Polymath-type Projects in the age of Formalized Mathematics

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Polymath

- Origins (~2009) in Gower's blog; social experiment to determine whether "massively collaborative mathematics [is] possible"
- Lots of successful projects, DHJ, Bounded Gaps between Primes (after Yitang Zhang, Maynard), etc   Caveat: "breakthroughs" due to individuals!..
- Idea: Instead of working in relative isolation and secrecy, with the hope of eventually proving something, writing it up, posting to arxiv (without getting scooped), etc…
- Announce intent "publicly" - no scooping - (helps to be Gowers/Tao), encourage participants; all can see the progress unfold
- Random people from around the world can suggest ideas, then project "leaders" (in practice) have to evaluate whether idea is likely to work, what to pursue, what to incorporate. Big Problem:

Polymath

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Big Problem:

- The more individuals contribute, the more you have to Verify for yourself that the idea works (costly, especially if Idea is from area far from your own expertise).
- As a result, you may sometimes be tempted instead to Trust that your collaborators know what they're doing, and that what they claim is correct. (Of course this is where lots of errors happen in practice…)
  Question: Do proportionately more co-authored papers have mistakes?

Trust Vs Verify

- I like to compare proving a big, complicated Theorem to building a Rocket taking people to Mars
- Lots of different components, all have to come together perfectly to successfully execute the mission
- For Rockets: we see right away whether the people arrive safely on Mars
- For Theorems: all the action happens inside the Brains of individuals!
- Imagine what kinds of lame Rockets we would have, if a single person had to manufacture every part themselves from scratch.
- That's what we currently do with math!

(Or At least that's what we tell ourselves! You're expected to understand (nearly) every bit of math that you use in a theorem…)

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Trust Vs Verify

- Buzzard: Does any human truly *know* the proof of FLT? (vs F. Calegari)
- Wiles/Taylor-Wiles uses a dizzying array of techniques from huge landscapes of mathematics (as in Kevin's lecture)
- Related: Many papers use Arthur Trace Formula, Classification of Finite Simple Groups, etc etc … (Makes people nervous)
- What if it didn't have to?
- When working Formally in a theorem prover (e.g., Lean), you can embrace the idea of "taking black boxes off a shelf" because their specs are extremely precise, and guaranteed to work as on the label!
- Imagine what kinds of Rockets (Theorems) we can build (prove) when we become truly collaborative in this way! Caveat: "breakthroughs" due to individuals?..
- Current ongoing project: PNT+

PNT+

- Co-organized with Terry Tao
- Goal: Fermat will need Chebotarev Density Theorem. Special case of that is Dirichlet's theorem (primes in progressions). Don't even have Prime Number Theorem in Lean. So let's get to work!

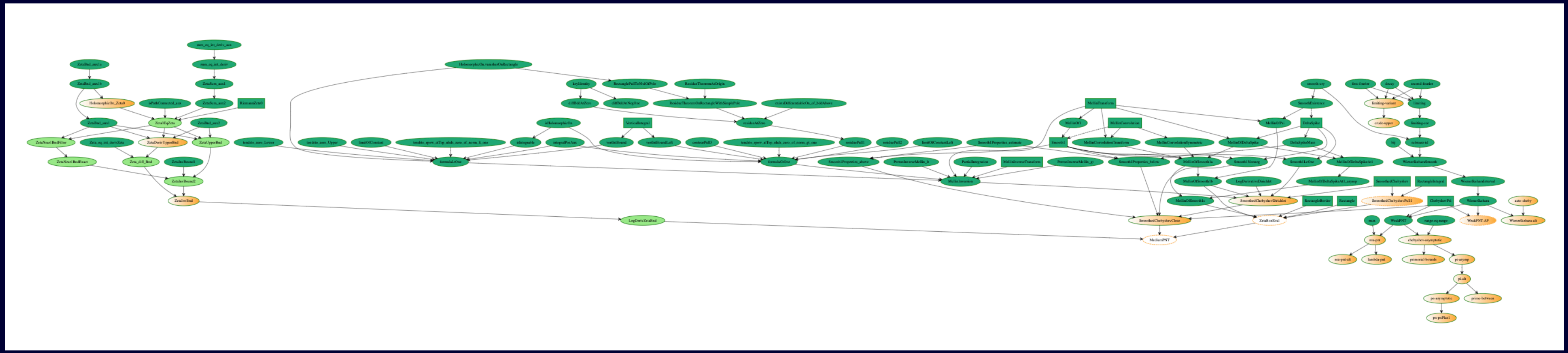
- Note: PNT has been formalized before, many times in fact!
- 2005: Avigad et al in Isabelle (Erdos-Selberg method); 2009: Harrison in HOL-light (Newman's proof); 2016: Carniero in Metamath (Erdos-Selberg); 2018: Eberl-Paulson in Isabelle (Newman)
- We will want to do in it a way that extends to much more general settings.

