Given a dynamical height h, let

$$S(h) := \{A \in PSL_2(\overline{\mathbb{Q}}) : h \circ A = h\}.$$

If $S(h)$ is finite, then there exists ϵ_h so that for all $PSL_2(\overline{\mathbb{Q}}) \setminus S(h)$,

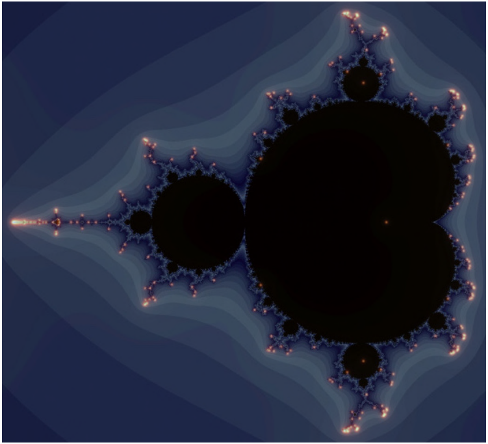
$$h \cdot (h \circ A) \geq \epsilon_h.$$

Question

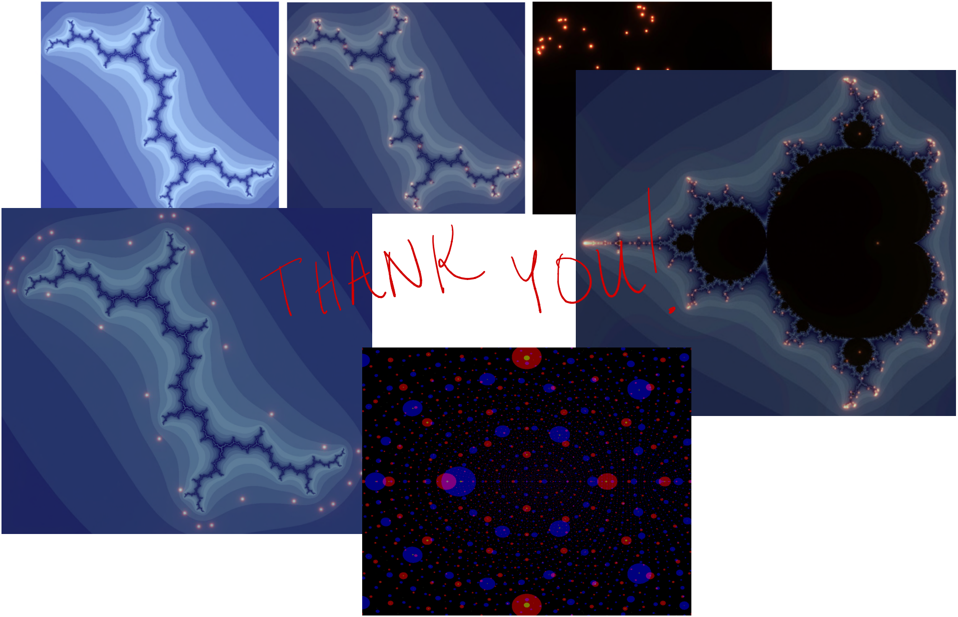
Fix $d \geq 2$. Does there exist a constant $\delta_d > 0$ so that for all f, g degree d morphisms of $\mathbb{P}^1$ with $S(h_f), S(h_g)$ finite,

$$h_f \cdot h_g \geq \delta_d?$$

Arithmetic equidistribution has provided the framework for a number of other conjectures and results in dynamics, one of which is a dynamical analogue of the André-Oort conjecture. Families of rational maps also can come with (less nice) height functions, and the geometry of small height points is very interesting.



The end



THANK YOU!