- Experiment in building a Rocket.

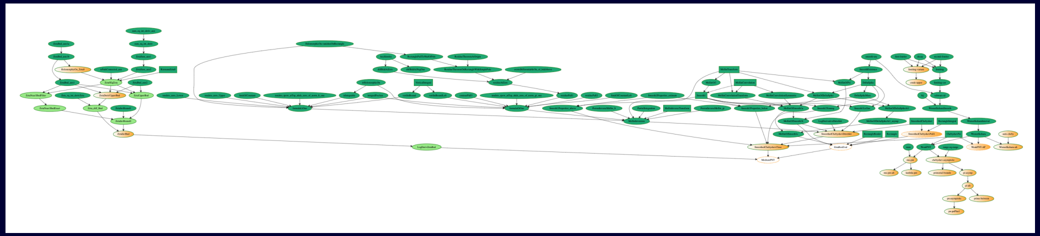
PNT+

- Experiment in building a Rocket.

- Terry had just completed leading a formalization of the Polynomial Frieman-Rusza conjecture (Gowers-Green-Manners-Tao)
- Building on many other similar projects: Sphere Eversion (Patrick Massot), Liquid Tensor Experiment, etc, etc
- Organizational infrastructure: Github + Blueprint + Zulip



PNT+



- Every week (or 3 or 5), I go through and select the “orange” tasks: ready to be completed but not yet done. I compile and post this list of Outstanding Tasks to a dedicated stream in Zulip.

PrimeNumberTheorem+ > Outstanding Tasks, V5

Alex Kontorovich EDITED

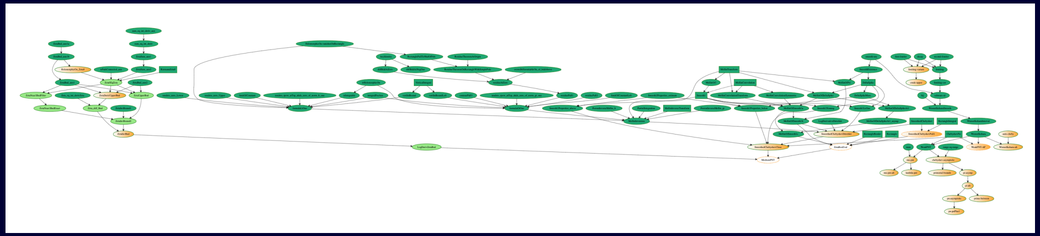
For V4, click here.

I've refactored the Second Approach (see the new file `ZetaBounds`), outlining a method that avoids any more complex analytic developments than those we already have (in particular, no Hadamard, not even Jensen). Just using very classical techniques (partial summation), this gives the error term $O(x\exp(-c(\log x)^{1/10})$ (whereas `StrongPNT` was meant to replace the exponent $1/10$ with $1/2$). I'm thus renaming `WeakPNT2` to `MediumPNT`. This error is basically good enough for just about any application, as it gets a rate that's better than $O(x/\log^C x)$, for any $C > 0$. As far as I'm aware (please correct me?), this will be the first time that an error of this quality is formalized. (We'll see if we even want to continue pushing for `StrongPNT`; we might want to move right on to Dirichlet/Chebotarev... I've temporarily (?) taken the `StrongPNT` section out of the blueprint, so we can see the dependency graph a bit clearer.) This also means we will no longer need any Selberg upper bound sieving (sorry @Arend Mellendijk !).

With those caveats, here's the next round of targets (I tried to break them into really rather small pieces, so there are a few more than usual):

1. sum_eq_int_deriv This is partial summation. I've already almost (modulo some annoying stuff mixing up `uIoo` and `uIcc` in `sum_eq_int_deriv_aux2`) got it proved on intervals of length at most one (`sum_eq_int_deriv_aux`), so now it's just a matter of stringing those together. **Done by me**
2. ZetaSum_aux1 is just an application of partial summation to the power function. Should not be too hard. **Done by me**
3. ZetaSum_aux1a is an auxiliary function, amounts to applying the triangle inequality inside an integral. Shouldn't be hard. **Done by @Paul Nelson**
4. ZetaSum_aux2 is taking `ZetaSum_aux1` and sending $b \to \infty$. **Done by @Heather Macbeth, @Vláďa Sedláček and me**
5. ZetaBnd_aux1 is a combination of 3. and 4., namely applying the triangle inequality inside an integral and sending $b \to \infty$. **Done by @Heather Macbeth, @Vláďa Sedláček and me**

PNT+



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- Once done, they PR (pull request) it in Github.
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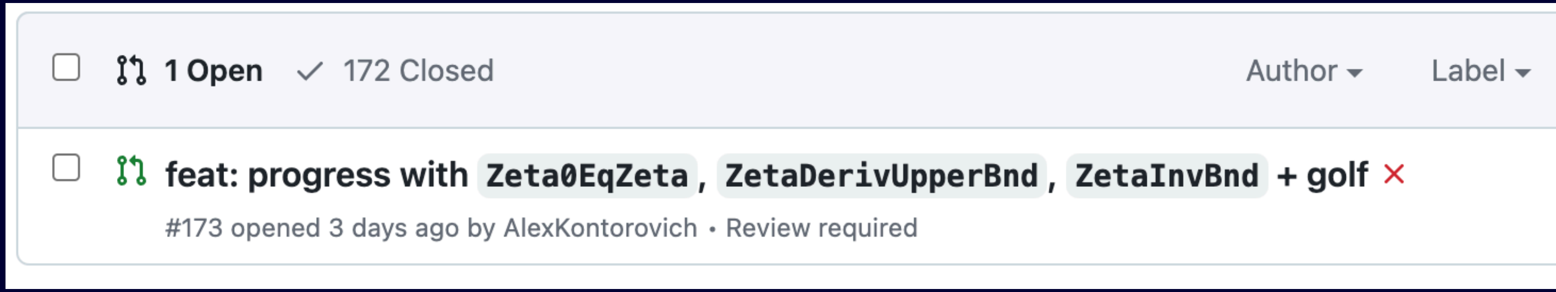

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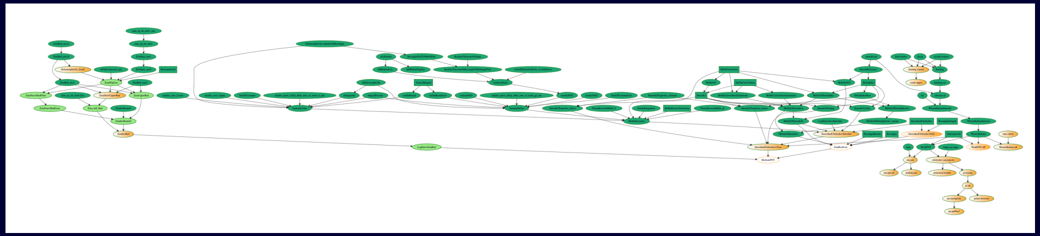

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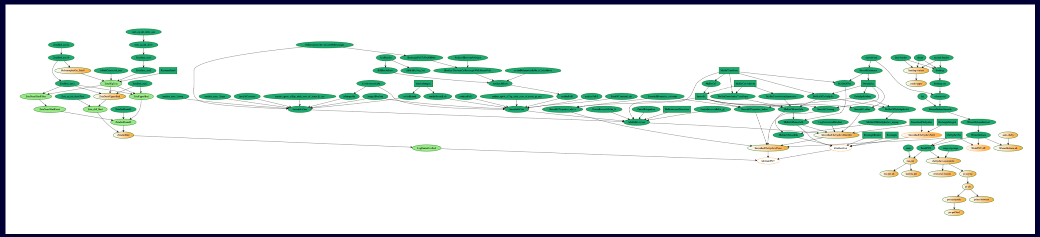
- *All* I have to check is that statement is correct; Lean does the rest!
- Moves very rapidly. Entire blueprint was not prepared in advance, and needed to be “refactored” several times.
- Like a jpeg loading, any time I make a new list of Outstanding Tasks, I need to fill in a lot of details of exactly how to do next steps.
- So, how’s it going?

PNT+



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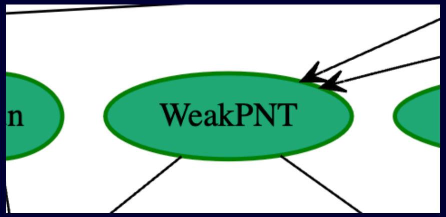
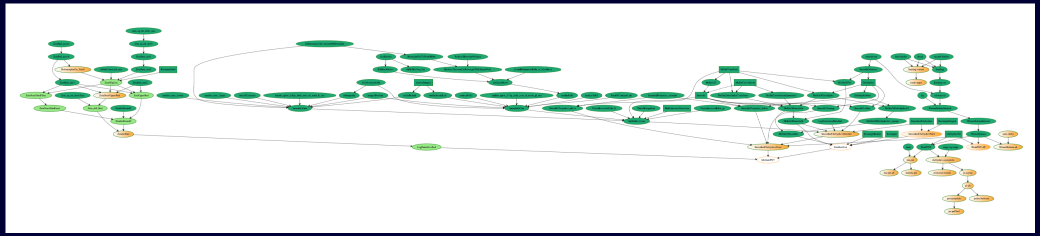


- Opened to the "public" on Jan 31
- Proof of PNT completed on Apr 8
- Not at all a race; we did much more than that in the meantime and since
- Original project comprised three attacks:
  - (I) using "Fourier" methods, Wiener-Ikehara Tauberian theorem. Work of Michael Stoll already reduced PNT to this. (Now done!)
  - (II) developing Mellin transform API (David Loeffler), pulling infinite vertical contours past poles, picking up residues. And
  - (III) Getting a "classical" error savings of $\exp(c(\log x)^{1/2})$ using Hadamard factorization (or local approximation, e.g., Landau)

PNT+



- So, how's it going?

- These were all a great excuse to get more analysis into Mathlib
- We didn't have Fourier inversion (now we do, Sebastian Gouzel)
- We didn't have that Fourier transform of Schwartz function is Schwartz, now we do (Gouzel + K-Loeffler-Macbeth + Beffara)

- We were also missing one of the least developed late undergrad / early grad areas of Mathlib (needed for lots of analytic number theory), namely: Complex Analysis

Complex Analysis

- I first heard about Lean in Sept 2000, when I was teaching Complex. So I thought: I can help!

Def:
$$\int_\gamma f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt$$
$$[z] = \gamma$$

- Lecture 1: Integrating holomorphic functions along curves.
- Problem: What kinds of curves?

- Enough to use piecewise once-continuously differentiable curves. (Mathlib will want something even more general, like differentiable curves whose derivatives are integrable…)

- Ok, so what is that, *exactly*???

Complex Analysis

- Ok, so what is that, *exactly*???
- Def: a curve is a map $\gamma : \mathbb{R} \to \mathbb{C}$ and a set $S \subset \mathbb{R}$ of points (finite? arbitrary? in a sequence, $S : \{0, \ldots, N\} \to \mathbb{R}$?) such that on each interval (s_i, s_{i+1}) (or is it closed interval?), γ is diff'ble and [...]?

- It's *so* easy to just say this as a human, and use it in a wide variety of contexts without even a moment's thought.
- But the decision of how exactly to set up this structure can drastically affect how usable it is in practice!

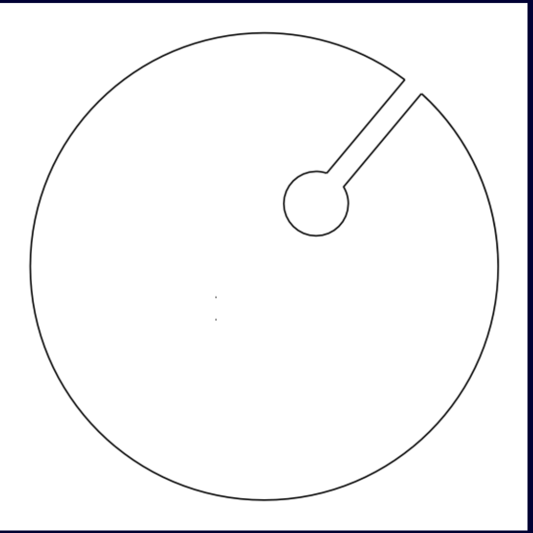
- Cool idea (Vincent Beffara): Can always avoid "piecewise"!

Complex Analysis

- Cool idea (Vincent Beffara): Can always avoid "piecewise"!
- If the curve you want has corners, compose with smooth functions to come to a complete stop at each corner. Then derivative is continuous!
- Next problem (well-known): Need to know "interior" of closed simple curve, i.e., Jordan Curve Theorem.
- This has also been formalized before (e.g., Hales + Mizar 2005)
- Analytic proof is notoriously tedious, 10,000 lines of code.
- Anyway, the "right" Mathlib proof will be a topological statement: if X is homeomorphic to k-sphere, and $Y = \mathbb{R}^n \backslash X$, then reduced integral homology groups $\tilde{H}_q(Y)$ are triv unless $q=n$ or $n-k$; else $\mathbb{Z}$
- So what to do, twiddle thumbs and wait?

Complex Analysis

- So what to do, twiddle thumbs and wait?
- Don’t need Jordan Curve Theorem, can do almost anything you want using explicit “Keyhole” contours
    - Do you really want to code up what this is, making sure to precompose every component with smoothing functions to stop at corners???



- Big Idea: Can do all the Complex Analysis we need just using Rectangles!

Complex Analysis

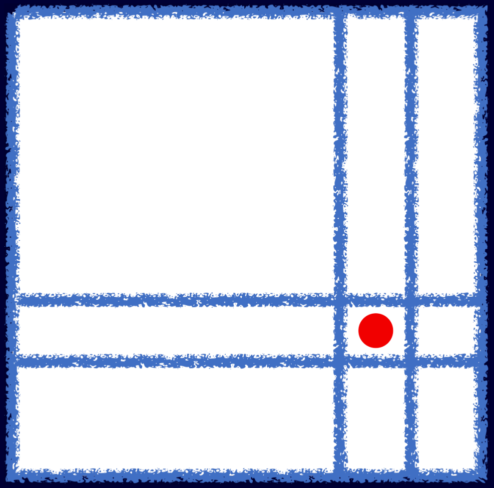
- Big Idea: Can do all the Complex Analysis we need just using Rectangles!
- We have Green's Theorem in Mathlib (Yury Kudryashov), so the integral of a holomorphic function over a rectangle is zero.

```
/-%%
\begin{definition}\label{Rectangle}\lean{Rectangle}\leanok
A Rectangle has corners $z$ and $w \in \C$.
\end{definition}
%%-/
/-- A `Rectangle` has corners `z` and `w`. -/
def Rectangle (z w : ℂ) : Set ℂ := [[z.re, w.re]] ×ℂ [[z.im, w.im]]
```

```
theorem HolomorphicOn.vanishesOnRectangle [CompleteSpace E] {U : Set ℂ}
    (f_holo : HolomorphicOn f U) (hU : Rectangle z w ⊆ U) :
    RectangleIntegral f z w = 0 :=
  integral_boundary_rect_eq_zero_of_differentiableOn f z w (f_holo.mono hU)
```

Complex Analysis

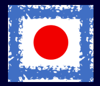
- Big Idea: Can do all the Complex Analysis we need just using Rectangles!
- We have Green's Theorem in Mathlib (Yury Kudryashov), so the integral of a holomorphic function over a rectangle is zero.
- And rectangles tile rectangles! (Not so with disks!)



- Chop big rectangle into 9 smaller rectangles
- 8 of them have integral zero!

- So you can zoom in as much as needed to a neighborhood of a pole

Complex Analysis

- So you can zoom in as much as needed to a neighborhood of a pole 
- Evaluating this boils down to: $$\int_{-\epsilon-i\epsilon}^{\epsilon+i\epsilon} \frac{ds}{s} = \int_{-\epsilon}^{\epsilon} \frac{dx}{x-i\epsilon} + \cdots$$
- Looks complicated! Need complex log??
- No, just add opposite sides together! Here is top + bottom:

$$\int_{-\epsilon}^{\epsilon} \frac{dx}{x-i\epsilon} - \int_{-\epsilon}^{\epsilon} \frac{dx}{x+i\epsilon} = 2i\int_{-1}^{1} \frac{dx}{x^2+1} = 2i(\arctan 1 - \arctan(-1)) = \pi i$$

- Same with left side + right side.
- What about pulling contours?

Complex Analysis

- What about pulling contours?
- Need things like Mellin transforms/inverses, e.g., Perron's formula:

$$\frac{1}{2\pi i}\int_{(2)} \frac{x^s}{s(s+1)}ds = \begin{cases} 0, & \text{if } x < 1 \\ 1 - \frac{1}{x}, & \text{if } x > 1 \end{cases}$$

- Human pf: if $x < 1$, pull contour to the left. Else pull right and pick up poles at s=0 and s=-1.
- Lean:

```
theorem Perron.formulaGtOne
        {x : ℝ} {σ : ℝ} (x_gt_one : 1 < x) (σ_pos : 0 < σ) :
    VerticalIntegral' (fun (s : ℂ) => ↑x ^ s / (s * (s + 1))) σ =
     1 - 1 / ↑x
```

- Pulling contours is adding/subtracting rectangles! + Limits

PNT+

- Pulling contours is adding/subtracting rectangles! + Limits
- Next goal: Using built up complex analysis, + “elementary” arguments (like partial summation), prove PNT with an error savings of $\exp(c(\log)^{1/10})$
- Recall: classical error replaces 1/10 by 1/2. (Best today: 3/5)
  Exercise: improve 3/5 to 1.
- Already better than any power of log, so plenty for any application
- After that: Dirichlet’s theorem, Chebotarev, …

Does any of this count as “research”??…

PNT+

- Want to contribute? Join us on Zulip, and/or Github, or just email me!
- Warning! This is a lot of fun. You might get sucked in...

Thank you!!